Space-charge-limited Flows

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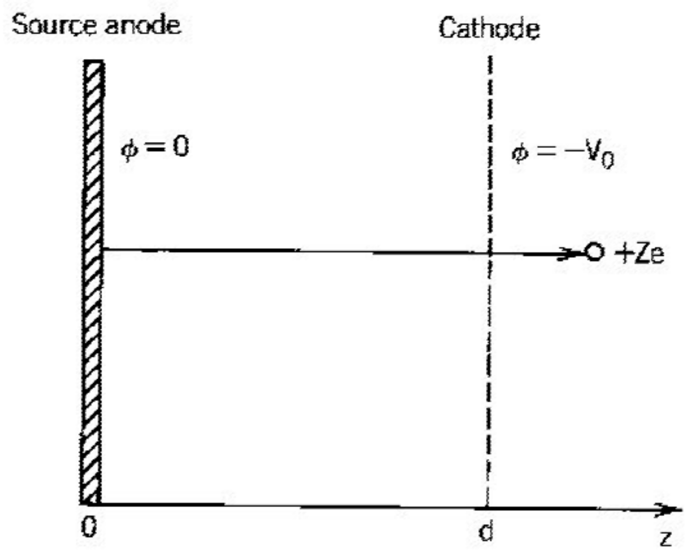
Child law: space-charge-limited (SCL) flow

- The Child law states the maximum current density that can be carried by charged particle flow across a one-dimensional extraction gap. The limit arises from the longitudinal electric fields of the beam space-charge.
- It is important because:
  - The Child limit gives the maximum current density from a charged particle extractor. Although the derivation applies to a specialized geometry, the results provide good estimates for a variety of high power beam devices.
  - The derivation of the Child law illustrates the calculation of a charged particle equilibrium with self-consistent space-charge fields.



Child law: space-charge-limited (SCL) flow

- Assumptions:
  - Particle motion is non-relativistic ($eV_0 \ll m_0 c^2$).
  - The source on the left-hand boundary supplies an unlimited flux of particles. Restrictions of flow result entirely from space-charge effects.
  - The transverse dimension of the gap is large compared with d. The only significant components of particle velocity and electric field are in the z direction.
  - The transverse magnetic force generated by current across the gap is small compared with the axial electric force. As a result, particles follow straight line trajectories across the gap. This assumption is valid for ion beams, but it is usually violated in high-current relativistic electron beam injectors.
  - Particles flow continuously — the electric fields and space-charge density at all positions in the gap are constant.
  - Ions are singly-charged.
  - Initial kinetic energy of particles at the emission surface is zero (or much smaller than the applied voltage).
  - Particles are collisionless in the gap.

Space-charge-limited (SCL) flow for an ideal diode

- The ion energy and flux conservation equations

$$\frac{1}{2}Mu^2(z) = -e\phi(z)$$

$$en(z)u(z) = J_0$$

$$\Rightarrow \quad n(z) = \frac{J_0}{e}\left(-\frac{2e\phi}{M}\right)^{-1/2}$$

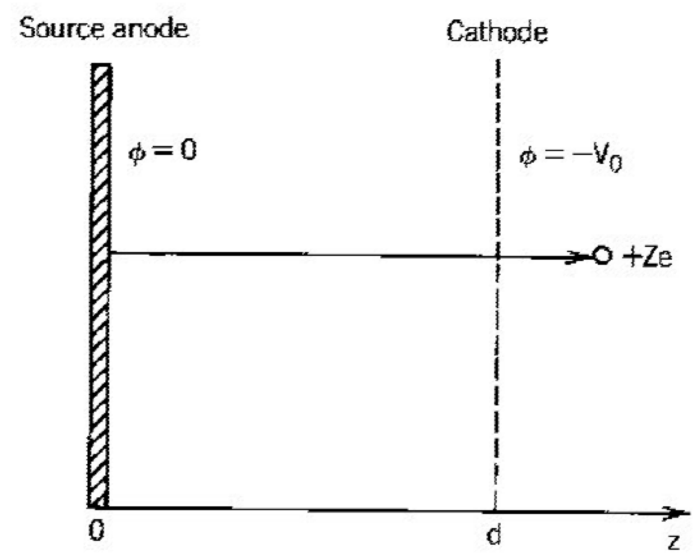
- Poisson's eq.

$$\frac{d^2\phi}{dz^2} = -\frac{e}{\epsilon_0}(n_i - n_e) = -\frac{J_0}{\epsilon_0}\left(-\frac{2e\phi}{M}\right)^{-1/2}$$

