$$\delta W = Q_i \delta q_i .(i = 1..n)$$

A generalized force Q_j contributes to δW only if the corresponding generalized coordinate q_j is given a virtual displacement. (independent !)

: Virtual work δW of the actual forces for each individual variation of only the generalized coordinates at a time.

Since the transformations are invertible, a single variation δq_j will induce a simultaneous variation of one or more of the physical coordinates.

A virtual displacement of a generalized coordinate in physical space ~

A combination of virtual displacements subjected to the constraints of

the system.

Generally, the corresponding virtual work done by the physical components of the forces can be computed and set equal to $Q_{(i)}\,\delta q_{(i)}$.

Example: Figs.2.9, 2.10

Consider a spring-loaded cart (Fig.2.4) with a swinging pendulum attached to it.

This system has two degrees of freedom. Chosen x and θ as generalized coordinates.

Since x and θ are independent variables,

$$\delta x = \delta x \, , \qquad \delta\theta = 0 \tag{2.34}$$

And

$$\delta x = 0 \, , \qquad \delta\theta = \delta\theta \tag{2.35}$$

are two sets of admissible virtual displacements.

Now compute the corresponding virtual work done by the external forces under each of the designated virtual displacements :

If $\delta x \neq 0$ **and** $\delta\theta = 0$

$$\delta W = -F_s \delta x \tag{2.36}$$

If $\delta x = 0$ **and** $\delta\theta \neq 0$ **(Fig. 2.10),**

$$\delta W = -mgl\sin\theta\delta\theta \qquad \textbf{(2.38)}$$

$$\therefore Q_{\theta} = -mgl\sin\theta \text{ (torque)} \qquad \textbf{(2.39)}$$

Then for an arbitrary combination of virtual displacements, the total virtual work is

$$\delta W = -kx\delta x - mgl\sin\theta\delta\theta \qquad \textbf{(2.40)}$$

Note :

<u>Physical interpretation</u> of a generalized force depends on the significance of the related generalized coordinate.

Once a given set of generalized coordinates are specified, the generalized forces can in principle always be determined, regardless of the physical interpretation of the generalized coordinates.

~ Holonomic systems, the computation of generalized forces is very simple. :

Virtual work done by holonomic constraint forces under a set of arbitrary virtual displacements compatible with the constraints is equal to zero.

Therefore, in the computation of generalized forces, only the applied forces need to be considered.

This results in a considerable benefit in the formulation of the equations of motion in terms of the generalized coordinates.

Special consideration may be given to conservative forces. Suppose that all the forces acting on a system of *N* particles are conservative. Each physical force is derivable from a potential function.

Suppose: a single potential function:

$$V = V(x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_N, y_N, z_N) \tag{2.41}$$

The force on the *i*-th particle may be obtained as

$$F_i = -\nabla_i V \tag{2.42}$$

Where the gradient ∇_i denote the operator

$$\nabla_i = \frac{\partial}{\partial x_i}\hat{\mathrm{i}} + \frac{\partial}{\partial y_i}\hat{\mathrm{j}} + \frac{\partial}{\partial z_i}\hat{\mathrm{k}}$$

Substituting the physical components of the forces (2.42) into Equation (2.31) results in the characterization of the virtual work as the negative of the variation of the potential function :

$$\delta W = -\sum_{i=1}^{N} \left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i \frac{\partial V}{\partial z_i}\delta z_i\right) = -\delta V \qquad \textbf{(2.44)}$$

Thus the virtual work done by a collection of conservative forces, under specified virtual displacements, is given as the negative of the variation of potential energy.

Principle of Virtual Work:

A conservative system is in static equilibrium iff the

total potential energy of the system is stationary

$$\delta V = 0 \qquad \textbf{(2.45)}$$

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Force on *i*-th particle may be obtained as

$$F_i = -\nabla_i V \qquad \textbf{(2.42)}$$

in here

$$\nabla_i = \frac{\partial}{\partial x_i}\hat{\mathrm{i}} + \frac{\partial}{\partial y_i}\hat{\mathrm{j}} + \frac{\partial}{\partial z_i}\hat{\mathrm{k}} \quad \equiv$$

Substitute Eqn.(2.42) into δW **: Eqn.(2.31) ,**

$$\delta W = -\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i \frac{\partial V}{\partial z_i}\delta z_i\right) = -\delta V \qquad \textbf{(2.44)}$$

: Virtual work done by conservative forces ~

Negative of the variation of potential energy.

