Beam-Cavity Interactions

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Introduction

- Resonant cavities are widely used to generate electromagnetic radiations via beam-particle interaction (klystron, magnetron) and accelerate charged particles.
- The category of resonant accelerators includes rf (radio-frequency) linear accelerators, cyclotrons, microtrons, and synchrotrons.
- Resonant accelerators have the following features in common:
  - Applied electric fields are harmonic. The continuous wave (CW) approximation is valid; a frequency-domain analysis is the most convenient to use. In some accelerators, the frequency of the accelerating field changes over the acceleration cycle; these changes are always slow compared to the oscillation period.
  - The longitudinal motion of accelerated particles is closely coupled to accelerating field variations.
  - The frequency of electromagnetic oscillations is often in the microwave regime. This implies that the wavelength of field variations is comparable to the scale length of accelerator structures. The full set of the Maxwell equations must be used.

Plane electromagnetic waves in lossy media

- The Maxwell's equations for a source-free lossy medium:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -j\omega \boldsymbol{B} \qquad \nabla \times \boldsymbol{B} = \mu \boldsymbol{J} + \mu\epsilon \frac{\partial \boldsymbol{E}}{\partial t} = \left(\mu\sigma + \frac{j\omega}{c^2}\right)\boldsymbol{E}$$

- By combining two equations, we obtain the following homogeneous Helmholtz's equation for a lossy medium:

Attenuation constant

$$\boldsymbol{\nabla}^2 \boldsymbol{E} - \gamma^2 \boldsymbol{E} = 0 \qquad \gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\sqrt{1 + \frac{\sigma}{j\omega\epsilon}}$$

Phase constant

- The solution for the linearly polarized wave in x-direction:

$$E_x = E_0 e^{-\gamma z} = E_0 e^{-\alpha z} e^{-j\beta z}$$

- For a good conductor

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4} = (1+j)/\sqrt{2}$$

$$\gamma = j\omega\sqrt{\mu\epsilon}\sqrt{1 + \frac{\sigma}{j\omega\epsilon}} \approx j\omega\sqrt{\mu\epsilon}\sqrt{\frac{\sigma}{j\omega\epsilon}} = \sqrt{j}\sqrt{\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}(1+j)$$

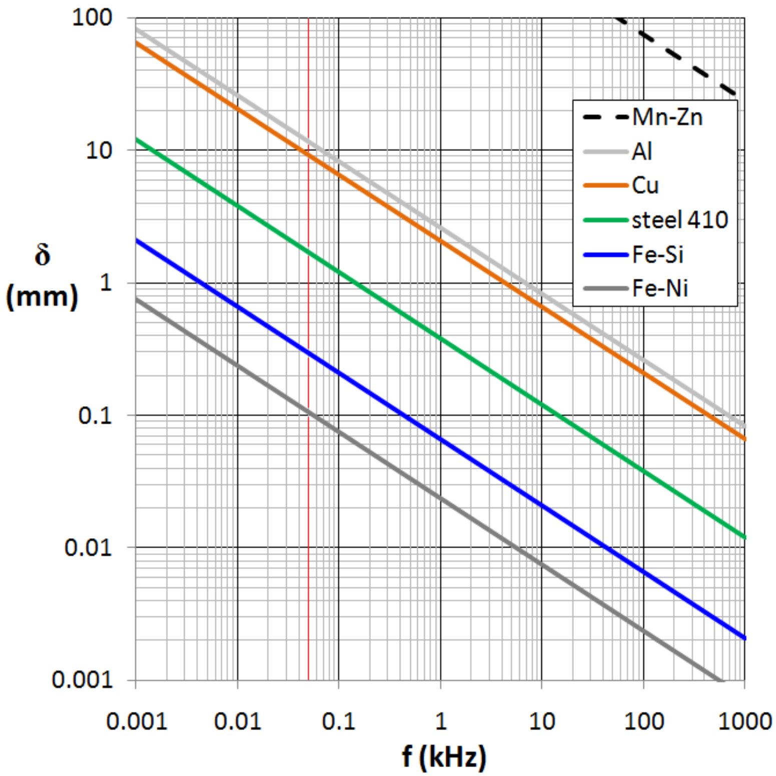
$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \equiv \frac{1}{\delta}$$

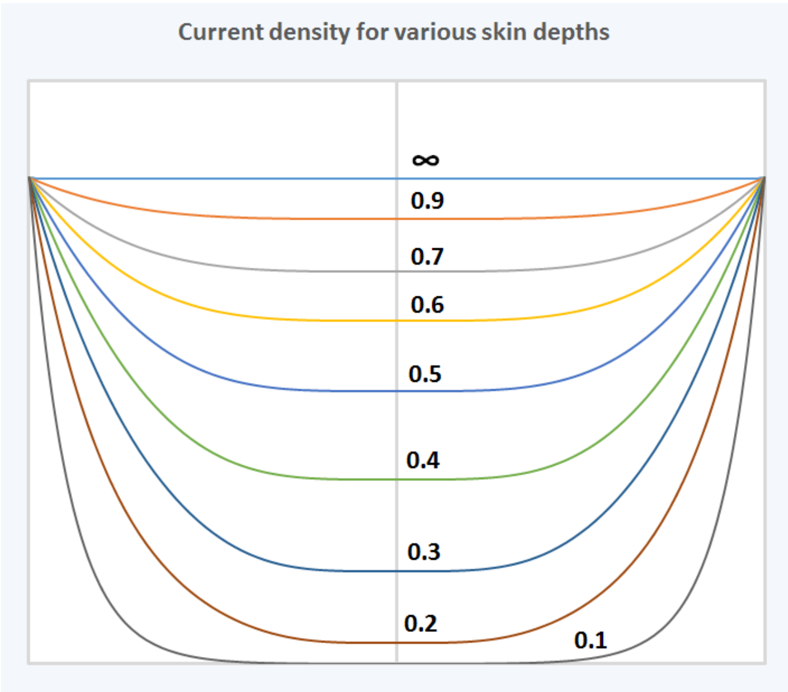
Skin depth

Skin depth

Skin Depths, δ in (mm), of Various Materials

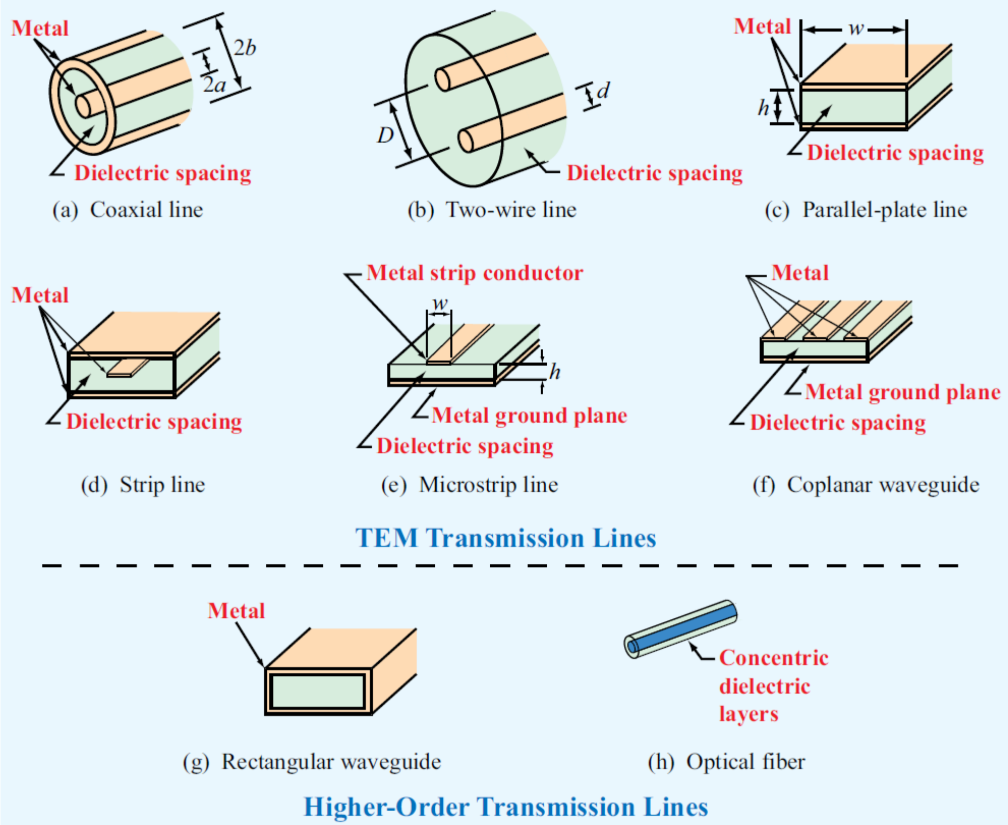
Material	σ (S/m)	f = 60 (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

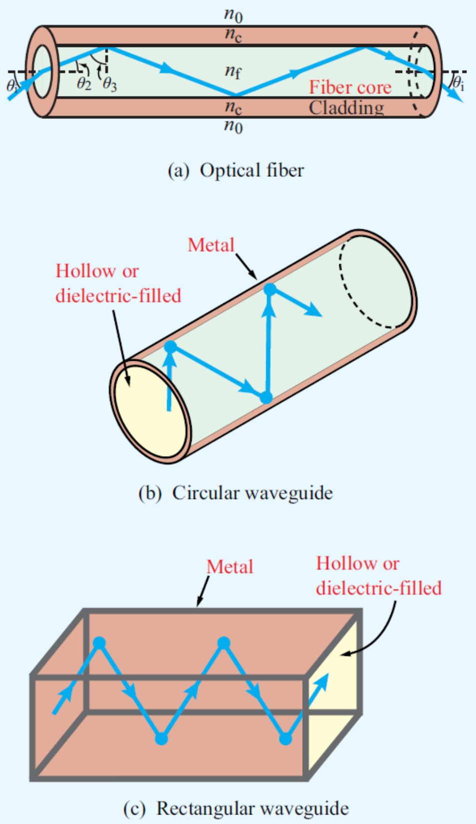




Higher-order transmission lines

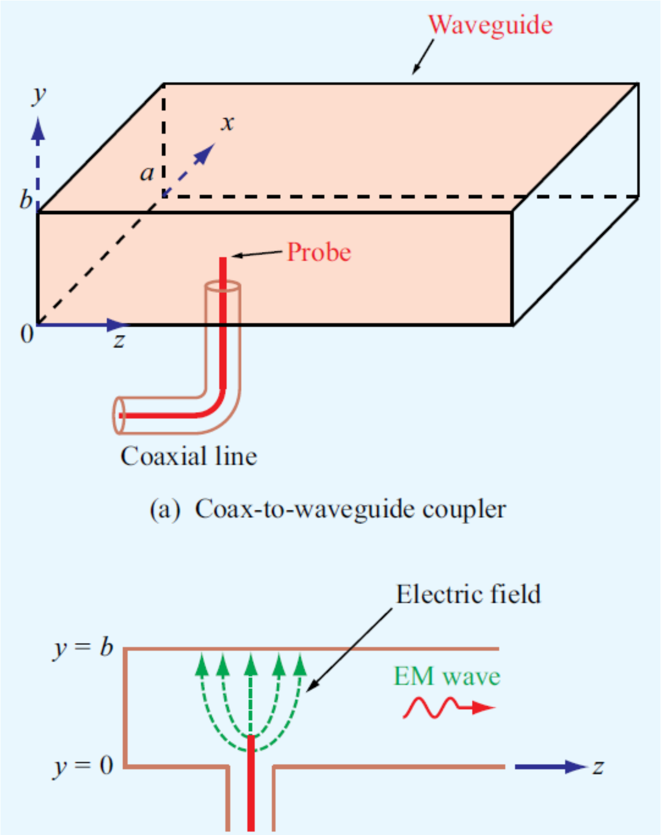
- TEM transmission lines including coaxial, two-wire, and parallel-plate lines, support E and H fields that are orthogonal to the direction of propagation.
- Fields supported by higher-order transmission lines may have E or H orthogonal to the direction of propagation k, but not both simultaneously. Thus, at least one component of E or H is along k.