- Defining $\zeta = z/d$ and $\Phi = -\phi/V_0$, we obtain

$$\Phi'' = \alpha/\sqrt{\Phi}$$

$$\alpha = \frac{J_0 d^2}{\epsilon_0 V_0 \sqrt{2eV_0/M}}$$

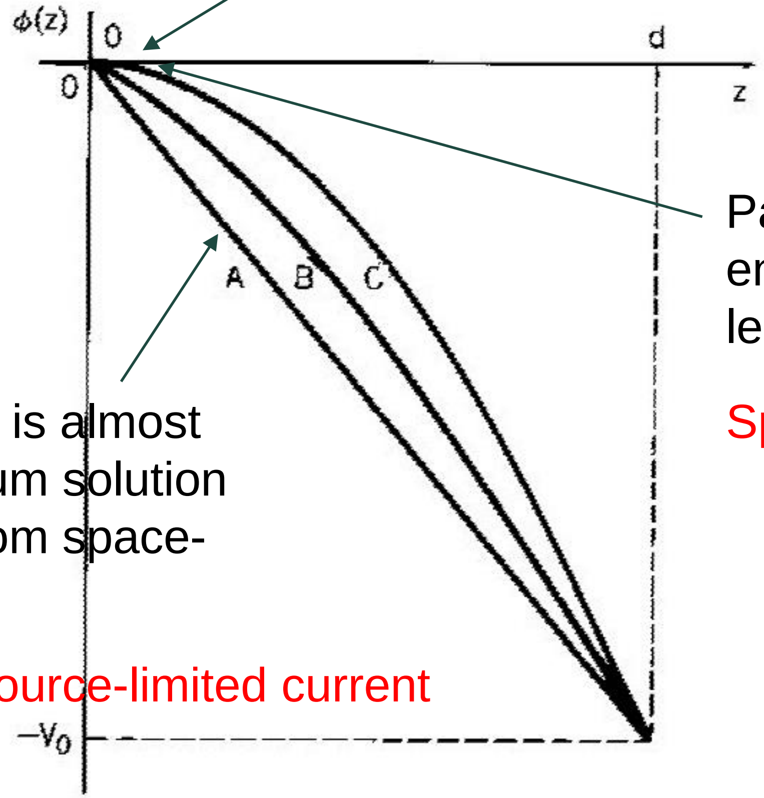


Space-charge-limited (SCL) flow for an ideal diode

- The boundary conditions for space-charge-limited current:

$$\Phi(0) = 0 \qquad \Phi(1) = 1 \qquad \boxed{\left.\frac{d\Phi}{d\zeta}\right|_{\zeta=0} = 0}$$

- The possible solutions:



Particles with low kinetic energy are just able to leave the source.

Space charge-limited current

The potential variation is almost the same as the vacuum solution with no contribution from space-charge.

Low current density: source-limited current

Space-charge-limited (SCL) flow for an ideal diode

- Multiplying by $2\Phi'$

$$\Phi'' = \alpha/\sqrt{\Phi} \quad \Longrightarrow \quad 2\Phi'\Phi'' = 2\alpha\Phi'/\sqrt{\Phi}$$

- Integrating both sides from 0 to 1 and applying the boundary conditions gives:

$$(\Phi')^2 = 4\alpha\sqrt{\Phi} \quad \Longrightarrow \quad \Phi' = \sqrt{4\alpha}\Phi^{1/4} \quad \Longrightarrow \quad d\Phi/\Phi^{1/4} = \sqrt{4\alpha}d\zeta$$

- Integrating the above equation, we obtain

$$\Phi^{3/4} = (3/4)\sqrt{4\alpha}\zeta \quad \xrightarrow[\Phi(1)=1]{} \quad \alpha = 4/9$$

- By definition,

$$\alpha = \frac{J_0 d^2}{\epsilon_0 V_0 \sqrt{2eV_0/M}} = \frac{4}{9}$$

- Finally, we obtain the space-charge-limited current density

$$\boxed{J_0 = \frac{4}{9}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{V_0^{3/2}}{d^2}}$$

Child law:
Space-charge-limited current in a plane diode

Space-charge-limited (SCL) flow for an ideal diode

- Potential, electric field and density within the gap

$$\phi = -V_0\left(\frac{z}{d}\right)^{4/3}$$

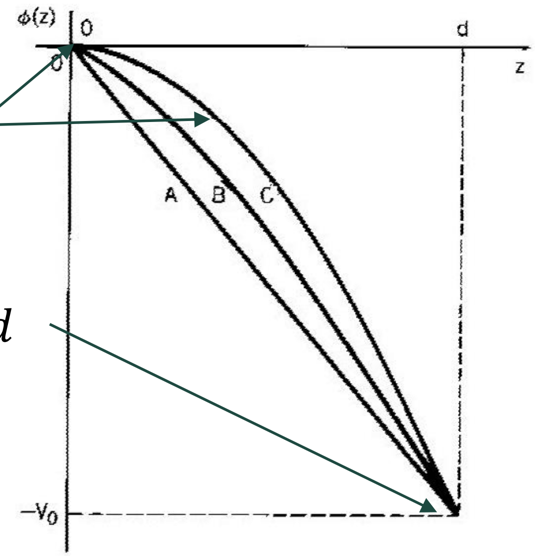
$$E = \frac{4}{3}\frac{V_0}{d}\left(\frac{z}{d}\right)^{1/3}$$

$$n = \frac{4}{9}\frac{\epsilon_0}{e}\frac{V_0}{d^2}\left(\frac{z}{d}\right)^{-2/3}$$

$$E_{min} = E(z=0) = 0$$

$$E_{max} = E(z=d) = 4V_0/3d$$

$$\boxed{J_0 = \frac{1}{4}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{E_{max}^2}{V_0^{1/2}}}$$



- Assuming that an ion enters the gap with initial velocity $u(0) = 0$

$$\frac{dz}{dt} = v_0\left(\frac{z}{d}\right)^{2/3}$$

where, v_0 is the characteristic ion velocity in the gap

$$v_0 = \left(\frac{2eV_0}{M}\right)^{1/2}$$

- Ion transit time across the gap

$$\frac{z(t)}{d} = \left(\frac{v_0 t}{3d}\right)^3 \implies \boxed{\tau_i = \frac{3d}{v_0}}$$

