FORCES OF CONSTRAINT

Lagrangian formalism :

Generalized coordinate **Minimum set of Eqns**

Part of the advantage : Constraint forces do no virtual work under a set of virtual displacements compatible with the constraints. Generally,

Constraints reduce the number of degrees of freedom.

The constraint forces <u>do not appear</u> in the eqns of motion : Symmetry of a system ?

~Holonomic systems <u>can be described</u> in terms of independent generalized coordinates free of constraints.

~ Non-holonomic constraints cannot be reduced to independent generalized coordinates.

Eqns of motion must be augmented by the Constraints

~> Forces of constraint are also established.

***Constraint forces in holonomic systems may also be analyzed.**

: Constraints are enforced by reacting forces in the directions normal to the constraint surfaces

Physically, a constraint must be imposed in the form of forces or moments. ~ > Constraints with additional generalized forces acting on the system. These forces depend on the motion and cannot be found prior to solving the eqns of motion.

: Should be solved simultaneously

Problems with or without constraint ?

~ Holonomic constraint can in principle be replaced by a *reacting constraint force*. - Additional dof may be introduced onto the problem by adding generalized coordinates (superfluous coordinates) corresponding to the violation of the constraints.

The generalized forces associated with the superfluous coordinates are the forces of constraint.

In case, <u>original coordinates and the extra coordinates are considered as independent, then the resulting eqns of motion will contain the constraint forces.</u>

These forces will only be in the eqns associated with the superfluous coordinates. After the eqns of motion are set up, the superfluous coordinates are set to constant values.

Setting up the problem this way results in eqns involving the constraint forces and also gives the values of these forces necessary to enforce the given constraints.

For non-holonomic constraints, the eqns of motion are formulated using Lagrange multiplier method.

Suppose : *n* generalized coordinates $q_1, q_2, \ldots, q_n$ **is restricted(?) by a non-holonomic constraint:**

$$A_1 dq_1 + A_2 dq_2 + \cdots + A_n dq_n + A_0 dt = 0$$

Since the variations take place without increment in time, $\delta t = 0$ **, the resulting eqn of constraint for the <u>virtual displacements</u> becomes**

$$A_1 \delta q_1 + A_2 \delta q_2 + \cdots + A_n \delta q_n = 0 \qquad \textbf{(2.69)}$$

<u>Geometrically,</u> Eqn (2.69) defines a direction orthogonal

to the virtual displacement δq.

Thus the constraint force is a ***scalar multiple*** **of the vector**

$(A_1, A_2, \ldots, A_n)$. **This scalar is a function of time** $\lambda(t)$.

Total generalized force acting on the generalized

coordinate q_k**, including applied and reacting forces, is**

$$Q_k + \lambda A_k$$

The resulting eqns of motion for non-holonomic systems

are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k+\lambda A_k, \qquad k=1,2,\ldots,n \qquad \textbf{(2.70)}$$

Eqns (2.70) together with (2.69) represent *n + 1* equations in *n + 1* unknowns, including the *Lagrange multiplier*. These eqns are solved simultaneously. In addition to solving for the generalized coordinates, the solution gives the component of the reacting *constraint force*.

Generalization: System is subjected to *J* non-holonomic constraints given by

$$A_{j1}\dot{q}_1 + A_{j2}\dot{q}_2 + \cdots + A_{jn}\dot{q}_n + A_{j0} = 0 \qquad \textbf{(2.71)}$$

or equivalently as

$$A_{j1}dq_1 + A_{j2}dq_2 + \cdots + A_{jn}dq_n + A_{j0}dt = 0$$

where *j* ranges from 1 to the number of such constraints *J*. Coefficients A_{jk} **<u>may be functions of the</u>**

generalized coordinates and time. Introduce *J* Lagrange

multipliers, $\lambda_j(t)$, one for each constraint eqn (2.71).

~ Total generalized force driving the *k*-th generalized

coordinate is

$$Q_k + \sum_{j=1}^{J} \lambda_j A_{jk}$$

Thus, eqn of motion for each generalized coordinate q_k :

$$\int\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k+\sum_{j=1}^{J}\lambda_j A_{jk}\right]\delta q_k \qquad \textbf{(2.72)}$$

The set of eqns (2.72) together with the *J* eqns of constraint (2.71) constitute *n* + *J* eqns in *n* + *J* unknowns. These eqns must be solved *simultaneously* for the generalized coordinates and the *J* Lagrange multipliers $\lambda_j(t)$.

The generalized constraint force reacting on the

coordinate q_k :

$$R_k = \sum_{j=1}^{J} \lambda_j A_{jk}$$

Method of Lagrange multipliers may also be applied to systems with holonomic constraints.

Recall that a holonomic constraint $f(q_1, \ldots . q_n, t) = const$

may be converted to differential form as

$$\frac{df}{dt} = \frac{\partial f}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial f}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial f}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial f}{\partial t} = 0$$

This is the same form as a non-holonomic constraint (2.71), with the coefficients

$$A_{jk} = \frac{\partial f}{\partial q_k} \qquad A_{j0} = \frac{\partial f}{\partial t}$$

Thus holonomic systems with constraints can also be analyzed, as well as systems having constraints of *both* types.

As an example, (p.120) : Consider the dynamics of a particle constrained to slide on a frictionless wire. This wire is in the shape of a parabola that is rotating about its axis of symmetry with constant angular velocity

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Cylindrical coordinates are intrinsic to this problem.