Waveguides

- If E is transverse to k but H is not, we call it a transverse electric (TE) mode, and if H is transverse to k but E is not, we call it a transverse magnetic (TM) mode.
- For guided transmission at frequencies below 30 GHz, the coaxial cable is by far the most widely used transmission line.
- At higher frequencies, however, the coaxial cable has a number of limitations:
  - (a) in order for it to propagate only TEM modes, the cable's inner and outer conductors have to be reduced in size to satisfy a certain size-to-wavelength requirement, making it more difficult to fabricate;
  - (b) the smaller cross section reduces the cable's power-handling capacity (limited by dielectric breakdown); and
  - (c) the attenuation due to dielectric losses increases with frequency.



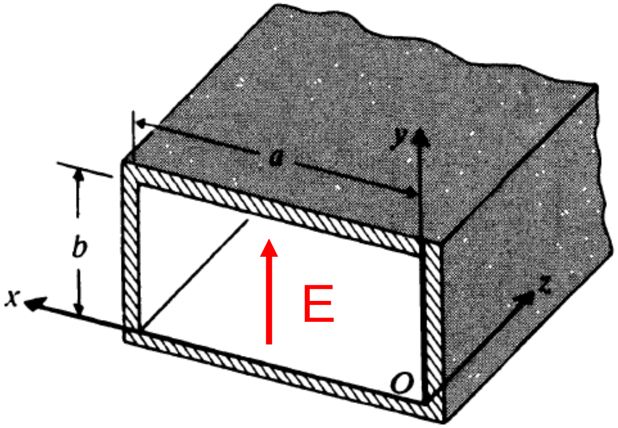
(a) Coax-to-waveguide coupler

(b) Cross-sectional view at $x = a/2$

Rectangular waveguides

- The Maxwell's equations for a lossless, source-free medium (such as the inside of a waveguide):

$$\boldsymbol{\nabla}^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0$$



- $\tilde{E}$ and $\tilde{H}$ of a wave traveling along the +z direction should exhibit a dependence on z of the form $e^{-j\beta z}$:

$$\boldsymbol{E} = E(x,z)\hat{\mathbf{y}} = E(x)e^{-j\beta z}\hat{\mathbf{y}}$$

- For $\mathrm{TE_{m0}}$ mode,

B.C.: $E(0) = E(a) = 0$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} + k^2 E = \frac{\partial^2 E}{\partial x^2} + (k^2 - \beta^2)E = \frac{\partial^2 E}{\partial x^2} + k_m^2 E = 0$$

- The solution:

$$E = E_0 \sin(k_m x) = E_0 \sin\left(\frac{m\pi}{a}x\right)$$

- Dispersion relation:

$$\omega^2 = \beta^2 c^2 + \left(\frac{m\pi}{a}\right)^2$$

Rectangular waveguides

- Dispersion relation:

$$\omega^2 = \beta^2 c^2 + \left(\frac{m\pi}{a}\right)^2$$

- Phase constant:

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2} = k\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

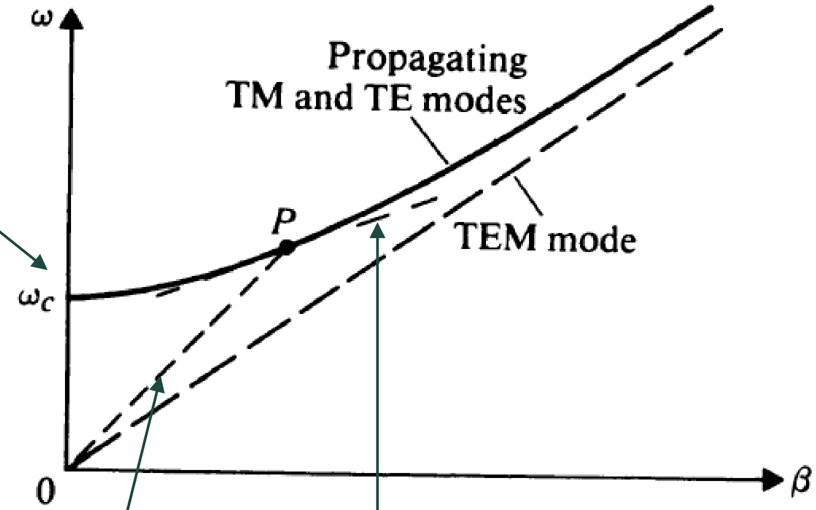
- Guide wavelength:

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} > \lambda$$

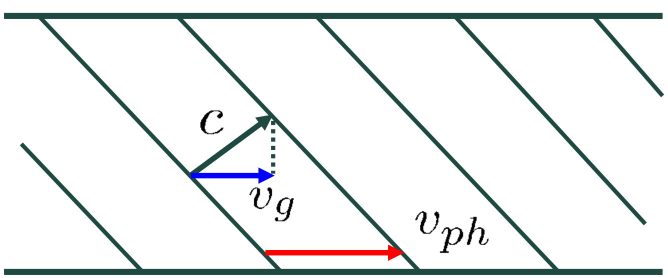
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega} c$$

- At cut-off ($\omega_c = \frac{m\pi}{a} c$), $\beta \to 0,$ $\lambda_g \to \infty$

$$\omega_c = \frac{m\pi}{a} c$$



$$v_p = \frac{\omega}{\beta} = \frac{\lambda_g}{\lambda} c > c \qquad v_g = \frac{d\omega}{d\beta} = \frac{\lambda}{\lambda_g} c < c$$



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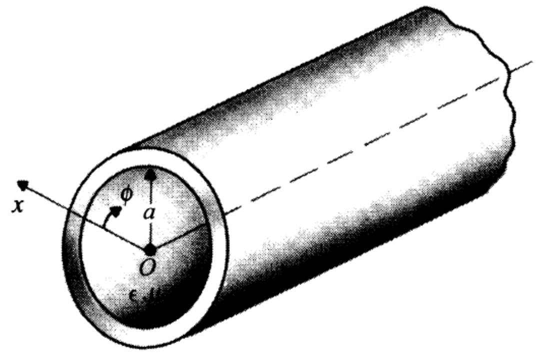
Circular waveguide

- The wave equation can be written as the sum of a transverse component and an axial component:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \qquad \mathbf{E} = \mathbf{E}_T + \mathbf{a}_z E_z$$

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0. \qquad \mathbf{H} = \mathbf{H}_T + \mathbf{a}_z H_z,$$



- For TM (transverse magnetic) wave, $H_z = 0, E_z \neq 0$, and all field components can be expressed in terms of $E_z = E_z^0(r,\phi)e^{-\gamma z}$, where $E_z^0(r,\phi)$ satisfies the homogeneous Helmholtz's equation:

$$\nabla^2_{r\phi}E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \qquad \gamma = \alpha + j\beta$$

$$\nabla^2_{r\phi}E_z^0 + h^2 E_z^0 = 0.$$

- In cylindrical coordinates, we obtain the following differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z^0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z^0}{\partial \phi^2} + h^2 E_z^0 = 0. \qquad E_z^0(r,\phi) = R(r)\Phi(\phi),$$

Circular waveguide

- The method of separation of variables gives:

$$\frac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0 \longrightarrow \Phi(\phi) \sim \cos n\phi \qquad n = 0,1,2,\cdots$$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(h^2 - \frac{n^2}{r^2}\right)R(r) = 0.$$

Bessel's differential equation

- The solution:

$$R(r) = C_n J_n(hr) + D_n \cancel{N_n}(hr),$$

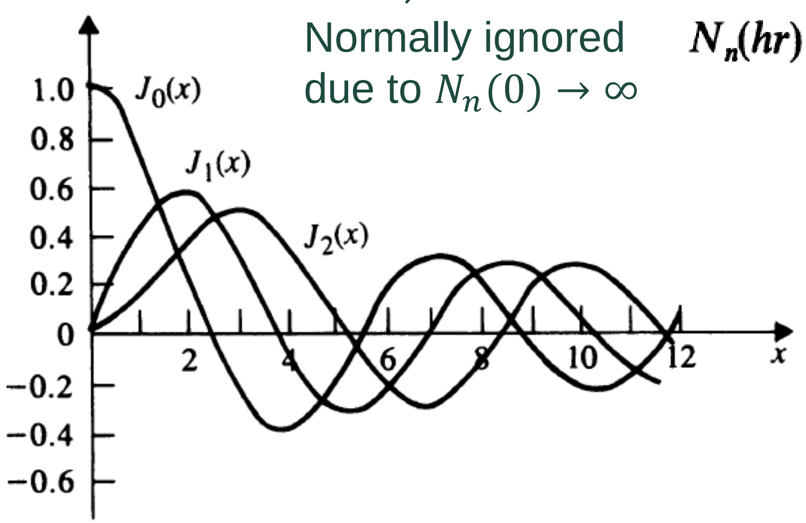
Normally ignored due to $N_n(0) \to \infty$

$$J_n(hr) = \sum_{m=0}^{\infty} \frac{(-1)^m (hr)^{n+2m}}{m!(n+m)!2^{n+2m}}$$

Bessel function of the first kind

$$N_n(hr) = \frac{(\cos n\pi)J_n(hr) - J_{-n}(hr)}{\sin n\pi}.$$

Bessel function of the second kind

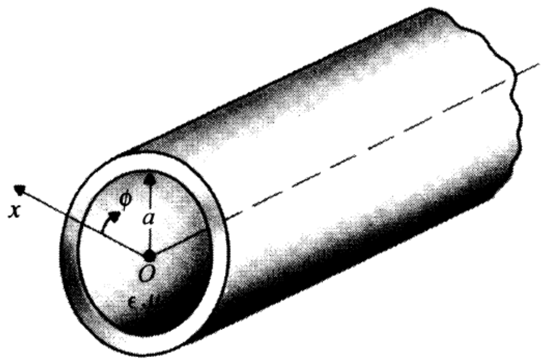


Zeros of $J_n(x)$, x_{np}

p \ n	$n = 0$	$n = 1$	$n = 2$
1	2.405	3.832	5.136
2	5.520	7.016	8.417

TM waves in circular waveguides

- For TM wave, $H_z = 0$. We write $E_z(r,\phi,z) = E_z^0(r,\phi)e^{-j\beta z}$, where $E_z^0(r,\phi)$ satisfies $\nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0$. The solution is



$$\boxed{E_z^0 = C_n J_n(hr)\cos n\phi. \qquad \text{(TM modes)}}$$

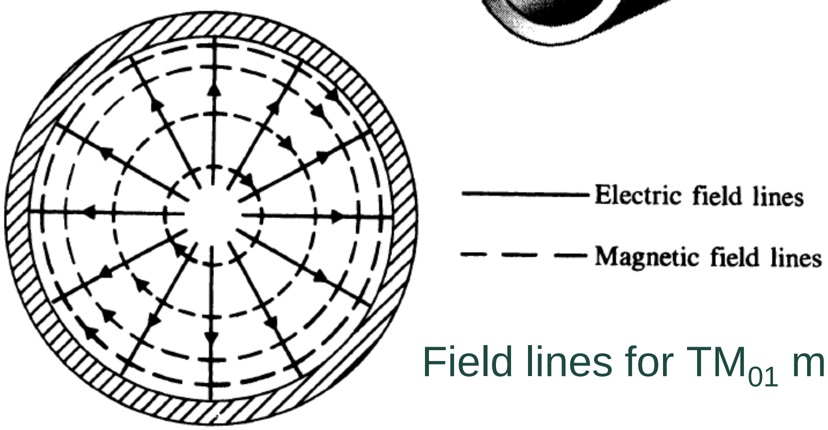
$$E_r^0 = -\frac{j\beta}{h} C_n J_n'(hr)\cos n\phi,$$

$$E_\phi^0 = \frac{j\beta n}{h^2 r} C_n J_n(hr)\sin n\phi,$$

$$H_r^0 = -\frac{j\omega\epsilon n}{h^2 r} C_n J_n(hr)\sin n\phi,$$

$$H_\phi^0 = -\frac{j\omega\epsilon}{h} C_n J_n'(hr)\cos n\phi,$$

$$H_z^0 = 0,$$



Field lines for TM_{01} mode

- The eigenvalues of TM modes (the admissible value of h) are determined from the boundary condition, $E_z^0(r=a) = 0$.

$$J_n(ha) = 0. \qquad \text{(TM modes)}$$

$$(h)_{\mathrm{TM}_{01}} = \frac{2.405}{a}, \qquad (f_c)_{\mathrm{TM}_{01}} = \frac{(h)_{\mathrm{TM}_{01}}}{2\pi\sqrt{\mu\epsilon}} = \frac{0.383}{a\sqrt{\mu\epsilon}}.$$

For the TM_{01} mode ($n = 0$), E_z^0, E_r^0, and H_ϕ^0 are the only nonzero field components.