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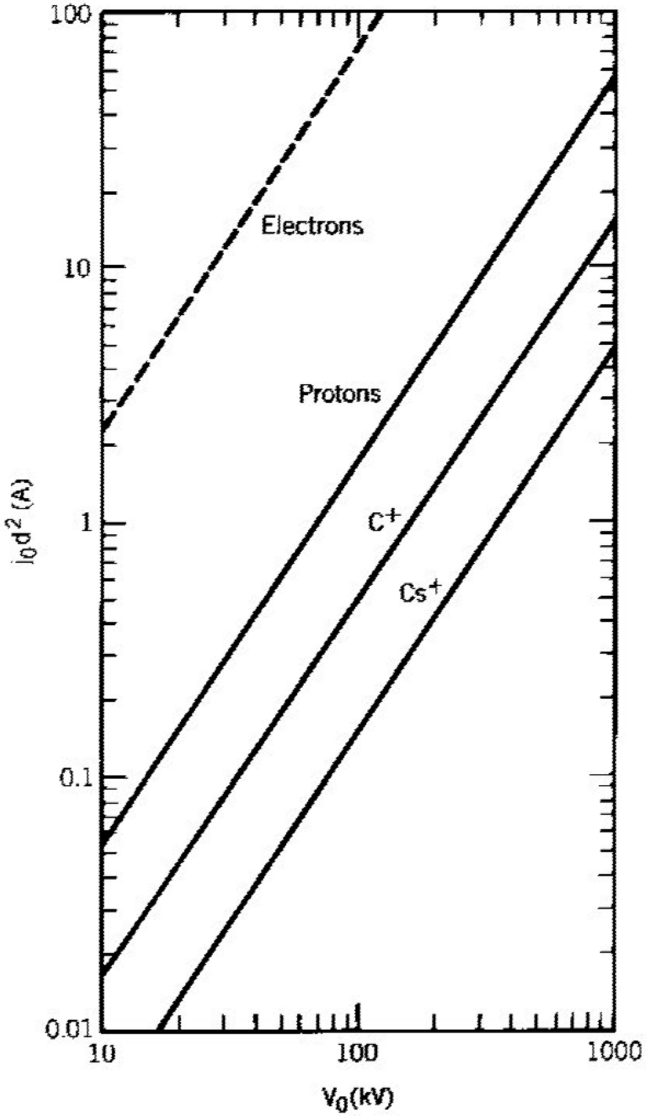
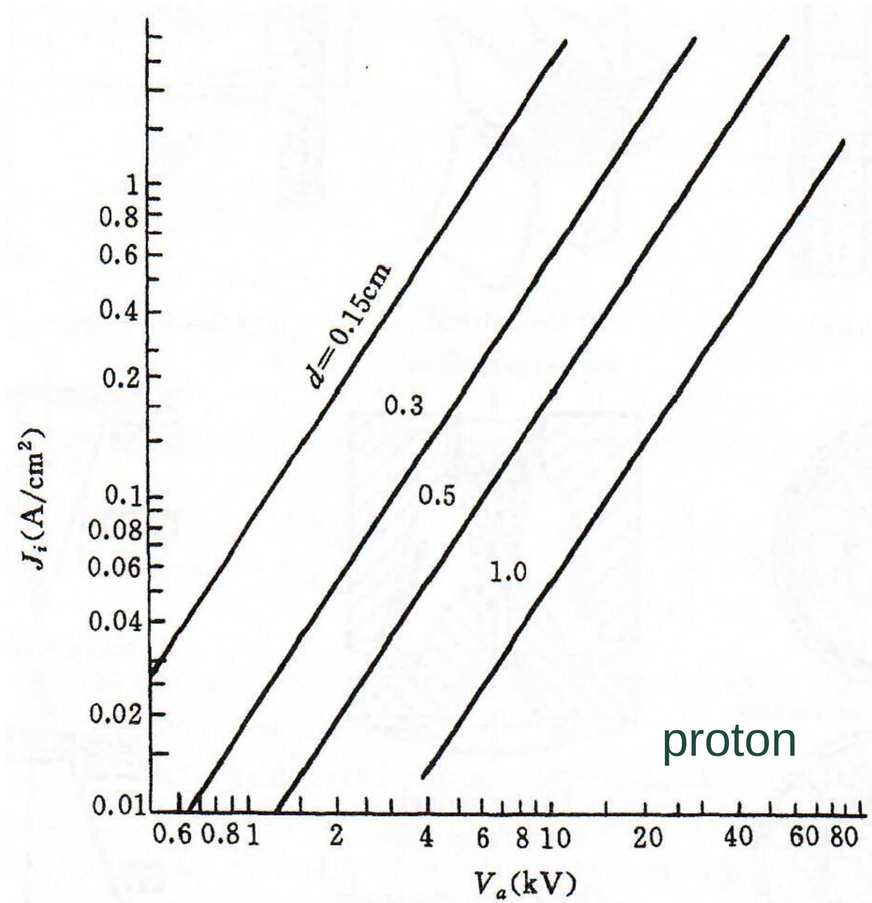
Extractable ion current density in a plane diode



$$J_0 = \frac{4}{9}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{V_0^{3/2}}{d^2}$$



SCL flow for an ideal diode: practical expression

- In practice,

$$J_i = 5.45 \times 10^{-8} \left(\frac{z}{A}\right)^{1/2} \frac{V_0^{3/2}}{d^2} \qquad (A/cm^2)$$

$$J_e = 2.34 \times 10^{-6} \frac{V_0^{3/2}}{d^2} \qquad (A/cm^2)$$

V_0: extraction voltage [V]
d: gap distance [cm]
z: ion charge
A: ion mass in amu

- The extractable currents from a round aperture with a radius a are given by

$$I_0 = \frac{4}{9}\pi\epsilon_0 \left(\frac{2e}{M}\right)^{1/2} \left(\frac{a}{d}\right)^2 V_0^{3/2}$$

$$I_i = 1.71 \times 10^{-7} \left(\frac{z}{A}\right)^{1/2} \left(\frac{a}{d}\right)^2 V_0^{3/2} \quad (A)$$

$$I_e = 7.35 \times 10^{-6} \left(\frac{a}{d}\right)^2 V_0^{3/2} \quad (A)$$

Space-charge-limited flow with multiple ion species

- Assuming that the multiple ion species with mass M_j, charge $q_j = z_j e$, and J_j, the Poisson's equation is given by

$$\nabla^2 V = -\frac{1}{\epsilon_0}\sum_j \rho_j = -\frac{1}{\epsilon_0}\sum_j J_j\sqrt{\frac{M_j}{2q_jV}} = -\frac{1}{\epsilon_0}J_t\sqrt{\frac{M_t}{2q_tV}}$$

