LAGRANGIAN SYSTEMS

Most dynamics problems ~ Holonomic !

(: Not all systems are conservative)

A conservative force - Derivable from a potential energy

(Depending only on the spatial coordinates of a system)

- Lagrangian can be constructed and the dynamics of the system is contained in the Lagrangian.

But there may still be a scalar function from which the generalized components of a force may be derived.

Suppose :

A scalar function $V(q,\dot{q},t)$ **for a generalized force** Q_k **as in Text !**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \ :$$

$$T \sim> \mathrm{V}(\mathrm{q}, \dot{\mathrm{q}}, \mathrm{t})$$

(2.65)

~ **We call** $V(q,\dot{q},t)$ **as a <u>generalized potential function</u>.**

Substituting the generalized force (2.65), then

Lagrange's equations (2.61) results in

$$\frac{d}{dt}\left[\frac{\partial T(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial T(q,\dot{q},t)}{\partial q_k}=\frac{d}{dt}\left[\frac{\partial V(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial V(q,\dot{q},t)}{\partial q_k} \qquad \textbf{(2.66)}$$

Now, we can still define a Lagrangian function in terms

of the kinetic energy of the system and the *generalized* potential function as $L(q,\dot{q},t) = T(q,\dot{q},t) - V(q,\dot{q},t)$

By setting all terms to the left-hand side,

$$\frac{d}{dt}\left[\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_k}\right] - \frac{\partial L(q,\dot{q},t)}{\partial q_k} = 0 \qquad \textbf{(2.67)}$$

: Identical to Lagrange's eqns for conservative systems.

Note : Unless the potential function depends *only* on the

generalized coordinates, the system governed by Equation (2.67) is *not* conservative.

Holonomic systems derivable from a generalized potential function $V(q, \dot{q}, t)$ are known as Lagrangian systems.

A well-known example of a velocity-dependent potential

- A charged particle in an electromagnetic field.

The force on the particle is given by

$$-e\nabla\phi - \frac{e}{c}\{\dot{A} - v \times curl A\}$$

: e **- charge carried by the particle,**

ϕ **scalar potential,**

A **vector potential of the field.**

The electromagnetic force field is derivable from the generalized potential

$$V(r,\dot{r}) = e\phi(r) - \frac{ev\cdot A}{c}$$

Not all systems are Lagranian, although all generalized forces ~ Conservative or Non-conservative

: Depending on the nature of the actual forces acting on a system.

The virtual work done by a generalized force Q_k under a

virtual displacement δq_k can be considered

$$Q_k \delta q_k = \delta W_k^{cons} + \delta W_k^{nc}$$

Resultant generalized force associated with a generalized coordinate q_k can thus be spilt into two contributions:

$$Q_k = Q_k^{cons} + Q_k^{nc}$$

Using conservative component such as potential function,

Then, each generalized force may be decomposed as

$$Q_k = -\frac{\partial V}{\partial q_k} + Q_k^{nc}$$

Construct the Lagrangian function $L = T - V$ **and formulate Lagrange's equations of motion, in hybrid form, as**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k^{nc} \qquad k = 1, 2, \ldots, n$$

Here Q_k^{nc} represent generalized forces *not derivable* from a potential function.

DISSIPATIVE SYSTEMS

Are all forces derivable from a potential function ?

~ Forces due to dissipation of energy :

friction force is non-conservative,

Nevertheless, some non-conservative generalized forces

may still be derivable from yet another scalar function.

- **Components proportional to the velocities of the particles**

$$F_{ix} = -c_{x_i}\dot{x}_i\,, \qquad F_{iy} = -c_{y_i}\dot{y}_i\,, \qquad F_{iz} = -c_{z_i}\dot{z}_i$$