TM_{0p} waves in circular waveguides

- Dispersion relation:

$J_0(x_{0p}) = 0$

$$\omega^2 = \beta^2 c^2 + \left(\frac{x_{0p}}{a}\right)^2$$

- Phase constant:

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{x_{0p}}{a}\right)^2} = k\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

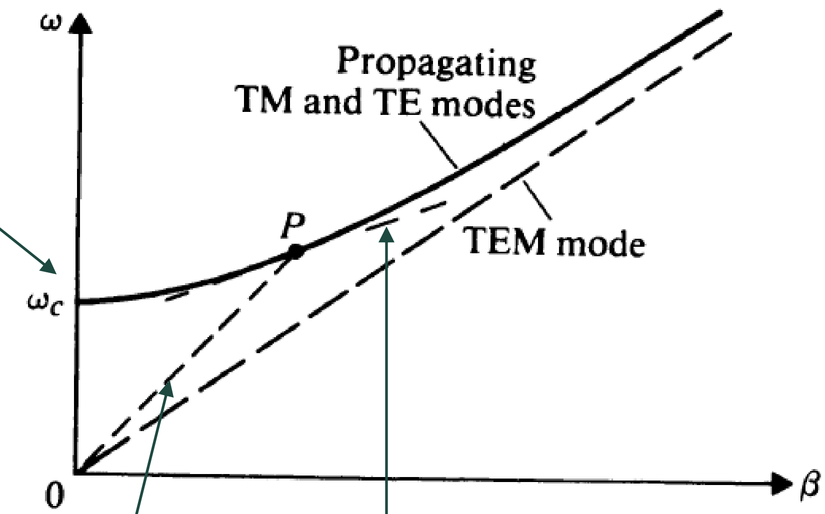
- Guide wavelength:

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} > \lambda$$

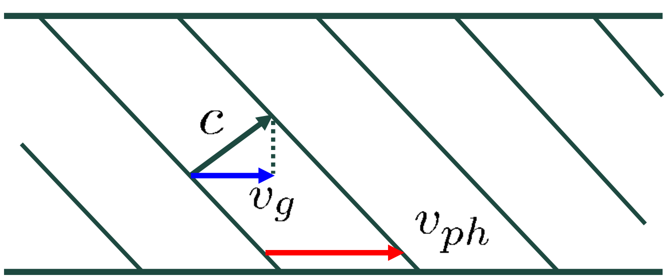
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega} c$$

- At cut-off ($\omega_c = \frac{x_{0p}}{a} c$), $\beta \to 0,$ $\lambda_g \to \infty$

$$\omega_c = \frac{x_{0p}}{a} c$$



$$v_p = \frac{\omega}{\beta} = \frac{\lambda_g}{\lambda} c > c \qquad v_g = \frac{d\omega}{d\beta} = \frac{\lambda}{\lambda_g} c < c$$



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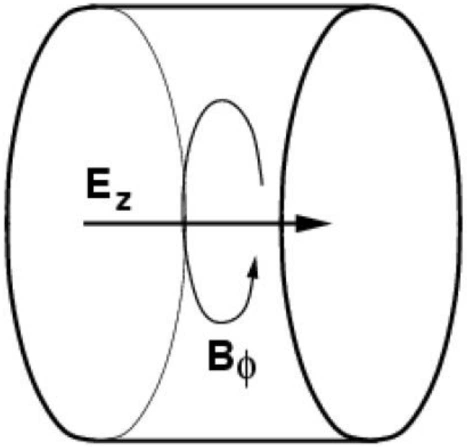
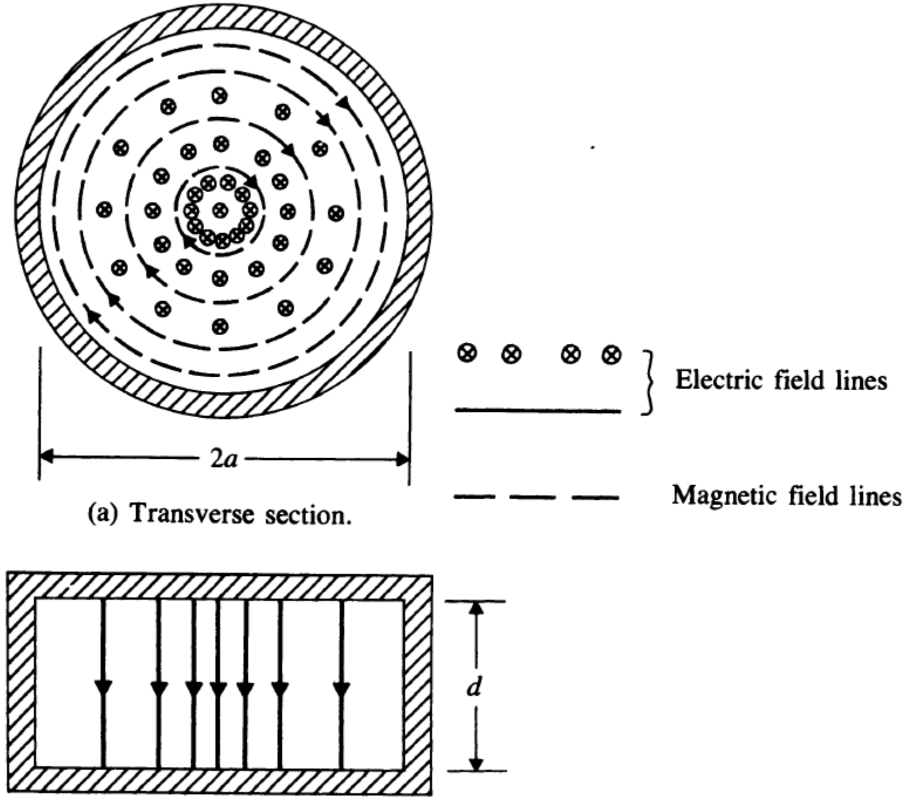
Circular cavity resonator (TM_{010} mode)

- A circular cylindrical resonator can be formed by placing conducting walls at both ends of a cylindrical waveguide. For TM_{01} mode, the ends of the waveguide are shorted by conducting plates at a distance d $(< 2a)$ apart.

$$E_z = C_0 J_0(hr) = C_0 J_0\left(\frac{2.405}{a} r\right),$$

$$H_\phi = -\frac{jC_0}{\eta_0} J_0'(hr) = \frac{jC_0}{\eta_0} J_1\left(\frac{2.405}{a} r\right),$$





(a) Transverse section.

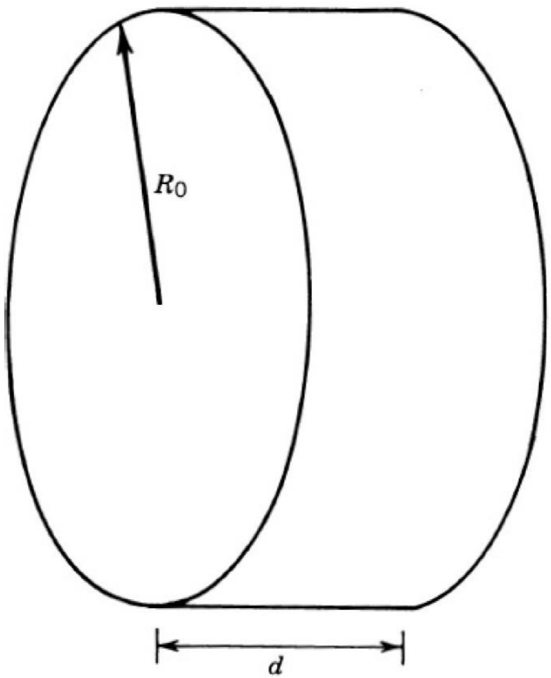
(b) Longitudinal section.

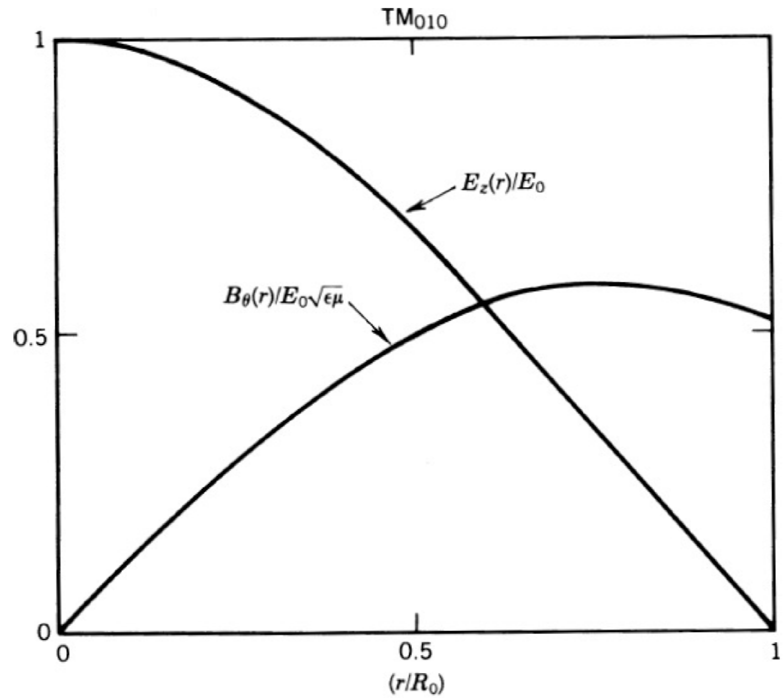
Pillbox cavity

- For a TM_{010} mode

$$E_z(r,t) = E_0 J_0\left(\frac{2.405r}{r_0}\right)\sin\omega_0 t \qquad \omega_0 = \frac{2.405c}{r_0}$$

$$B_\theta(r,t) = \frac{jE_0}{c} J_1\left(\frac{2.405r}{r_0}\right)\cos\omega_0 t$$





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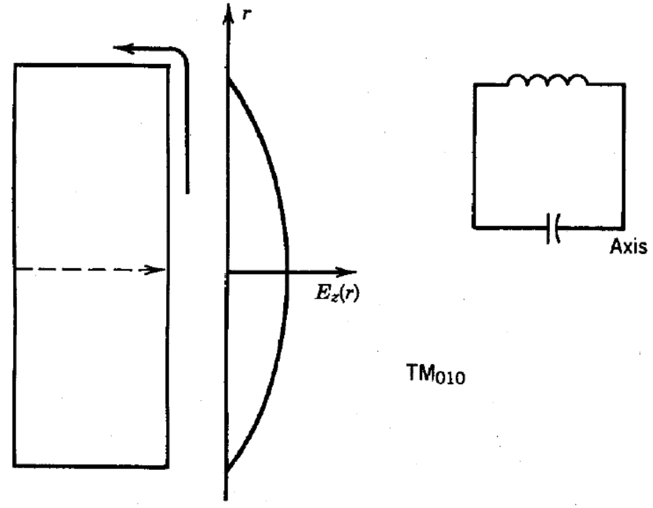
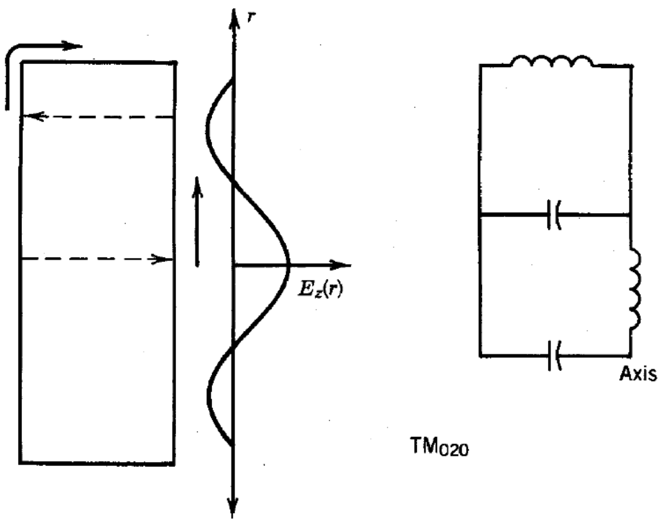
TM modes

- TM_{0n0} modes are optimal for particle acceleration.
- TM_{0n0} mode depends only on R_0, not d.
- The longitudinal electric field is uniform along the propagation direction of the beam and its magnitude is maximum on axis.
- The transverse magnetic field is zero on axis; this is important for electron acceleration where transverse magnetic fields could deflect the beam.