- The reduced relationship is given by

$$\sqrt{\frac{M_t}{q_t}} = \sum_j a_j\sqrt{\frac{M_j}{q_j}} \qquad a_j = \frac{J_j}{J_t} = \frac{J_j}{\sum_j J_j}$$

- For plasma ion sources, the proportion of the extracted current of different species is not the same as that of the ion density of the different species in the plasma:

$$J_j \approx 0.4 n_j z_j e\sqrt{2z_j kT_e/M_j} \propto n_j z_j\sqrt{z_j/M_j}$$

- Total current density for the SCL emission is given by

$$J = 5.45\times 10^{-8}\left(\sum_j \mu_j\sqrt{\frac{z_j}{A_j}}\right)\frac{V_a^{3/2}}{d^2} \qquad (A/cm^2) \qquad \mu_j = \frac{n_j z_j}{\sum_j n_j z_j}$$

High-voltage sheath: SCL flow between plasma and negatively-biased electrode

- For a plasma $J_0 = en_s u_B$

$$\boxed{en_s u_B = \frac{4}{9}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{V_0^{3/2}}{s^2}}$$

- Child law sheath $\boxed{s = \frac{\sqrt{2}}{3}\left(\frac{\epsilon_0 \mathrm{T_e}}{en_s}\right)^{1/2}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{3/4} = \frac{\sqrt{2}}{3}\lambda_{Ds}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{3/4}}$

- Potential, electric field and density within the sheath

$$\Phi = -V_0\left(\frac{x}{s}\right)^{4/3} \qquad E = \frac{4}{3}\frac{V_0}{s}\left(\frac{x}{s}\right)^{1/3} \qquad n = \frac{4}{9}\frac{\epsilon_0}{e}\frac{V_0}{s^2}\left(\frac{x}{s}\right)^{-2/3}$$

- Assuming that an ion enters the sheath with initial velocity $u(0) = 0$

$$\frac{dx}{dt} = v_0\left(\frac{x}{s}\right)^{2/3}$$

where, v_0 is the characteristic ion velocity in the sheath

$$v_0 = \left(\frac{2eV_0}{M}\right)^{1/2}$$

- Ion transit time across the sheath

$$\frac{x(t)}{s} = \left(\frac{v_0 t}{3s}\right)^3 \Longrightarrow \boxed{\tau_i = \frac{3s}{v_0}}$$

Space-charge-limited flow with an initial injection energy

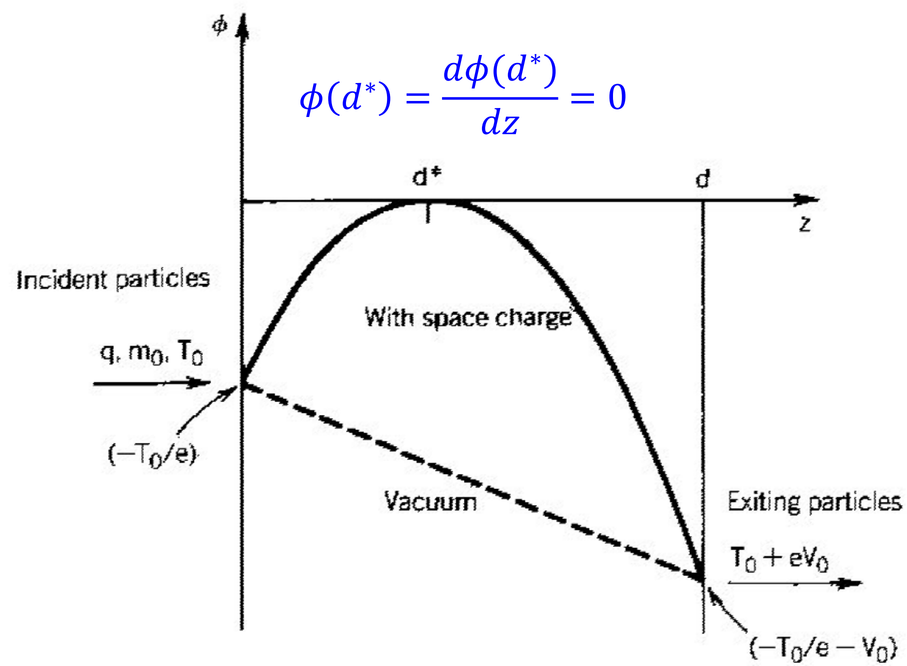
- We can extend the one-dimensional Child limit to describe acceleration gaps where particles enter with a non-zero energy. The results have application to the gaps of high-current multistage accelerators and the flow of electrons through grid-controlled devices like vacuum triodes.
- Because of their initial energy, particles can cross the gap even if the electric field at the entrance is negative.
- The solution is as following:

$$j_0 = \frac{4}{9}\epsilon_0 \left(\frac{2Ze}{m_0}\right)^{1/2} \frac{V_0^{3/2}}{d^2} F(\chi)$$