There is only one degree of freedom, namely the position

of the mass on the wire. The two constraints are $z = br^2$**,**

where *b* is some constant, and $\dot{\theta} = \omega$**. These constraints**

are holonomic, which imply constraints of the form (2.71)

as

$$\lambda_1(\delta z - 2br\delta r) = 0 \text{ and } \lambda_2 \delta\theta = 0$$

The coefficients in (2.71) are seen in matrix form as

$$\begin{bmatrix} -2br & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence there will be <u>two Lagrange multipliers</u> – one for each constraint. The Lagrangian of the system is

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

The equations of motion are

$$m\ddot{r} - mr\dot{\theta}^2 = -2b\lambda_1 r$$

$$\frac{d}{dt}(mr\dot{\theta}^2) = \lambda_2$$

$$m\ddot{z} + mg = \lambda_1$$

Now since $\dot{\theta} = \omega$, there are four unknowns $r(t)$, $z(t)$, $\lambda_1(t)$, and $\lambda_2(t)$. : Three eqns of motion and the constraint eqn $z = br^2$.

<u>Eliminating the multiplier</u> $\lambda_1(t)$ results in the equation

$$\ddot{r} - r\omega^2 = -2br(\ddot{z} + g)$$

Differentiation of the constraint $z = br^2$ **results in**

$$\ddot{z} = 2b\dot{r}^2 + 2br\ddot{r}$$

and so we end up with the single differential equation for

$r(t)$ **as**

$$\ddot{r}(1 + 4b^2r^2) + 4b^2r\dot{r}^2 = r(\omega^2 + 2bmg) \qquad \textbf{(2.73)}$$

The entire analysis reduces to the solution of Eqn (2.73).

The coordinate $z(t)$ **is obtained from the constraint**

eqn. The two Lagrange multipliers are also given in terms of *r* and *z* from the eqs of motion.

Finally, the *torque* required to maintain the uniform rotation is

$$\lambda_2 = 2m\omega r\dot{r}$$

And the components of the reacting *constraint force*

exerted by the wire on the mass are

$$R_r = -2b\lambda_1(t)r(t) \quad \text{and} \quad R_z = \lambda_1(t)$$

Practice !!

INTEGRALS OF MOTION

Up to now: Concern on formulating the eqs of motion.

Lagragian formalism for a systematic way to apply

Newton's laws of motion using generalized coordinates.

What is the next step ? Actually analyze the dynamics based on the eqns of motion.

~ Eqns consist of a system of *n* O.D.E, each of the 2nd order! ~ Typically nonlinear.

Except, the eqns of motion are linear, Eqns of motions are sometimes linearized based on the small displace-

ments assumptions.

This may have some utility in stability analysis, but

linearization typically destroys the applicability of the

eqns of motion.

Generally, eqns of motion are too complicated !

➔ by integration based on elementary methods.

For specified initial conditions, the eqns of motion are

usually integrated numerically.

Example, Runge-Kutta algorithms : good accuracy.

Drawback is that the resultant numerical solution is only

valid for *one* set of initial conditions.

Aim of analytical mechanics ?

~Analysis of the eqns of motion themselves,

without actually solving the system of eqns.

Such qualitative analysis was introduced in Chapter 1

with the energy analysis of conservative systems.

Conservative systems are distinguished by conservation

of total mechanical energy.

- Allowed the partial integration of eqn of motion.

This concept is readily extended to general systems.

Suppose that a certain combination of the generalized

coordinates and velocities remains *invariant* during the evolution of the system. : If there exists some function $G(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$ that remains constant over time, then

$$G(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) = C \tag{2.74}$$

or equivalently

$$\frac{dG}{dt} = 0$$

The relation (2.74) is called an integral of the motion.

***C* is called a *constant of motion*. ~ An integral of motion represents a quantity that is conserved during the motion.**

There are only first derivatives in an integral of motion, so each integral of motion ~ a partial integration of the original system : used as reduction of the order of the system.

Lagrange's eqns represent *n* second-order (partial) diff. eqns. ~ Ideally, the solution of Lagrange's equations consists of finding 2*n* integral of motion (2.74), each containing *only* the generalized coordinates.(?)

This is typically not possible, but certain systems do admit some integrals of motion.

For example, in a *conservative system*, the total

mechanical energy is an *invariant* of the system.

$$T(q,\dot{q},t) \;+\; V(q) \;=\; \text{const}$$

is an integral of the motion. Value of the constant of motion is determined by initial conditions.

A conservative system is a special case of a Lagrangian system.

~ Eqs of motion for a Lagrangian system :

$$\frac{d}{dt}[\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_k}] - \frac{\partial L(q,\dot{q},t)}{\partial q_k} = 0 \qquad \textbf{(2.75)}$$

Total time derivative of $L(\mathrm{q},\dot{\mathrm{q}},t)$ **is**

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$(\sim L, q_k \bullet \dot{q}_k + L, \dot{q}_k \bullet \ddot{q}_k + \frac{\partial L}{\partial t}. for.k = 1..n)$$

*** Lagrange's Eqns (2.75) we have**

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

this means

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_{k=1}^{n} \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial t}$$

Therefore

$$\frac{d}{dt}\left(\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = -\frac{\partial L}{\partial t} \quad ..(2.76)$$

Dimension of *L* is energy, the quantity in the parentheses

is known as the Jacobi energy function

$$h(\mathrm{q},\dot{\mathrm{q}},t)=\sum_{k=1}^{n}\dot{q}_k\frac{\partial L}{\partial \dot{\mathrm{q}}_k}-L$$

Eqn (2.76) can be written :

$$\frac{dh}{dt}=-\frac{\partial L}{\partial t}$$

~ If the Lagrangian does not contain time *t* explicitly,

then the Jacobi energy function is *invariant* during the motion. ~ Energy function is an integral of the motion with

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = h = \text{cosnt}$$