Mode	k_n	ω_n
TM_{010}	$2.405/R_0$	$2.405/\sqrt{\varepsilon\mu}\, R_0$
TM_{020}	$5.520/R_0$	$5.520/\sqrt{\varepsilon\mu}\, R_0$
TM_{030}	$8.654/R_0$	$8.654/\sqrt{\varepsilon\mu}\, R_0$
TM_{040}	$11.792/R_0$	$11.792/\sqrt{\varepsilon\mu}\, R_0$



(a)



(b)

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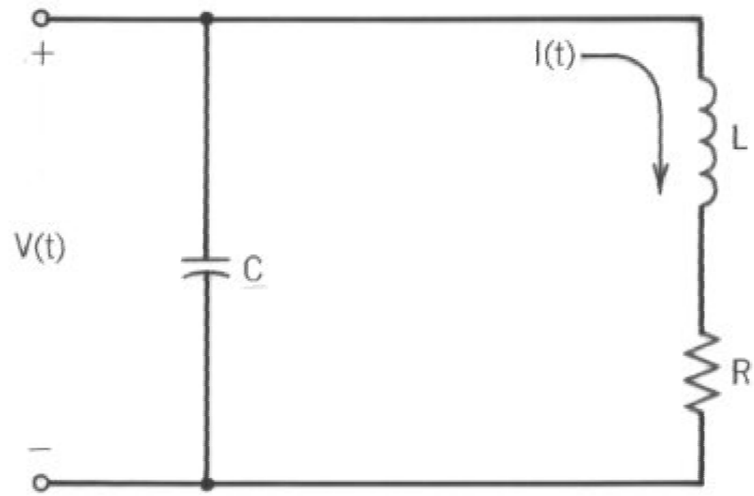
Equivalent circuit of TM_{010} mode



- The resonance frequency:

$$\omega_0 = \frac{2.405c}{r_0} = \frac{1}{\sqrt{LC}}$$

- The cavity voltage at $r = 0$:

$$V(t) = \int_0^d E_z(z,t)dz = E_0 d \sin\omega_0 t$$

- The cavity current:

$$I(t) = \int_0^{r_0} 2\pi r j_{dz}(r,t)dr = \int_0^{r_0} 2\pi r\epsilon_0 \frac{\partial E_z(r,t)}{\partial t}dr = 2\pi r\epsilon_0 E_0 r_0 c J_1(2.405)\cos\omega_0 t$$

- The capacitance and inductance:

$$C = \frac{I(t)}{dV(t)/dt} = \left(\frac{\epsilon_0 \pi r_0^2}{d}\right)\left(\frac{2J_1(2.405)}{2.405}\right)$$

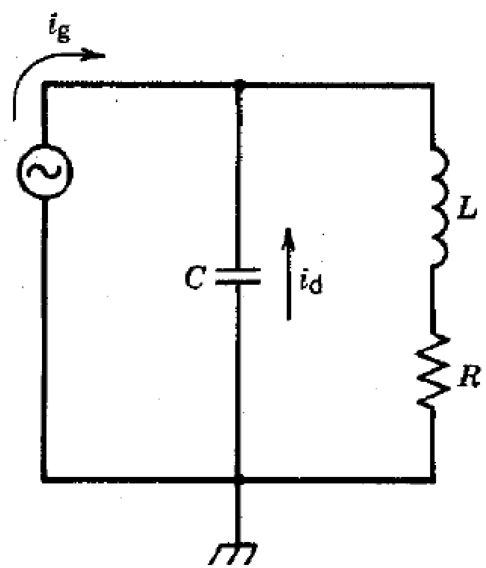
$$L = \frac{1}{\omega_0^2 C} = \left(\frac{\mu_0 d}{2\pi}\right)\left(\frac{1}{2.405 J_1(2.405)}\right)$$

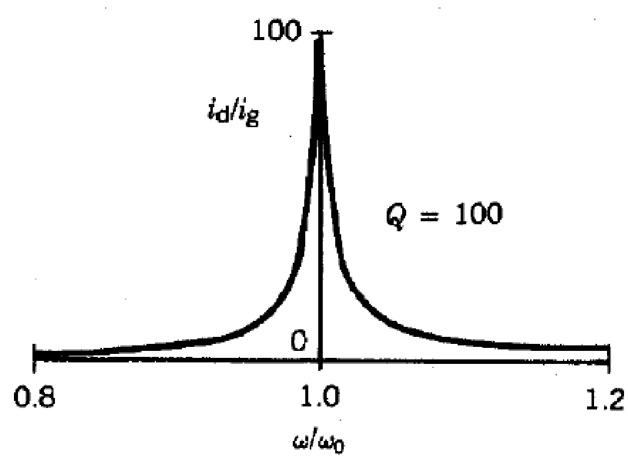
$$Q = \frac{\omega_0 L}{R} = \frac{\sqrt{L/C}}{R}$$

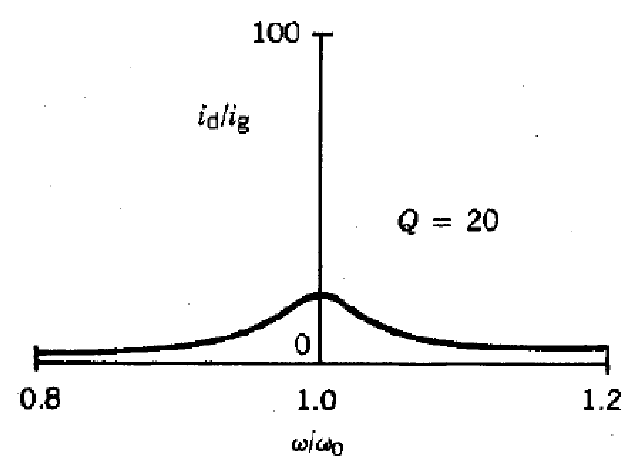
Cavity Q (TM_{010} mode)

- A cavity resonator stores energy in the electric and magnetic fields for any particular mode pattern. In any practical cavity the walls have a finite conductivity: that is, a non-zero surface resistance, and the resulting power loss causes a decay of the stored energy.
- The quality factor, or Q, of a resonator is a measure of the bandwidth of the resonator.

$$Q = 2\pi \frac{\text{Time} - \text{average energy stored at a resonant frequency}}{\text{Enegy dissipated in one period of this frequency}} = \frac{\omega W}{P_L}$$

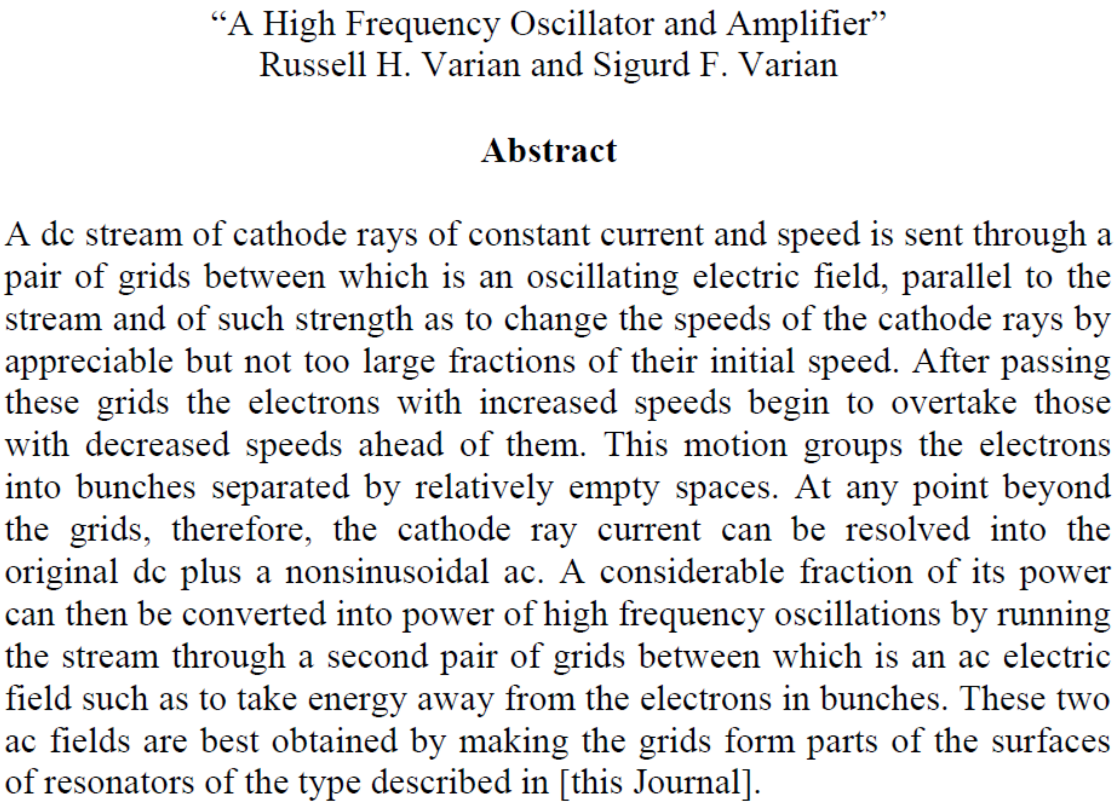






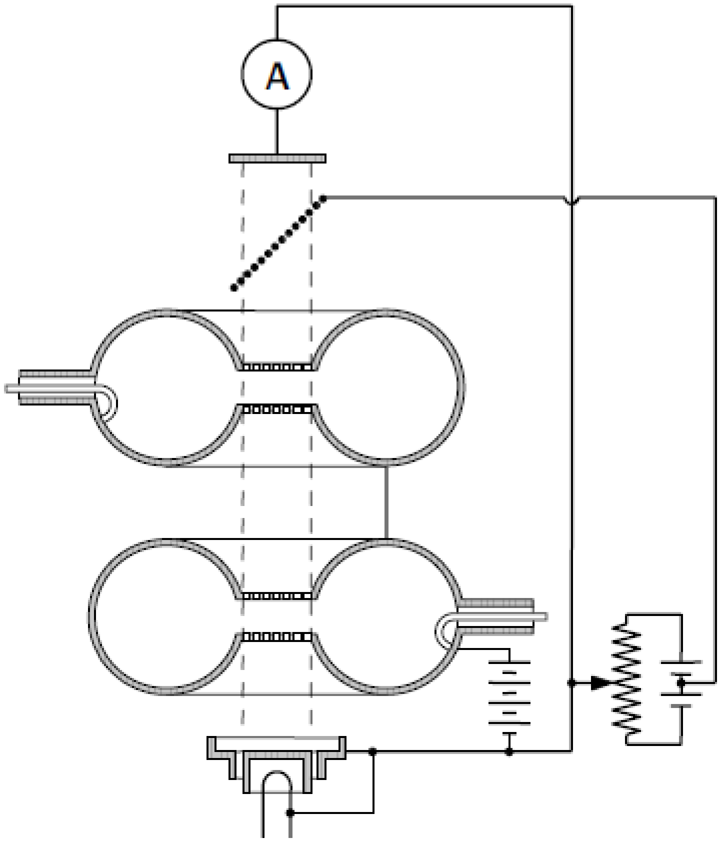
Klystron

- Invented by Varian brothers [JAP 10, 321 (1939)], replacing the lumped element tuned circuit with resonant cavities.
- The klystron is the most widely used microwave source for high-energy particle accelerators due to it function as high-gain amplifier with excellent frequency stability.