- The correction factor is

$$F(\chi) = \chi^{3/2}\left[\left(1 - \frac{1}{\chi}\right)^{3/4} + 1\right]^2$$

$$\chi = -\phi_2/(\phi_1 - \phi_2)$$



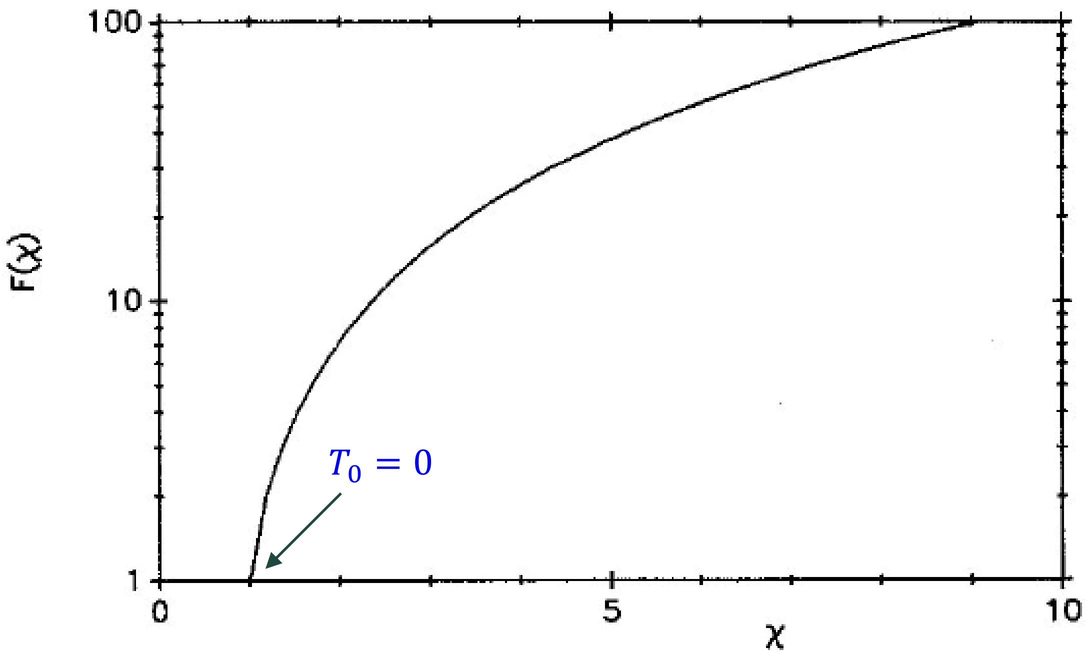
$\phi_1 = \phi(z = 0) = -T_0/e$ $\phi_2 = \phi(z = d) = -T_0/e - V_0$

Space-charge-limited flow with an initial injection energy

- The solution reduces to the standard Child law when the injection energy approaches zero ($\chi = 1$).
- The function $F(\chi)$ grows rapidly for increasing χ.
- The figure emphasizes that the longitudinal space-charge limit drops rapidly as particles accelerate. For example, the space-charge limit in a post-acceleration gap that doubles the energy of particles ($\chi = 2$) is 7.2 times the Child limit.

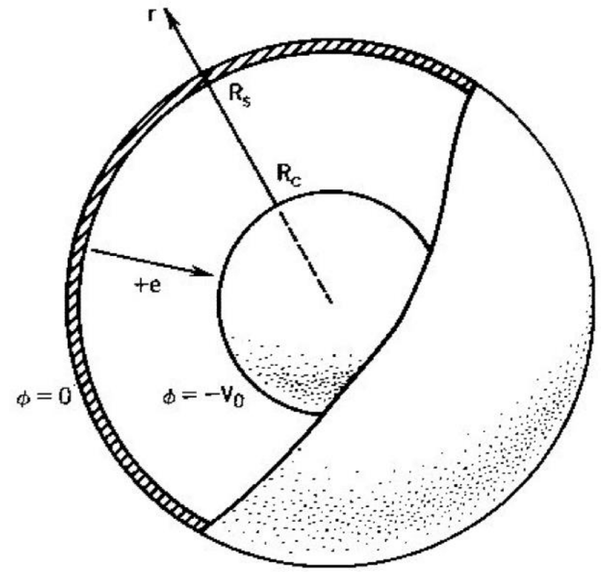
$$j_0 = \frac{4}{9}\epsilon_0 \left(\frac{2Ze}{m_0}\right)^{1/2} \frac{V_0^{3/2}}{d^2} F(\chi)$$

$$\chi = -\frac{\phi_2}{\phi_1 - \phi_2} = \frac{T_0/e + V_0}{V_0}$$



Space-charge-limited flow in spherical geometry

- The quantity R_s is the source radius, R_c is the collector radius, and r represents a radial position between the two. We can find two types of solutions: inward flow ($R_s > R_c$) and outward flow ($R_s < R_c$).
- For convenience we treat a steady-state flow of positive non-relativistic particles with mass m_0. We take the source potential equal to zero while the collector has bias voltage $-V_0$.



- The Poisson equation has the following form for a spherically symmetric potential:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{\rho}{\epsilon_0}$$

- Conservation of energy implies that the ion velocity is related to ϕ by:

$$v = \sqrt{-2e\phi/m_0}$$

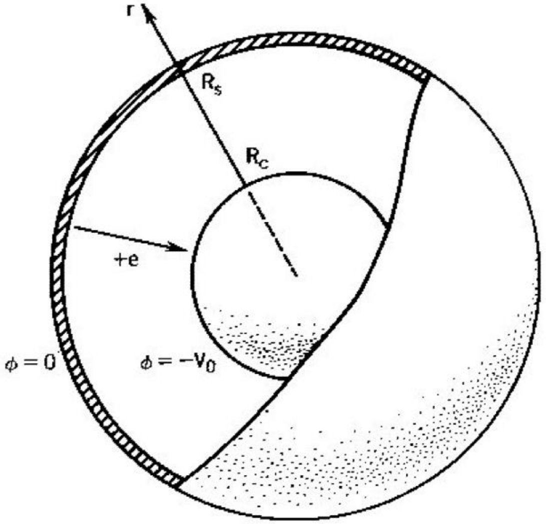
- In equilibrium, the radial current I is constant at all radii. This condition implies that the space-charge density is related to the velocity by

$$\rho = \frac{I}{4\pi r^2 v}$$

Space-charge-limited flow in spherical geometry

- We obtain the following self-consistent form for the Poisson equation:

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{I}{4\pi\epsilon_0}\sqrt{\frac{m_0}{-2e\phi}}$$



- We define the dimensionless potential and radius as

$$\Phi = -\frac{\phi}{V_0} \qquad R = \frac{r}{R_s}$$

- The Poisson equation becomes

$$\frac{d}{dR}\left(R^2\frac{d\Phi}{dr}\right) = -\frac{A}{\sqrt{\Phi}} \qquad A = \frac{I}{4\pi\epsilon_0 V_0^{3/2}}\sqrt{\frac{m_0}{2e}}$$

- The boundary conditions:

$$\Phi(R=1) = 0 \qquad \Phi(R = R_c/R_s) = 1 \qquad \left.\frac{d\Phi}{dR}\right|_{R=1} = 0$$

Space-charge-limited flow in spherical geometry

- Langmuir and Blodgett [Phys. Rev. 24, 49 (1924)] developed a well-known numerical solution. They assumed the following form for the electrostatic potential:

$$\Phi(R)^{3/2} = \frac{9A}{4}\alpha(R)^2 \qquad\qquad \alpha(R)\text{: Langmuir function}$$

- Defining the variable $\gamma = \ln R$, α can be obtained by solving the equation:

$$3\alpha\frac{d^2\alpha}{d\gamma^2} + \left(\frac{d\alpha}{d\gamma}\right)^2 + 3\alpha\frac{d\alpha}{d\gamma} - 1 = 0$$

- The series solution for α:

$$\alpha = \gamma - 0.3\gamma^2 + 0.075\gamma^3 - 0.0143182\gamma^4 + 0.0021609\gamma^5 - \cdots$$

- The total current:

$$I = \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m_0}}\frac{4\pi V_0^{3/2}}{\alpha^2} \qquad\qquad \alpha = \alpha(R_c/R_s)$$

Table: Langmuir function

Table 6.1. Langmuir function versus normalized radius — converging beam

$R = r/R_s$	α^2	$R = r/R_s$	α^2
1.0000	0.0000	0.1923	8.636
0.9524	0.0024	0.1852	9.135
0.9091	0.0096	0.1786	10.01
0.8696	0.0213	0.1724	10.73
0.8333	0.0372	0.1667	11.46
0.8000	0.0571	0.1538	13.35
0.7692	0.0809	0.1429	15.35
0.7407	0.1084	0.1333	17.44
0.7143	0.1396	0.1250	19.62
0.6897	0.1740	0.1176	21.89
0.6667	0.2118	0.1111	24.25
0.6250	0.2968	0.1053	26.68
0.5882	0.394	0.1000	29.19
0.5556	0.502	0.0833	39.98
0.5623	0.621	0.0714	51.86
0.5000	0.750	0.0625	64.74
0.4762	0.888	0.0556	78.56
0.4545	1.036	0.0500	93.24
0.4348	1.193	0.0333	178.2
0.4167	1.358	0.0250	279.6
0.4000	1.531	0.0200	395.3
0.3846	1.712	0.0167	523.6
0.3704	1.901	0.0143	663.3
0.3571	2.098	0.0125	813.7
0.3448	2.302	0.0111	974.1
0.3333	2.512	0.0100	1144
0.3125	2.954	0.0083	1509
0.2941	3.421	0.0071	1907
0.2778	3.913	0.0063	2333
0.2632	4.429	0.0056	2790
0.2500	4.968	0.0050	3270
0.2381	5.528	0.0040	4582
0.2273	6.109	0.0033	6031
0.2174	6.712	0.0029	7610
0.2083	7.334	0.0025	9303
0.2000	7.976	0.0020	13015

Table 6.2. Langmuir function versus normalized radius — diverging beam

$R = r/R_s$	α^2	$R = r/R_s$	α^2
1.0	0.0000	6.5	1.385
1.05	0.0023	7.0	1.453
1.1	0.0086	7.5	1.516
1.15	0.0180	8.0	1.575
1.2	0.0299	8.5	1.630
1.25	0.0437	9.0	1.682
1.3	0.0591	9.5	1.731
1.35	0.0756	10	1.777
1.4	0.0931	12	1.938
1.45	0.1114	14	2.073
1.5	0.1302	16	2.189
1.6	0.1688	18	2.289
1.7	0.208	20	2.378
1.8	0.248	30	2.713
1.9	0.287	40	2.944
2.0	0.326	50	3.120
2.1	0.364	60	3.261
2.2	0.402	70	3.380
2.3	0.438	80	3.482
2.4	0.474	90	3.572
2.5	0.509	100	3.652
2.6	0.543	120	3.788
2.7	0.576	140	3.903
2.8	0.608	160	4.002
2.9	0.639	180	4.089
3.0	0.669	200	4.166
3.2	0.727	250	4.329
3.4	0.783	300	4.462
3.6	0.836	350	4.573
3.8	0.886	400	4.669
4.0	0.934	500	4.829
4.2	0.979	600	4.960
4.4	1.022	800	5.165
4.6	1.063	1000	5.324
4.8	1.103	1500	5.610
5.0	1.141	2000	5.812
5.2	1.178	5000	6.453
5.4	1.213	10000	6.933
5.6	1.247	30000	7.693
5.8	1.280	100000	8.523