The virtual work done by these dissipative forces under a set of virtual displacements is

$$\delta W = \sum_i \mathrm{F} \bullet \delta \mathrm{r} = -\sum_{i=1}^{N} (c_{x_i} \dot{x}_i \delta x_i + c_{y_i} \dot{y}_i \delta y_i + c_{z_i} \dot{z}_i \delta z_i)$$

$$= -\sum_{i=1}^{N} \left[\sum_{k=1}^{n} (c_{x_i} \dot{x}_i \frac{\partial x_i}{\partial q_k} + c_{y_i} \dot{y}_i \frac{\partial y_i}{\partial q_k} + c_{z_i} \dot{z}_i \frac{\partial z_i}{\partial q_k}) \delta q_k \right]$$

$$= -\sum_{k=1}^{n} \left[\frac{1}{2} \sum_{k=1}^{N} \frac{\partial}{\partial \dot{q}_k} (c_{x_i} \dot{x}_i^{\,2} + c_{y_i} \dot{y}_i^{\,2} + c_{z_i} \dot{z}_i^{\,2}) \right] \delta q_k$$

Generalized forces associated with the dissipation forces

$$Q_k^{nc} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial \dot{q}_k} (c_{x_i} \dot{x}_i^2 + c_{y_i} \dot{y}_i^2 + c_{z_i} \dot{z}_i^2)$$

$$= -\frac{1}{2}\frac{\partial}{\partial \dot{q}_k}\sum_{i=1}^{N} \left(c_{x_i}\dot{x}_i^2 + c_{y_i}\dot{y}_i^2 + c_{z_i}\dot{z}_i^2\right)$$

…

Now, define a scalar function for *generalized velocities*

$$D = \frac{1}{2}\sum_{i=1}^{N}\left(c_{x_i}\dot{x}_i^2 + c_{y_i}\dot{y}_i^2 + c_{z_i}\dot{z}_i^2\right)$$

Thus the dissipative generalized forces in terms of *D* :

$$\delta W = \sum_{k=1}^{n} Q_k^{nc} \delta q_k = -\sum_{k=1}^{n} \frac{\partial D}{\partial \dot{q}_k} \delta q_k$$

D **: Rayleigh's Dissipation Function**

Finally, the most general form of Lagrange's equations of motion as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k^* \quad : (2.68)$$

where $L = T - V$ **: Lagrangian,** ***D*** **: Dissipation function**

Q_k^* **: Generalized force** ***not derivable from a potential function or a dissipation function*****.**

Note : Rayleigh's dissipation function ~ one-half the rate at dissipated energy : average loss of power in a non-conservative system.

Ex : A simple spring-mass system as in Fig.2.13

Additional loading with a viscous damper, a harmonica-lly applied forcing function

Lagrangian of the system ? $L = T - V$

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2$$

The dissipation function for viscous damper:

$$D = \frac{1}{2} c\dot{q}^2$$

Another generalized force ~ applied harmonic force

Substituting into Eqn (2.68) :

$$\frac{d}{dt}(m\dot{q}) - (-kq) + c\dot{q} = A\cos\omega_f t$$

which can be put into standard form as

$$m\ddot{q} + c\dot{q} + kq = A\cos\omega_f t$$

Ex: Simple pendulum with pin friction as in Fig 2.14

Assume : Pin exerts a *resisting* moment proportional to the angular velocity of the pendulum:

$$M_f = -\nu\dot{\theta}$$

Instantaneous rate of energy loss

$$P = M_f \dot{\theta}$$

Dissipation function (average power lost) is

$$D = \frac{1}{2}\nu\dot{\theta}^2$$

Since $Q_\theta^* = 0$**, the equation of motion is**

$$\frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) + \nu\dot{\theta} = 0$$

FORCES OF CONSTRAINT

Lagrangian formalism:Highlighted by two main features

It has been demonstrated !

Part of the advantage: Constraint forces do no virtual work under a set of virtual displacements compatible with the constraints.

<u>Constraints reduce the number of degrees of freedom.</u>

The constraint forces themselves do not appear in the equations of motion : Symmetry of a system ??

Holonomic systems can be described by a set of

independent generalized coordinates free of constraints.

Systems with non-holonomic constraints cannot be reduced to independent generalized coordinates.

The equations of motion must be augmented by the Constraints ~> Forces of constraint are also established.

Constraint forces in holonomic systems may also be

analyzed.