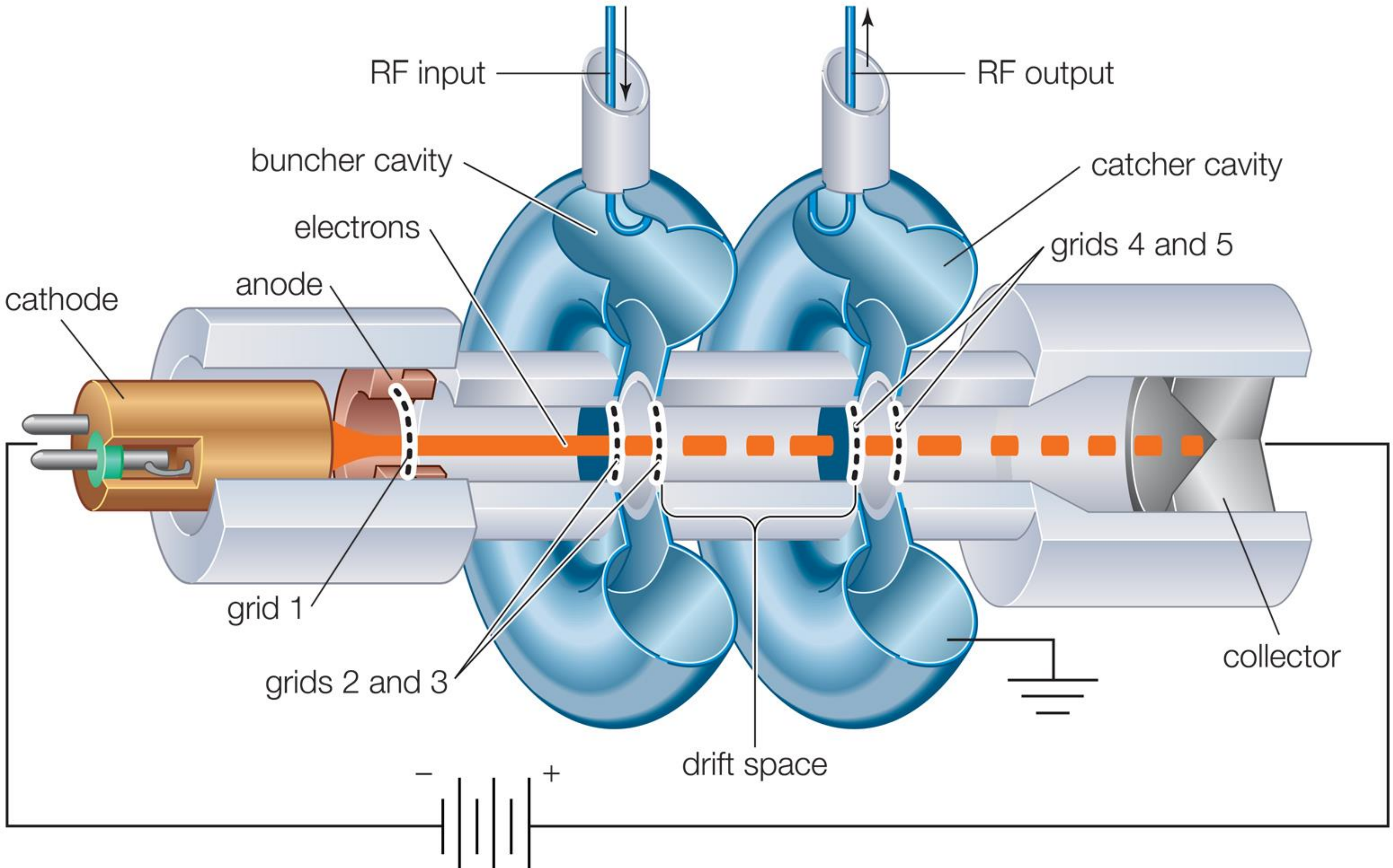

"A High Frequency Oscillator and Amplifier"
Russell H. Varian and Sigurd F. Varian

Abstract

A dc stream of cathode rays of constant current and speed is sent through a pair of grids between which is an oscillating electric field, parallel to the stream and of such strength as to change the speeds of the cathode rays by appreciable but not too large fractions of their initial speed. After passing these grids the electrons with increased speeds begin to overtake those with decreased speeds ahead of them. This motion groups the electrons into bunches separated by relatively empty spaces. At any point beyond the grids, therefore, the cathode ray current can be resolved into the original dc plus a nonsinusoidal ac. A considerable fraction of its power can then be converted into power of high frequency oscillations by running the stream through a second pair of grids between which is an ac electric field such as to take energy away from the electrons in bunches. These two ac fields are best obtained by making the grids form parts of the surfaces of resonators of the type described in [this Journal].



Klystron


RF input
RF output
buncher cavity
catcher cavity
electrons
grids 4 and 5
anode
cathode
grid 1
grids 2 and 3
collector
drift space
−
+

Driving resonant cavities with electron beams: general features

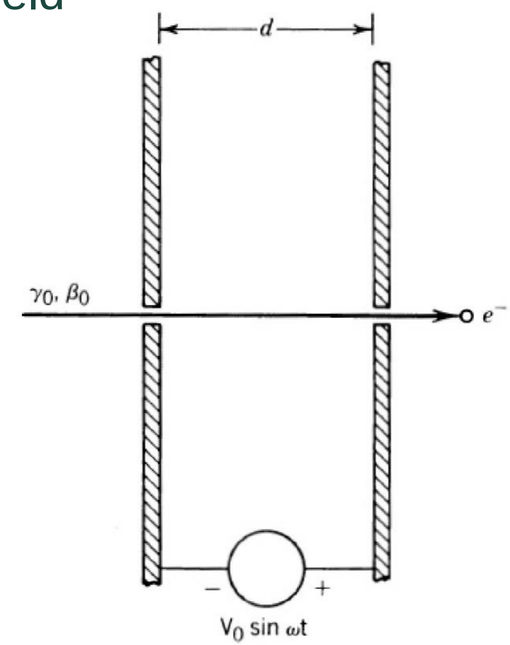
- Resonant cavities are often used to extract energy from charged particle beams for microwave radiation generation.
- We shall study some features of resonant cavities and develop lumped circuit element models that give a simplified description of beam interactions with resonant modes.
- Electron beams are used because they are easier to generate, accelerate, and transport than ion beams of equal power.
- In comparison with direct conversion devices like the inverse diode, resonant cavities have two important capabilities: frequency selection and impedance transformation:
  - While an inverse diode responds to all frequencies, energy transfer from a beam to a resonant cavity is strong only at specific frequencies. As a result, resonant cavities can generate highly monochromatic radiation from a modulated beam.
  - The charged particle beams in many devices have high kinetic energy and low current. On the other hand, most microwave loads have low AC impedance. The resonant cavity can act as a transformer, converting beam energy at high voltage and low current to electromagnetic energy at low voltage and high current.

Interaction between a single charged particle and a harmonic field variation

- Under the assumption of $eV_0 \ll (\gamma_0 - 1)m_e c^2$, the kinetic energy of an electron that crosses the gap changes by an amount

$$\Delta T_e = \int eE_z dz \cong \frac{eV_0}{d}\beta_0 c \int_{\frac{\phi}{\omega_0}-\frac{d}{2\beta_0 c}}^{\frac{\phi}{\omega_0}+\frac{d}{2\beta_0 c}} \sin(\omega_0 t)\, dt$$

$\phi = \omega_0 t_0$: Phase of the particle with respect to the oscillating gap field



- We obtain

$$\frac{\Delta T_e}{eV_0} = \left[\left(\frac{2\beta_0 c}{\omega_0 d}\right)\sin\left(\frac{\omega_0 d}{2\beta_0 c}\right)\right]\sin\phi = T\sin\phi$$

- Transit time factor in terms of the electron transit time $\Delta t = d/\beta_0 c$

$$T = \left(\frac{2}{\omega_0 \Delta t}\right)\sin\left(\frac{\omega_0 \Delta t}{2}\right)$$

- The energy transferred from the cavity to the beam over an interval t_0:

$$\Delta E = \int_0^{t_0} V_0 \sin(\omega_0 t)\, I(t)\, T\, dt$$

Energy exchange between the gap and the beam occurs only if the current varies at frequency ω_0

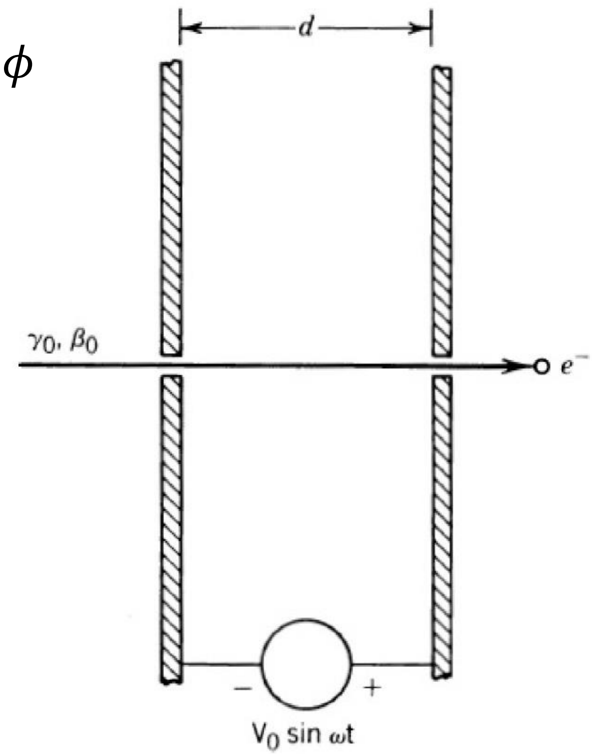
Interaction between an electron beam and a harmonic field variation

- One possible form for the current of a modulated beam is

$$I(t) = I_0[1 + \cos(\omega_0 t - \phi)]$$

- The time-averaged power transferred from the cavity to the harmonic component of an electron beam is:

$$\langle P \rangle = \int_0^{\frac{2\pi}{\omega_0}} \frac{TV_0 \sin(\omega_0 t)\, I_0 \cos(\omega_0 t - \phi)}{2\pi/\omega_0} dt = \frac{V_0 I_0 T}{2} \sin\phi$$



- In the phase range $-\pi \leq \phi \leq 0$, the beam drives voltage oscillations in the gap.
- If we add an external circuit to absorb energy from the gap voltage, we have the basis for a microwave generator.
- The power extracted from a beam with a given I_0 depends on the amplitude of gap voltage, the transit-time factor, and the phase between the beam current and the gap voltage.

