● Constraint Eqn.:

Given a system with generalized coordinates q_1, q_2 **and the constraint equation.**

$$\left(3q_1 \sin q_2 + \frac{q_2^2}{q_1} + 2\right)\delta q_1 + \left(q_1^2 \; \cos q_2 + 2q_2\right)\delta q_2 = 0$$

$$\equiv \frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 = 0$$

Constraint equation is holonomic, if there exist an integrating factor ***g***(q_1,q_2) **such that**

$$\frac{\partial f}{\partial q_1} = \left(3q_1 \sin q_2 + \frac{q_2^2}{q_1} + 2\right)g \qquad \frac{\partial f}{\partial q_2} = \left(q_1^2 \; \cos q_2 + 2q_2\right)g$$

if ***g***$(q_1,q_2) = q_1$

$$f = \left(q_1^{\,3} \sin q_2 + q_1 q_2^2 + q_1^2 + h_1(q_2) + C_1\right)$$

and

$$. \; f = \left(q_1^{\,3} \sin q_2 + q_1 q_2^2 + h_2(q_1) + C_2\right)$$

where h_1 *and* h_2 **are functions as a result of integration over**

q_1, q_2 **and** C_1 **and** C_2 **are constant.**

Comparing the terms:

$$h_2(q_1) = q_1^2, h_1(q_2) = 0..and..C_1 = C_2$$

Finally,

$$f = \left(q_1^{\ 3} \sin q_2 + q_1 q_2^2 + q_1^2 + C\right) = 0$$

Generally, there are no set guidelines for finding the integrating factor.

- **A block of mass *m* is attached a cord of original length L and is rotating about a thin hub :**

Find the constraint force if

(a)The cord is not wrapping around the hub.

(b)The cord is not wrapping around the hub.

Sol)

(a) Constraint is holonomic

$$f(x, y) = x^2 + y^2 - L^2 = 0$$

Mass is keep rotating <u>without doing a work with the constraint.</u>

(b) Assume that the hub radius is very small, then the tension in the rope directed toward point 0.

Thus, $\sum M_o = 0$

So that **<u>angular momentum is conserved:</u>**

Let the polar coordinate (r, θ)**:** $r = L - r_0\theta$ **(1)**

Angular momentum $H_0 = mr^2\dot{\theta} = C$

$r^2\dot{\theta} = h$ **(2)**

Remember Eq.(1) : $\dot{\theta} = -\dfrac{\dot{r}}{r_o}$

Then, $r^2\dot{\theta} = r^2(-\dfrac{\dot{r}}{r_o}) = h$

$$r^2\dot{r} = -r_o h \equiv C$$

$$\frac{1}{3}r^3 = Ct + D$$

Considering $r^2 = x^2 + y^2$

$$f(x, y, t) = \frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - \frac{L^3}{3} - Ct = 0$$

● Particle on the surface of a cone :

Consider the motion of a particle of mass m which is to move on the surface of a cone of half-angle α and which is subjected to a gravitation force g.

Let the axis of the cone correspond to the $z-axis$ and let the apex of the cone be located as the origin.

(a) For general system, state the categories of the constraint.

(b) Obtain the constraint for this problem. And, choose a category in (a)

(c) Obtain Lagrange's equation for radius *r*, and determine the general solution.

Sol) **Choose (r,θ,z) as the generalized coordinate due to the cylindrical symmetry of the model.**

Constraint : $z\tan\alpha = r$ **or** $z = r\cot\alpha$

~ two degrees of freedom system

We can eliminate either z or r : Choose z

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2 = \dot{r}^2 + (r\dot{\theta})^2 + (\dot{r}\cot\alpha)^2 = (\dot{r}\csc\alpha)^2 + (r\dot{\theta})^2$$

And also, if we choose *U* = 0 at *z* = 0

The potential energy

$$U = mgz = mgr\cot\alpha$$

Then, *L*=*T* – *U*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \ldots\ldots \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = C$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$\ddot{r} - r\dot{\theta}\sin^2\alpha + g\sin\alpha\cos\alpha = 0$$

In case $\omega = \Omega$

- **The position of a particle of mass *m* is given by the Cartesian coordinates** (x, y, z). **Assuming a potential energy function** $V = \frac{1}{2}k(x^2 + y^2 + z^2)$ **and a constraint described by the equation**

  $2\dot{x} + 3\dot{y} + 4\dot{z} + 5 = 0$ **find**

  (a) Differential equation of motion.

  (b) Velocity of the moving constraint.

Sol)

Kinetic energy in Cartesian coordinate system :

$$T = \frac{1}{2}\mathrm{m}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\mathrm{L} = \mathrm{T} - \mathrm{V}$$

***Rheonomic system : 2dx+3dy+4dz+5dt=0 : Elimination ?**

Integrable ?

- Holonomic system

$y = 3x^2$: $6x\dot{x} - \dot{y} = 0 : 6xdx - dy = 0$

Elimination

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$\mathrm{V} = \mathrm{mgy}$$

- **Systems with Time-dependent Coefficient. Mathieu's Equation**

$$T = \frac{1}{2}m[(L\dot{\theta}\cos\theta)^2 + (\dot{u} + L\dot{\theta}\sin\theta)^2]$$
$$= \frac{1}{2}m(L^2\dot{\theta}^2 + 2L\dot{u}\dot{\theta}\sin\theta + \dot{u}^2]$$

$$\mathrm{V} = \mathrm{mg}[\mathrm{L}(1-\cos\theta) + u]$$

and $\delta W = F\delta u$

Let $q_1 = \theta, q_2 = u$

$$\delta W = \Theta\delta\theta + U\delta u$$
$$\Theta = 0, U = F$$

$(\theta, u) : 2DOF$ **:---→ Eqns of motion !**

For neighborhood of $\theta \simeq \mathrm{O}$ **:**

$$\ddot{\theta} + (\frac{g}{L} + \frac{\ddot{u}}{L})\theta = 0$$

$$\ddot{u} + g = \frac{F}{m}$$

Assume $u = A\cos\omega t$

Then

$$F = m(g - A\omega^2 \cos\omega t)$$

$$\ddot{\theta} + (\frac{g}{L} - \frac{A\omega^2}{L}\cos\omega t)\theta = 0$$

For convenience,

$$\theta = x, \frac{g}{L} = \delta, -\frac{A\omega^2}{L} = 2\alpha$$

Finally, Mathieu equation

$$\ddot{x} + (\delta + 2\varepsilon\cos 2t)x = 0 \ldots\ldots \varepsilon \ll 0$$

For x-directional motion instead of y-direction ?

T,V ?