<u>Principle of Virtual Work:</u> **(MINI ? MAX ?)**

$$\delta V = 0$$

Principle of Stationary Potential Energy ~

Necessary and sufficient condition for static equilibrium

of a conservative system.

Transforming to generalized coordinates :

Total P.E. of a conservative system as

$$V = V(q_1, q_2, \cdots, q_n) \qquad \textbf{(2.46)}$$

Consequently, the variation of the P.E. function in terms

of δq_j **is : Eqn(2.47)**

or

$$\delta W = -\sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \equiv -V,_{q_j} \delta q_j \quad (\,for\ j=1..n) \qquad \textbf{(2.48)}$$

For a conservative system :

Generalized forces ~ also derivable from a potential

function in terms of the generalized coordinates q_j .

That is,

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{2.49}$$

Therefore the determination of generalized forces for conservative systems is very easy (?)

Using transformation of coordinates

~ As a final step in the derivation of equations of motion

After the change of variables has been consummated, we will only need to keep the final result.

LAGRANGE'S EQUATIONS OF MOTION

Up to now, we consider the connection between physical variable and generalized coordinates based on the geometric configuration of a system (admissible !). Especially, generalized coordinates compatible with the constraints make the kinematics much more manageable for holonomic systems

We are now in a position to make the transition between vector mechanics and analytical mechanics.

Instead of using free - body diagrams :

Based on the variation of energy and the minimum number of coordinates needed to characterize the dynamics of the system (always possible ?).

: Lagrangian dynamics !!

Kinetic energy, potential energy, and virtual work are all scalar quantities. Thus, the transformation of these quantities is rather straightforward.

Based on a system q_j instead of the physical coordinates r_i.

- A unified approach in a way that is *independent* of any

particular coordinate system or set of generalized coordinates.

For a system of *N* particles subjected to only holonomic constraints. The more general case will be considered later.

Assume a system with *n* degrees of freedom and that there is a transformation :

For the *i* th particle in a vector form as

$$m_i \mathbf{a}_i = \mathbf{F}_i \tag{2.50}$$

or

$$\frac{d\mathrm{p}_i}{dt} = F_i \tag{2.51}$$

: linear momentum of the *i*-th particle as

$$\mathrm{p}_i = m_i \dot{\mathrm{r}}_i \tag{2.52}$$

Find out how the equations of motion transform under

the transformation to generalized coordinates.

$\frac{d}{dt}(..)$ **? :**

Generalized momentum corresponding to the *k* th generalized coordinate is given by

$$\dot{p}_k = \frac{d}{dt}(p_k) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) \qquad \textbf{(2.53)}$$

By definition, the total kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \qquad \textbf{(2.54)}$$

then the generalized momentum p_k as

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_k} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_k}\right) \qquad \textbf{(2.55)}$$

Remember the chain rule :

$$\dot{x}_i = \sum_{j=1}^{N} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t}$$

Then take derivative wrt $\dot{q}_k$:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k} \qquad \textbf{(2.57)}$$

Thus, each component p_k can be expressed as Eqn.(2.58)

Taking the total time derivative of Eqn (2.58) and applying the product rule to the terms in the summation

(Remember : $\frac{d}{dt}(x\,\dot{y}) = \dot{x}\,\dot{y} + x\,\ddot{y}$ **)**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_{i=1}^{N} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k}\right)$$

$$+\sum_{i=1}^{N} m_i \left[\dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) + \dot{y}_i \frac{d}{dt}\left(\frac{\partial y_i}{\partial q_k}\right) + \dot{z}_i \frac{d}{dt}\left(\frac{\partial z_i}{\partial q_k}\right)\right]$$

Remind the terms in the first summation as

the Newton's Second Law

$$m_i \ddot{x}_i = F_{ix} \qquad m_i \ddot{y}_i = F_{iy} \qquad m_i \ddot{z}_i = F_{iz}$$

Thus the terms can be rewritten as

$$\sum_{i=1}^{N} m_i (\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k}) = \sum_{i=1}^{N} (F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k})$$

where the right-hand side ~ generalized force Q_k given by the transformation equations.

To interpret the second summation terms in Eqn(2.59),

note that

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 x_i}{\partial t \partial q_k}$$

$$= \frac{\partial}{\partial q_k}\left[\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}\right] = \frac{\partial}{\partial q_k}\left[\dot{x}_i\right] \equiv \dot{x}_{i,k}$$

Thus the time rate of change of the *k*th generalized momentum is given by Eqn(2.60)

Finally, the equations of motion in terms of q_k :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.61)}$$

: General form of Lagrange's Equations of Motion

There is one equation corresponding to each q_k.

The system of equations represents a coupled system of