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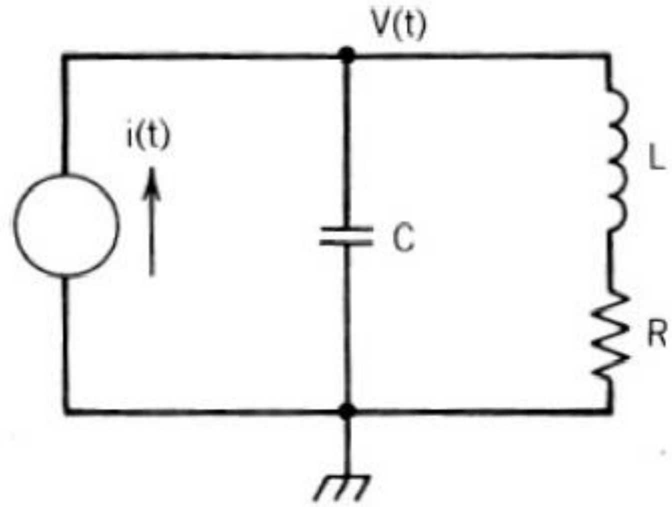
Beam-cavity interaction

- Suppose that the beam is the only source of power for the cavity and that the cavity walls are the only power sink. We model the beam as an ideal current source of $I_0 \sin \omega_0 t$.

- The cavity presents the following impedance to the beam for the TM_{010} mode:



$$Z(\omega) = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega RC}$$

- When $\omega = \omega_0$, the cavity impedance is resistive:

$$Z(\omega_0) = \frac{R + j\omega L}{j\omega RC} \approx \frac{L/C}{R}$$

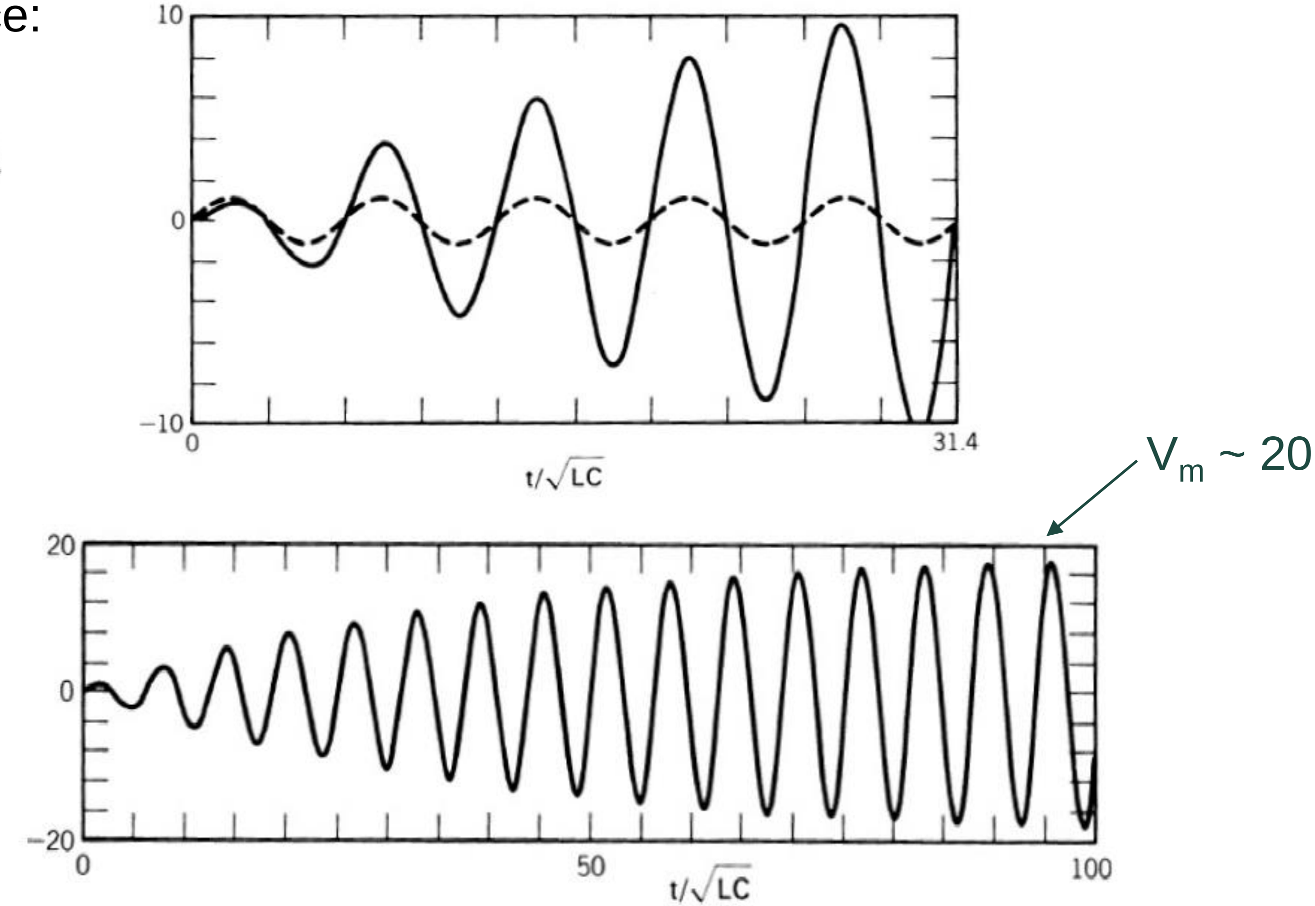
- The cavity voltage in steady state is 180° out of phase with the beam current. In other words, the cavity oscillation assumes a phase to extract the maximum power from the beam.

- Since $Z(\omega_0) \gg R$, the cavity acts as a transformer. Thus, a small beam current transfers energy at high voltage to the large circulating current of the resonant mode.

Beam-cavity interaction: time-dependent solution

- Cavity parameters: L = 1, C = 1, R = 0.05, Q = 20
- Beam current source:

$$i(t) = 0 \qquad (t < 0),$$
$$i(t) = \sin(t) \qquad (t \geqslant 0).$$



- The time to saturation is called the cavity fill time. The fill time is approximately Q/ω_0.

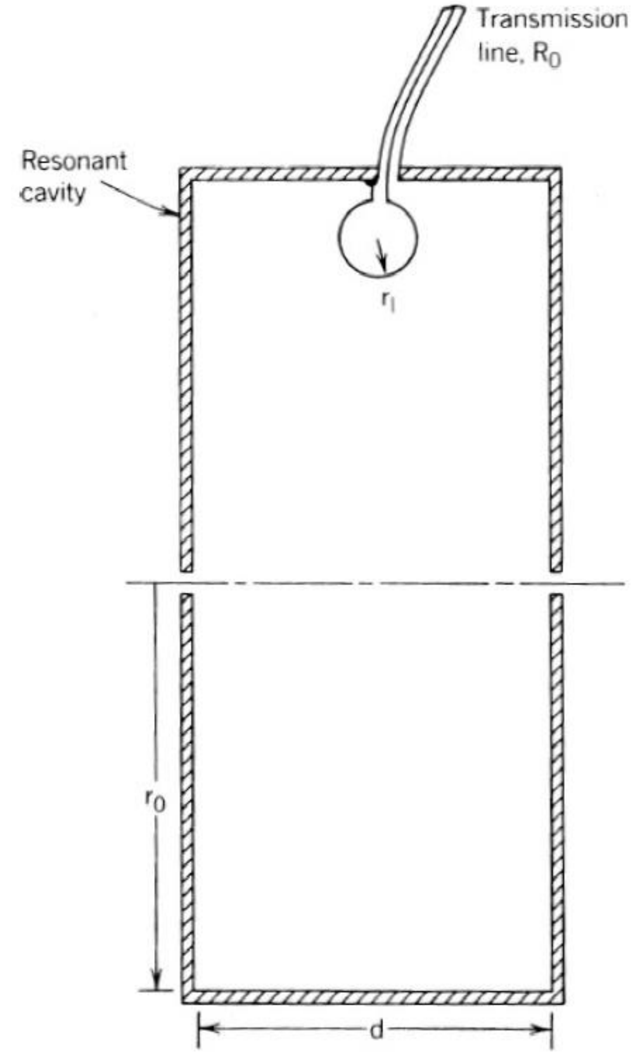
Energy extraction from a resonant cavity

- The center conductor of a transmission line penetrates the cavity near the outer radius, where it forms a loop that connects to the cavity wall.
- The area outlined by the loop is normal to the magnetic field of the TM_{010} mode. The oscillating field induces a voltage on the transmission line.
- Under assumption that the magnetic field is unperturbed by the presence of the loop, we can estimate the transmission input voltage by applying Faraday's law:

$$V_{line}(t) \cong (\pi r_1^2)\omega_0 \left(\frac{E_0}{c}\right) J_1(2.405)\sin(\omega_0 t)$$

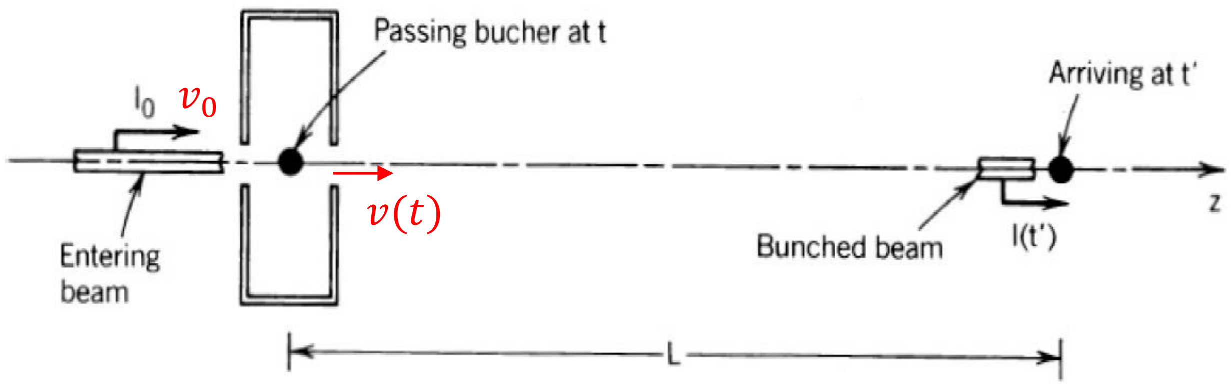
- If the loop drives a transmission line with matched termination R_0, then the time-averaged power leaving the cavity on the transmission line is

$$P_{out} = V_{line}^2/2R_0$$



Longitudinal beam bunching

- A charged particle beam interacts strongly with a resonant cavity only if it has a current modulation at the resonant frequency.
- The longitudinal compression of a steady-state beam into periodic pulses is called beam bunching.
- Suppose an incident monoenergetic beam has kinetic energy T_0 and uniform current I_0.