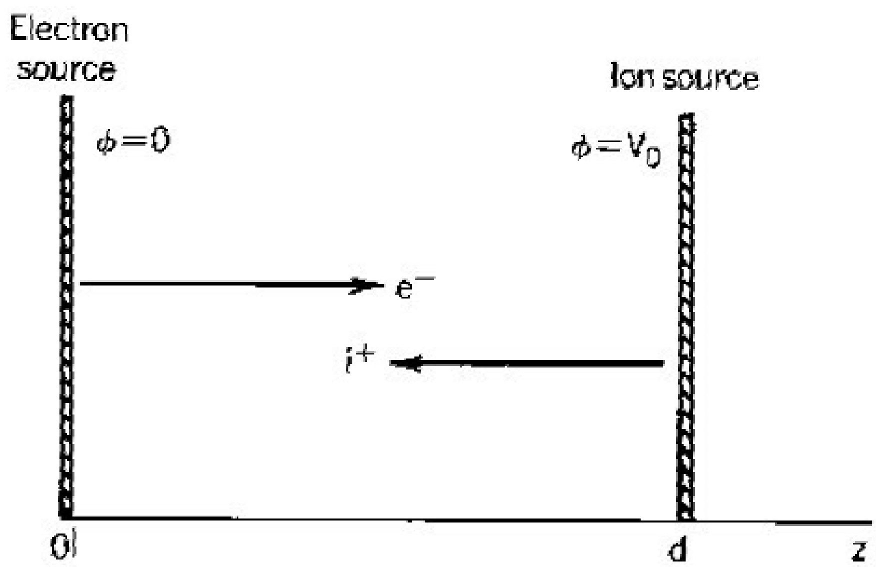
Bipolar flow

- The term bipolar flow refers to the simultaneous space-charge-limited flow of ions and electrons emitted from opposite sides of an acceleration gap.

- Charge conservation for ions and electrons:

$$n_e(\phi) = \left(\frac{j_e}{e}\right)\sqrt{\frac{m_e}{2e\phi}}$$

$$n_i(\phi) = \left(\frac{j_i}{e}\right)\sqrt{\frac{m_i}{2e(V_0 - \phi)}}$$



- In terms of dimensionless variables:

$$\Phi = \frac{\phi}{V_0} \qquad Z = \frac{z}{d}$$

- The Poisson equation:

$$\frac{d^2\Phi}{dZ^2} = A\left[\frac{1}{\sqrt{\Phi}} - \frac{1}{\sqrt{1-\Phi}}\left(\frac{j_i}{j_e}\sqrt{\frac{m_i}{m_e}}\right)\right] \qquad A = \frac{j_e d^2}{\epsilon_0\sqrt{2e/m_e}V_0^{3/2}}$$

Bipolar flow

- Multiply both sides by $2\Phi'$ and integrating the equation from 0 to Z gives the expression:

$$\left(\frac{d\Phi}{dZ}\right)^2 = 4A\left[\sqrt{\Phi} + (\sqrt{1-\Phi} - 1)\left(\frac{j_i}{j_e}\sqrt{\frac{m_i}{m_e}}\right)\right]$$

$$\alpha = \frac{j_i}{j_e}\sqrt{\frac{m_i}{m_e}}$$

- The boundary conditions:

$$\Phi(Z=0) = 0 \qquad \Phi(Z=1) = 1 \qquad \left.\frac{d\Phi}{dZ}\right|_{Z=0} = \left.\frac{d\Phi}{dZ}\right|_{Z=1} = 0$$

- The solution for space-charge-limited bipolar flow:

$$j_{e,bipolar} = \frac{4\epsilon_0}{9}\left(\frac{2e}{m_e}\right)^{1/2}\frac{V_0^{3/2}}{d^2}\left[\frac{3}{4}\int_0^1 \frac{d\Phi}{\sqrt{\Phi^{1/2} + (1-\Phi)^{1/2} - 1}}\right]^2$$

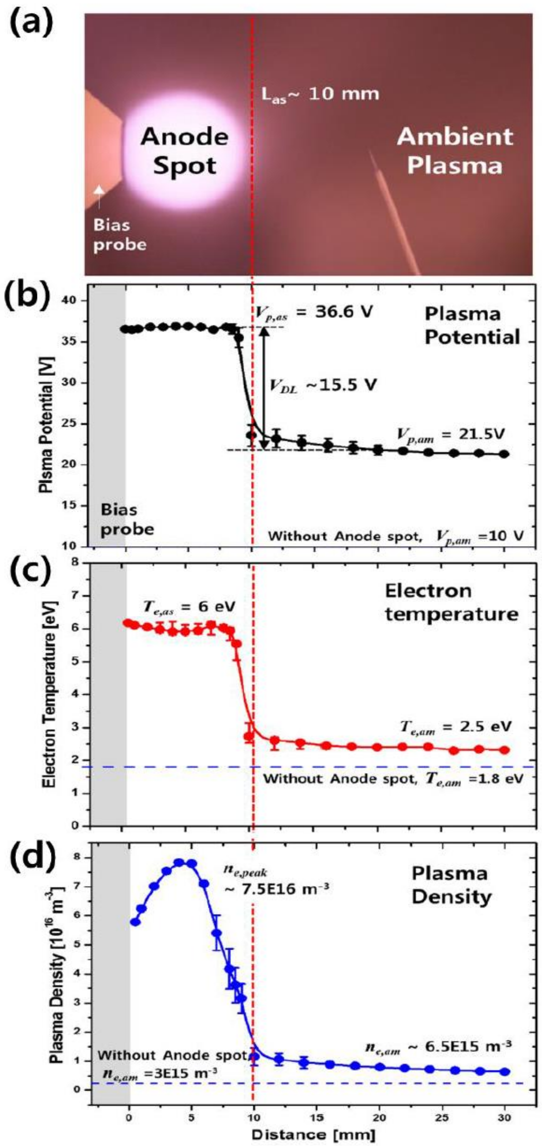
$j_i/j_e = \sqrt{m_e/m_i}$ → Langmuir condition