Only realize that constraints are enforced by reacting forces in the directions normal to the constraint surfaces

Physically, a constraint must be imposed in the form of forces or moments. Thus we associate constraints with additional generalized forces acting on the system.

These forces depend on the motion and cannot be found prior to solving the equations of motion.

Each holonomic constraint can in principle be replaced by a *reacting constraint force*. Additional degrees of freedom may be introduced onto the problem by adding generalized coordinates corresponding to the violation of

the constraints.

These additional coordinates are called superfluous coordinates. The generalized forces associated with the superfluous coordinates are the forces of constraint. If the original coordinates and the extra coordinates are considered as independent then the resulting equations of motion will contain the constraint forces.

1.(5)(a) Systematically state the advantages & drawback of the 'Newtonian Dynamics' and 'Analytical Dynamics'.

(5) (b) Explain, 'Actual' and 'Virtual' displacements with a diagram.

(5) (c) Explain with a physical meaning, 'Lagrange's equation is invariant under a coordinate transformation for system.'

2. Given a system with generalized coordinates q_1, q_2 and the constraint equation.

(15) $$\left(3q_1 \sin q_2 + \frac{q_2^2}{q_1} + 2\right) dq_1 + \left(q_1^2 \cos q_2 + 2q_2\right) dq_2 = 0$$

Determine whether the constraint is holonomic or not.

3. A position of a particle of mass m is given by the Cartesian coordinates (x, y, z).

Assuming a potential energy function $V = \frac{1}{2}k(x^2 + y^2 + z^2)$ and a constraint described by the equation $2\dot{x} + 3\dot{y} + 4\dot{z} + 5 = 0$, Find,

(5)(a) Differential equations of motion.

(20)(b) Velocity of the moving constraint.

4. Consider the motion of a particle of mass m which is constrained to move on the surface of a cone of half-angle α and which is subject to a gravitational force g. Let the axis of the cone correspond to the $z-axis$ and let the apex of the cone be located at the origin as in the Figure.

(5)(a) For <u>general</u> system, state the categories of constraints.

(5)(b) Obtain the constraint for this problem. And, choose a category in (a).

(10)(c)Obtain Lagrange's equation for radius r, and determine general solution.

1. (5)(a) Draw arbitrary two curves for t and $t+\Delta t$. State 'actual force' and 'virtual force' acting on a body, and state brief comments for each force.

(10)(b) Lagrangian dynamics considers <u>all</u> forces acting on a body as a whole using scalar quantities T and V. Explain this concept using equations of motion for particles.

2. Fig.1 shows absolute coordinate system X-Y-Z, and a body-fixed coordinate system x-y-z to describe a point P. Using a body fixed coordinate system, express :

(3)(a) Absolute velocity

(12)(b)Absolute acceleration and indicate the direction of each vector components.

3. A block of mass m is attached a cord of original length L and is rotating about a thin hub as shown in Fig.2. Find the constraint force if

(10) (a) the cord is not wrapping around the hub.

(15) (b) the cord is wrapping around the hub.

4. As shown in Fig. 3, a particle m moves in the fixed plane x,y under the influence of the attractions $-C_1x, -C_2y$ from the coordinate axes and a force μr perpendicular to the radius vector and proportional to the distance from the origin O which point is again the only equilibrium position. All quantities should be based on x, y coordinate system.

(5) (a) Obtain potential energy of the forces.

(10) (b) Obtain differential equations of motion for a particle.

5. Under the gravity field g, consider a particle of mass m which is constrained to move on the surface of a cone of half-angle α as in Fig.4.

(5)(a) Express constrain equation. Also, obtain T and V.

(10)(b) Derive Lagrange equation of motion for r.

(10) 6. Use $[T]_{r,\theta,z}^{x,y,z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[T]_{r,\theta,z}^{R,\theta,\varphi} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$ derive $[T]_{x,y,z}^{R,\theta,\varphi}$.

(15) 7.Using two systems as in Fig.5,. *s*how that 'Lagrangian system is invariant under coordinate transformation'.

Fig.1

Fig.2

Fig.3

Fig.4