- Charge conservation implies that $I(t)$, the current at the observation point for particles that leave the injector at time t, is given by

$$I(t) = I_0(t)\left(\frac{dt}{dt'}\right) \qquad\qquad t' = t + L/v(t)$$

Exit velocity from the buncher gap

Longitudinal beam bunching

- The exit velocity from the buncher gap can be expressed by

$$v(t) = v_0\sqrt{1 + eV(t)/T_0}$$

- We obtain

$$t' \cong t + \left(\frac{L}{v_0}\right)\left(1 - \frac{eV(t)}{2T_0}\right) \quad \Rightarrow \quad \frac{dt'}{dt} = 1 - \left(\frac{L}{v_0}\right)\left(\frac{e}{2T_0}\right)\frac{dV(t)}{dt}$$

- For a harmonic voltage, the energy shift at the cavity exit is

$$eV(t) = eV_0T\sin\omega_0 t$$

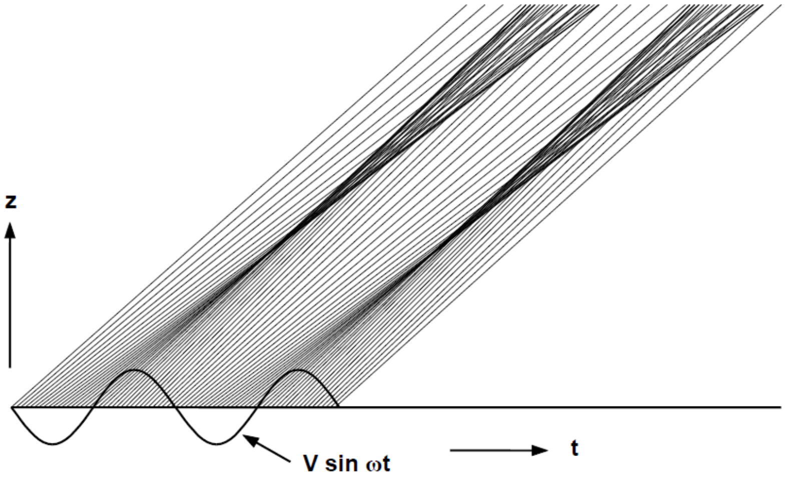
- The ratio of the interval at the observation point to that at the buncher is

$$\frac{dt'}{dt} = 1 - \left(\frac{L\omega_0}{v_0}\right)\left(\frac{eV_0T}{2T_0}\right)\cos\omega_0 t$$

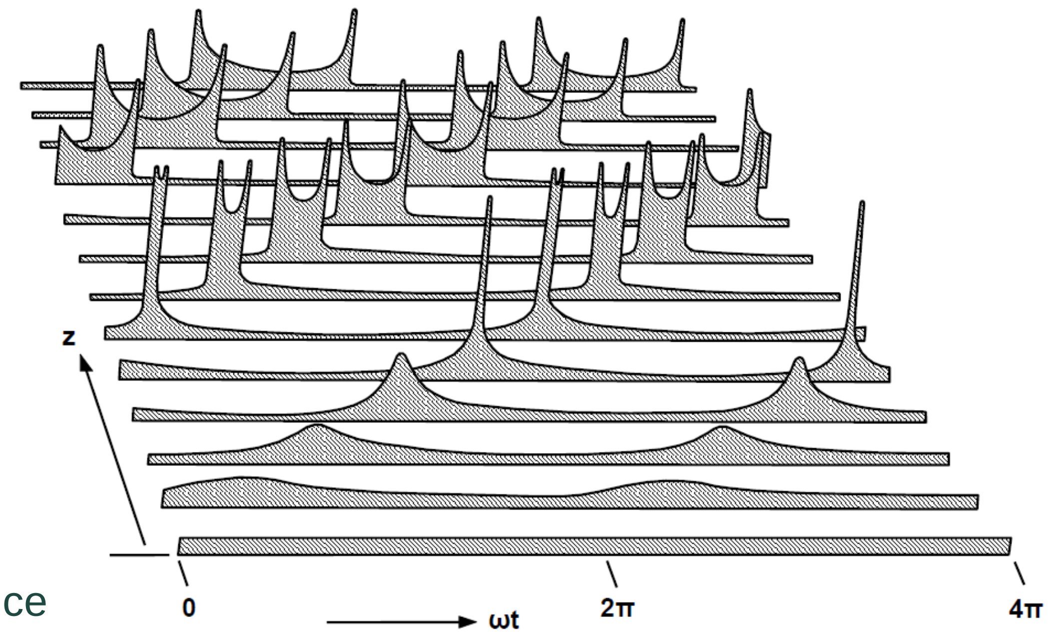
- The beam current at the observation point for particles that leave the injector at time t

$$I(t) = \frac{I_0}{1 - \chi\cos\omega_0 t} \qquad \chi = \left(\frac{L\omega_0}{v_0}\right)\left(\frac{eV_0T}{2T_0}\right)$$ Harmonic bunching parameter

Longitudinal beam bunching: Applegate diagram



Distance-time curves for the electrons in a velocity-modulated beam in a field-free drift space (Applegate diagram)

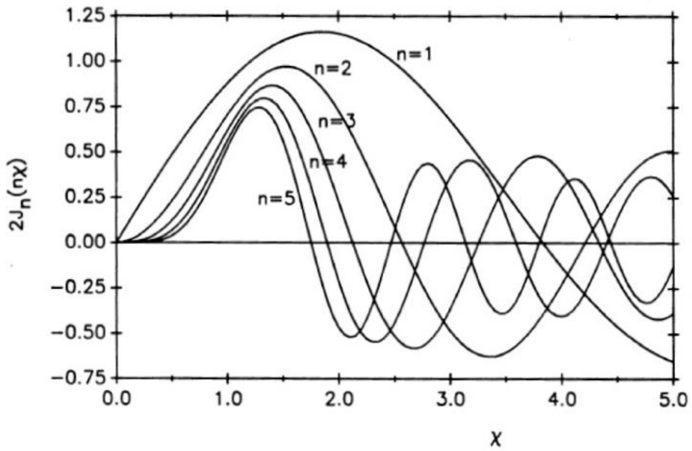


Current waveforms as a function of distance

Longitudinal beam bunching

- We seek a Fourier decomposition of the downstream current as a function of t', the arrival time:

$$I(t') = a_0 + \sum_{n=0}^{\infty} [a_n \cos(n\omega_0 t') + b_n \sin(n\omega_0 t')]$$

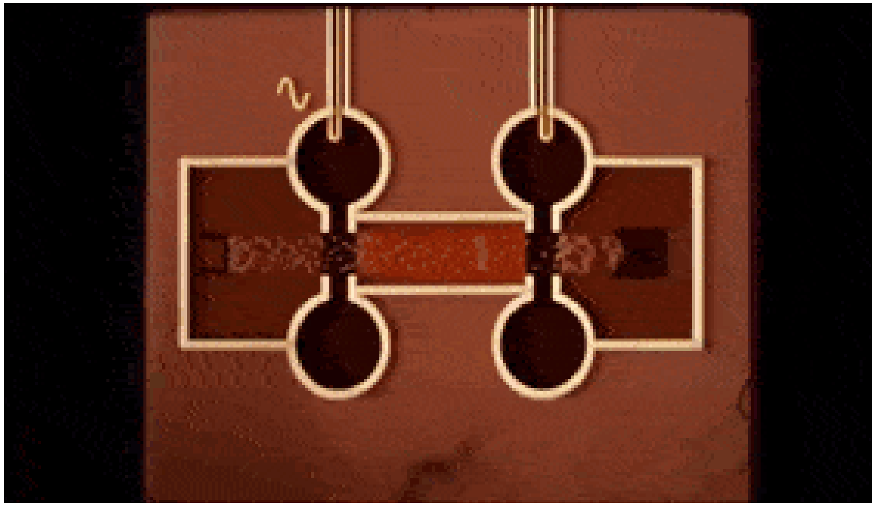


- The downstream current is given by:

$$I(t') = I_0 \left\{ 1 + \sum_{n=0}^{\infty} 2J_n(n\chi) \cos\left[n \left(\omega_0 t' - \frac{L\omega_0}{v_0} \right) \right] \right\}$$

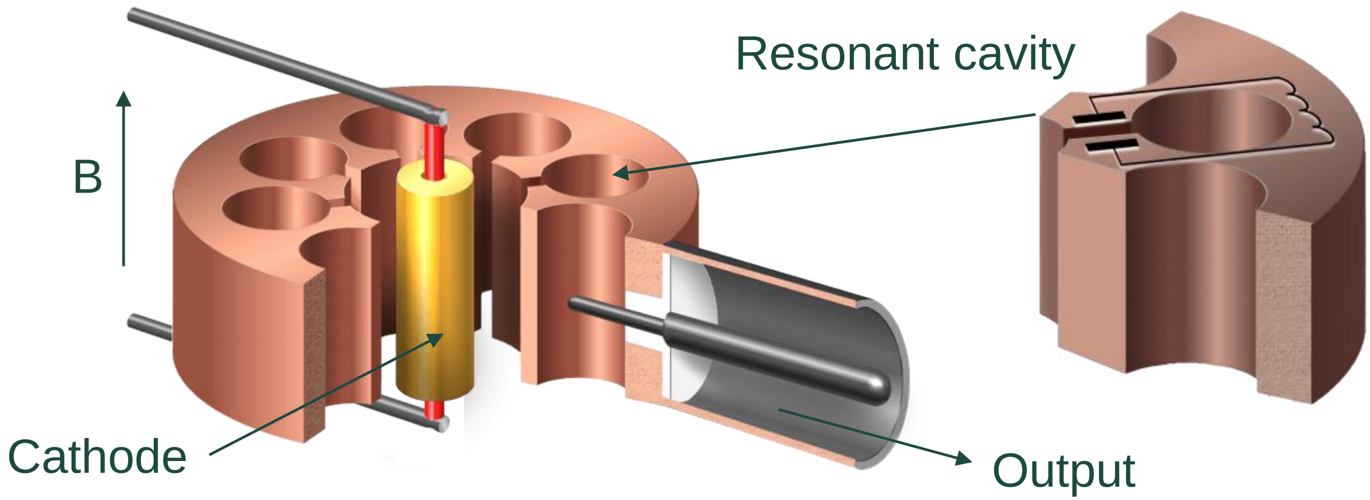
- The amplitude of the fundamental mode component reaches a maximum value at $\chi = 1.84$—at this point, $I_1 = 1.16 I_0$.
- To generate radiation at frequency f_0, the best location for the load cavity of a klystron is

$$L = 1.84 \left(\frac{v_0}{\omega_0} \right) \left(\frac{2T_0}{eV_0} \right)$$



Magnetron

- The magnetron was the first successful source of high-power microwave radiation. It is still a widely used microwave oscillator, with applications in industrial processing, microwave ovens, and radar.
- The magnetron converts the energy of a sheet electron beam to microwave radiation. A radial electric field between the electrodes causes an azimuthal drift of electrons emitted from the cathode.
- The anode is a complex structure, interrupted periodically by resonant cavities. The cavities support electromagnetic oscillations—the electric field of the modes points in the azimuthal direction and has maximum amplitude where the cavities connect to the anode-cathode gap.

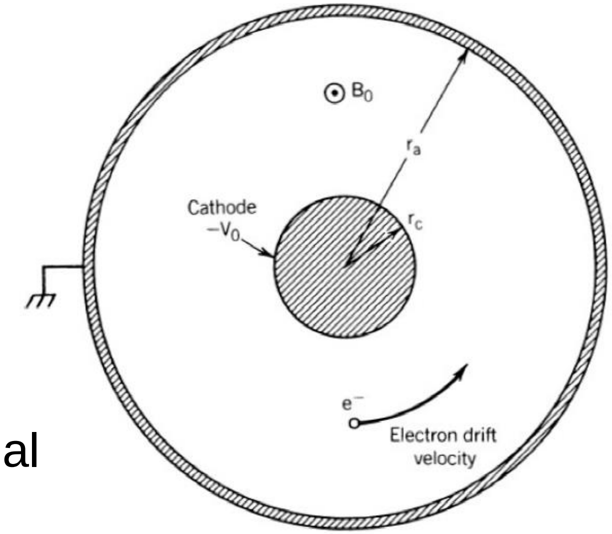


Electron equilibrium in a magnetically-insulated gap between two cylinders

- Conservation of canonical angular momentum gives the angular velocity of electrons as a function of radius:

$$v_\theta(r) = eA_\theta(r)/m_e \qquad B_z = \frac{1}{r}\frac{\partial}{\partial r} rA_\theta(r) = B_0 \qquad A_\theta(r) = B_0(r^2 - r_c^2)/2r$$

$$\frac{d\theta}{dt} = \left(\frac{eB_0}{2m_e}\right)\left(1 - \frac{r_c^2}{r^2}\right) = \left(\frac{\omega_g}{2}\right)\left(1 - \frac{r_c^2}{r^2}\right)$$



- When electron flow is laminar (Brillouin flow; $v_r = 0$), energy conservation has the form:

$$-e\phi(r) + m_e r^2 \left(\frac{d\theta}{dt}\right)^2 = 0$$

- We obtain the self-consistent radial variation of potential for Brillouin equilibrium:

$$\phi(r) = \left(\frac{eB_0^2}{8m_e}\right)\left(1 - \frac{r_c^2}{r^2}\right)^2 \qquad \phi(r_c) = 0, \qquad \phi(r_a) = V_0$$

- Hull cut-off condition for electron cloud extending almost to the anode ($r = r_a$)