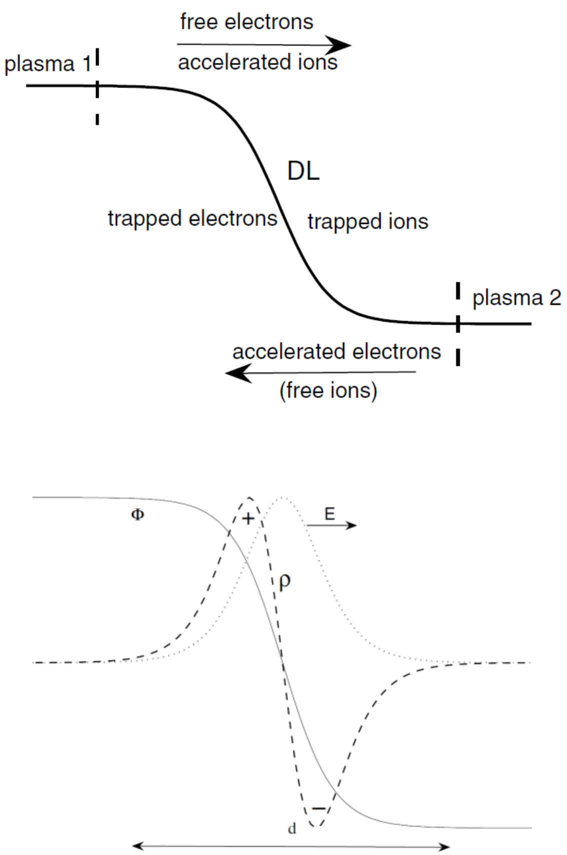
- Or simply:

Enhancement factor

$$j_{e,bipolar} = 1.86\, j_{e,Child} \qquad j_{i,bipolar} = 1.86\, j_{i,Child}$$

Double layer: bipolar flow between two plasmas





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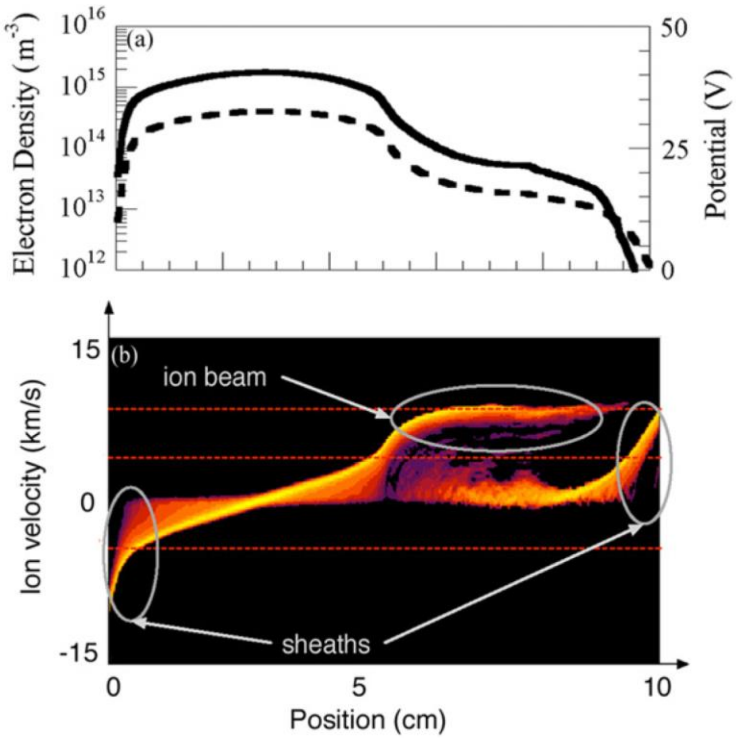

Y. Lee, Ph.D. Dissertation, SNU (2017)

C. Charles, PSST 16, R1 (2007)

Space-charge-limited flow of relativistic electrons

- Single gap electron extractors have been operated in the megavolt range ($eV_0 > m_e c^2$). These extractors are driven either by electrostatic pulsed power generators or by stacked induction linac cavities.
- Consider a one-dimensional gap with applied voltage V_0 and width d. If we set the electrostatic potential ϕ equal to zero at the cathode, conservation of energy gives the following expression for the relativistic gamma factor for electrons in the extractor:

$$\gamma(z) = 1 + e\phi(z)/m_e c^2$$

- For electron motion in the z direction, the axial velocity and electron density are related to γ by

$$v_z = c\sqrt{\gamma^2 - 1}/\gamma \qquad n_e(\gamma) = \gamma j_e/(ec\sqrt{\gamma^2 - 1})$$

- We obtain the one-dimensional Poisson equation:

$$\frac{d^2\gamma}{dz^2} = \left[\frac{ej_e}{\epsilon_0 m_e c^3}\right]\frac{\gamma}{\sqrt{\gamma^2 - 1}}$$

- B.C.s: $\gamma(z = 0) = 1 \qquad \gamma(z = d) = \gamma_0 = 1 + eV_0/m_e c^2 \qquad \left.\frac{d\gamma}{dz}\right|_{z=0} = 0$

Space-charge-limited flow of relativistic electrons

- The space-charge-limited current density for relativistic electrons:

$$j_e = \left[\frac{\epsilon_0 m_e c^3}{e d^2}\right] \frac{G(\gamma_0)}{2} \qquad G(\gamma_0) = \int_1^{\gamma_0} \frac{d\zeta}{(\zeta^2 - 1)^{1/4}}$$

- Ultra-relativistic limit ($eV_0 \gg m_e c^2$):

$$\frac{d^2\phi}{dz^2} = \frac{e n_e}{\epsilon_0} \approx \frac{j_e}{\epsilon_0 c}$$

$$\Rightarrow \quad j_e = \frac{2\epsilon_0 c V_0}{d^2}$$

- Non-relativistic limit ($eV_0 \ll m_e c^2$):

$$j_e = \frac{4}{9}\epsilon_0 \left(\frac{2e}{m_e}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$