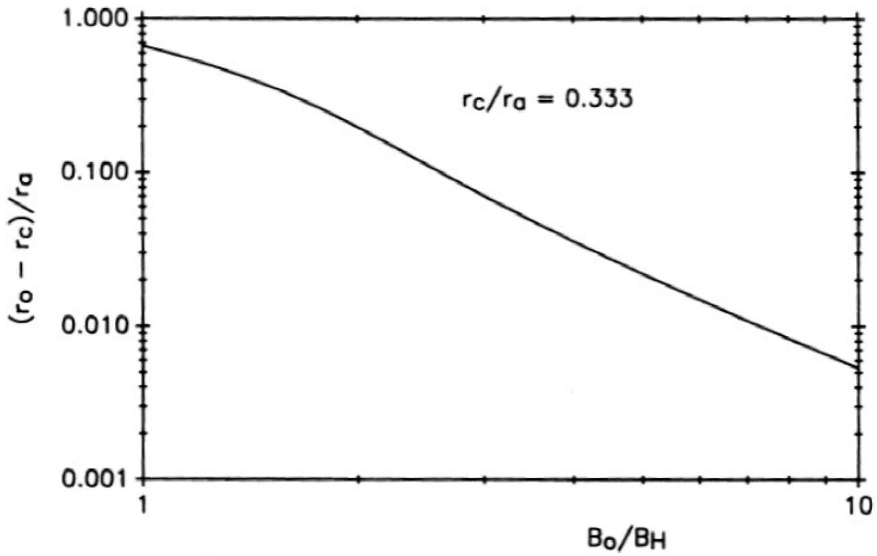
$$B_0 \geq \left(\frac{8m_e V_0}{e}\right)^{1/2}\left(\frac{r_a}{r_a^2 - r_c^2}\right) = B_H$$

Brillouin cloud motion

- The region filled by electrons ($r_c \le r \le r_0$) is often called the Brillouin cloud.

$$\left[\frac{(r_0/r_a)^2-(r_c/r_a)^2}{(r_0/r_a)}\right]^2+\left[\frac{(r_0/r_a)^4-(r_c/r_a)^4}{(r_0/r_a)^2}\right]\ln\left(\frac{r_a}{r_0}\right)^2-\left(\frac{B_H}{B_0}\right)^2\left[1-\left(\frac{r_c}{r_a}\right)^2\right]^2=0.$$

$$\Rightarrow \quad r_0 \sim f\left(\frac{B_0}{B_H}\right)$$



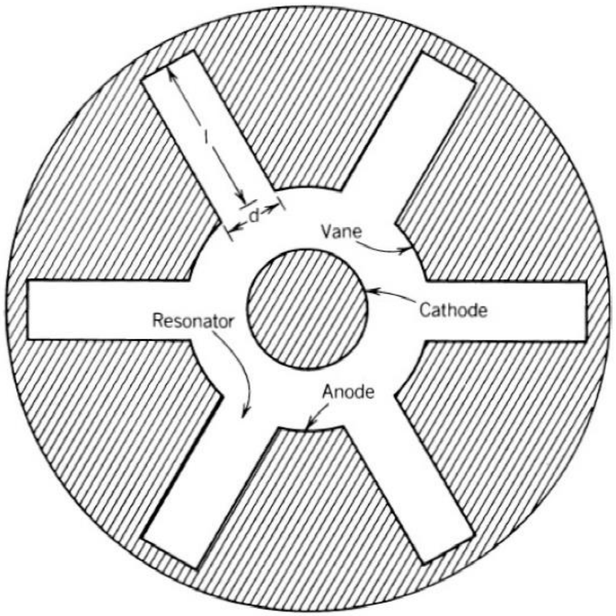
- The most important property of the electron distribution for the operation of the magnetron is the azimuthal velocity of electrons on the Brillouin cloud surface. The velocity should closely match the phase velocity of the slow wave supported by the resonant structure.

$$\frac{v_\theta}{c}=\frac{(2eV_0/m_ec^2)^{1/2}}{(1-r_c^2/r_a^2)}\left[\frac{B_0}{B_H}\frac{r_0}{r_a}\left(1-\frac{r_c^2/r_a^2}{r_0^2/r_a^2}\right)\right].$$

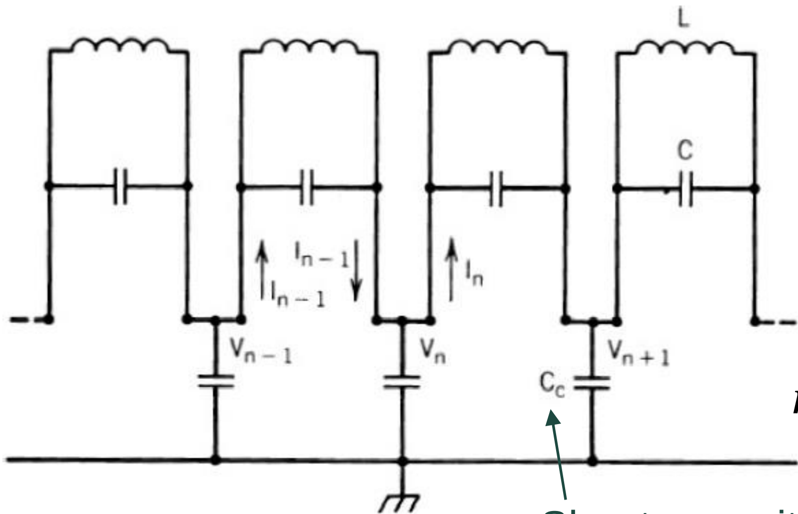
As an example, for $V_0 = 70$ kV, $r_c/r_a = 0.5$, and $B_0/B_H = 2$, then $r_0/r_a = 0.67$ and $v_\theta/c = 0.41$.

Coupled-cavity resonator

- In the magnetron, the resonators are not independent because adjacent cavities share a vane.



$$\omega_0 = \frac{1}{\sqrt{LC}}$$



$$\kappa = \frac{C_c}{C}$$

- From circuit calculation, we obtain the frequency of a coupled mode with phase advance μ.

$$\Omega = \frac{\omega}{\omega_0} = \left(\frac{1-\cos\mu}{1-\cos\mu+\kappa/2}\right)^{1/2}$$

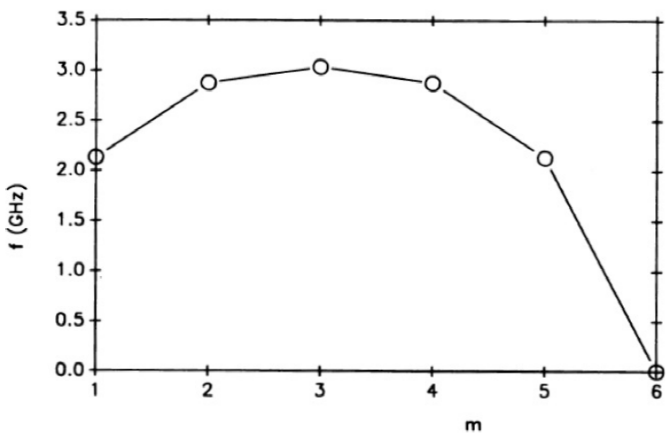
$$\mu = m\frac{2\pi}{N}, \qquad m = 1,2,3,\cdots$$

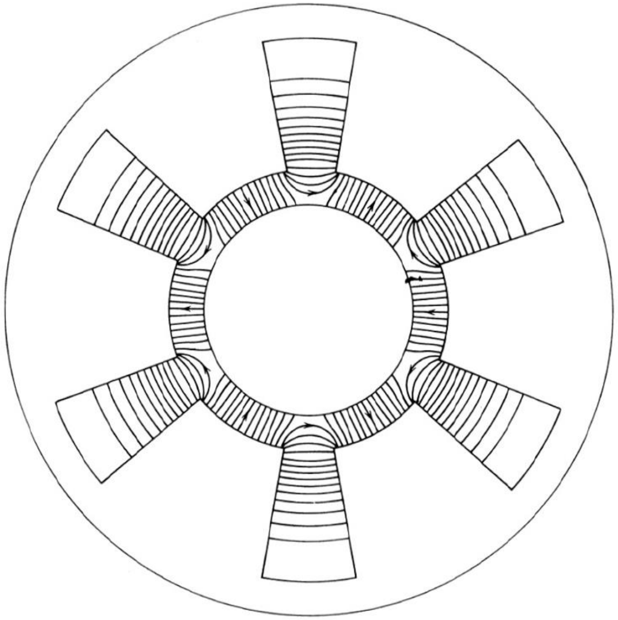
No. of cavities

Coupled-cavity resonator

- The π mode is the standard oscillation mode for a magnetron—it has the highest frequency. In the π mode, the voltage polarity reverses between adjacent cavities.

$$\Omega_{N/2} = \frac{\omega}{\omega_0} = \left(\frac{1}{1+\kappa/4}\right)^{1/2}$$



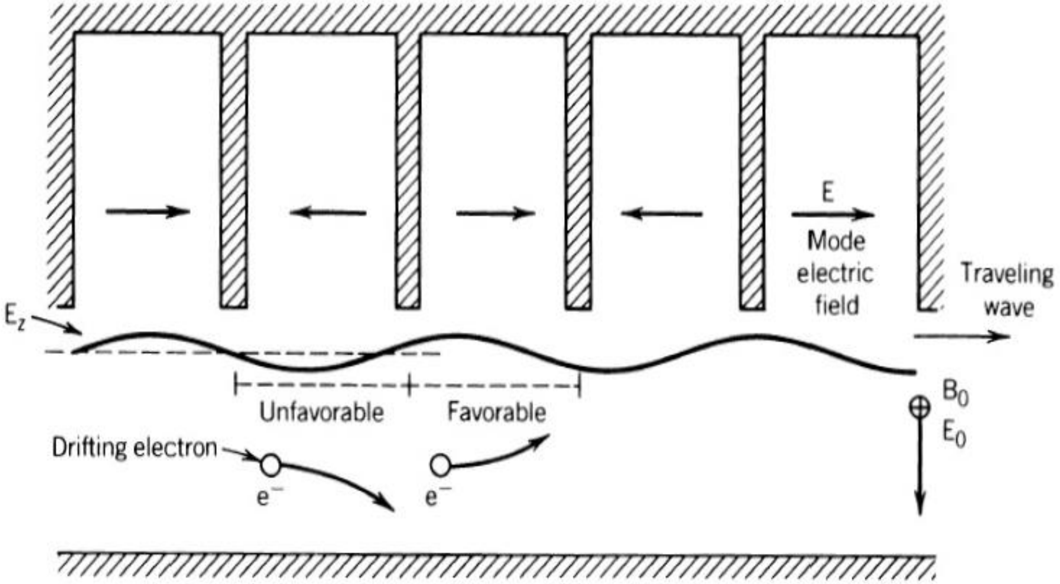


- For the best power extraction, the drift velocity of electrons at the surface of the Brillouin cloud should match the phase velocity of the π mode fields. Electrons experience a nonzero average deceleration (or acceleration) if they move a distance $2\pi r_0/N$ in a time $1/2f_0$, i.e.

$$\omega/k\ (\pi\ \text{mode}) = 4\pi r_c f_0/N$$

Electron interaction with electromagnetic waves in magnetron

- Electrons moving at the same velocity as the traveling wave experience a constant azimuthal electric field. This field superimposes a radial ExB drift on the azimuthal motion. In the region of decelerating electric field, the electrons drift radially outward. Conversely, electrons drift toward the cathode in regions of accelerating electric field.
- The electrons that yield energy to the wave move to a position where they interact more strongly—these electrons are called favorable-phase electrons. Electrons that absorb energy from the wave toward the cathode where the azimuthal electric field is weak—they are called unfavorable-phase electrons.



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Electron interaction with electromagnetic waves in magnetron

- Because of the radial electron displacement, the drifting beam gives up more energy to the wave than it absorbs.
- We should note that the mechanism that concentrates electrons in phase does not depend on longitudinal bunching—instead, it results from the radial electron drift and the variation in wave amplitude across the anode-cathode gap.