● **The position of a particle of mass *m* is given by the Cartesian coordinates** (x, y, z). **Assuming a potential energy function** $V=\frac{1}{2}k(x^2+y^2+z^2)$ **and a constraint described by the equation**

$2\dot{x}+3\dot{y}+4\dot{z}+5=0$ **find**

(a) Differential equation of motion.

(b) Velocity of the moving constraint.

Sol)

Kinetic energy in Cartesian coordinate system :

$$T=\frac{1}{2}\mathrm{m}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$\mathrm{L}=\mathrm{T}-\mathrm{V}$$

***Rheonomic system : 2dx+3dy+4dz+5dt=0 : Elimination ?**

Integrable ?

(a) Equations of motion

$$a_{1x}=2, a_{1y}=3, a_{1z}=4, a_{1t}=5$$

From $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=\sum_{j=1}^{m}\lambda_j c_{ji}$$

$$m\ddot{x} + kx = 2\lambda ---(1)$$
$$m\ddot{y} + ky = 3\lambda ---(2)$$
$$m\ddot{z} + kz = 4\lambda ---(3)$$

(b) Velocity of the moving constraint

$(1)\times 2+(2)\times 3+(3)\times 4$

$$m(2\ddot{x}+3\ddot{y}+4\ddot{z})+k(2x+3y+4z)=29\lambda$$

For constraint eqn

$(2\ddot{x}+3\ddot{y}+4\ddot{z})=0,$ and

integrating..the..moving..constraint : $(2x+3y+4z)+5\mathrm{t}=a$

$a:cons\tan t$

$\sim k(2x+3y+4z)=k(a-5t)\equiv 29\lambda$

$->$

$$\lambda=\frac{k(a-5t)}{29}$$

Eqns(1),(2),(3)

*Particular..solution..*related..with..λ

$$x_p=-\frac{2(a-5t)}{29}k,\ y_p=...z_p=...$$

$$\sim \dot{x}_p=-\frac{-10}{29}k,,,,,,,$$

$v=$~~~~

● Holonomic system (Gravity: g)

$y = 3x^2 \quad : \quad 6x\dot{x} - \dot{y} = 0 : 6xdx - dy = 0$

Elimination (x)

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$
$$V = mgy$$

(a) **Equations of motion for *x, y***

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j c_{ji}$$

Constraint : one

$$m\ddot{x} = \lambda_{1x}, m\ddot{y} + mg = \lambda_{1y}$$

From the constraint eqn:

$$y = 3x^2 : \quad 6x\delta x - \delta y = 0 :$$
$$a_{1x} = 6x, a_{1y} = -1$$
$$m\ddot{x} = 6x\lambda_1, m\ddot{y} + mg = -\lambda_1$$

(b) Find the maximum constraint force

for $\dot{y}(0) = 0, y(0) = y_0$

$$|C| = \sqrt{(\lambda_1 a_{1x})^2 + (\lambda_1 a_{1y})^2} = |\lambda_1| \sqrt{a_{1x}{}^2 + a_{1y}{}^2} = |\lambda_1| \sqrt{36x^2 + 1^2}$$

$$\frac{dC}{dt} = 0!!!$$

$$C = mg(12y_o + 1)$$

INTEGRALS OF MOTION

Up to now: Concern on formulating the eqs of motion.

<u>Lagragian formuation :</u>

<u>A systematic way to apply Newton's laws of motion</u> using generalized coordinates.

What is the next step ?

Actually analyze the dynamics based

on the eqns of motion.

~ Eqns consist of a system of *n* ODE, each of the 2nd order!

~ Typically nonlinear.

Except, the eqns of motion are linear.

Eqns of motions are sometimes linearized based on the small displacements assumptions.

This may have some utility in stability analysis, but linearization typically destroys the applicability of the eqns of motion.

Generally, eqns of motion are too complicated !

- ➔ **by integration based on elementary methods.**

For specified initial conditions, the eqns of motion are usually integrated numerically.

Example, Runge-Kutta algorithms : good accuracy.

Drawback is that the resultant numerical solution is only valid for *one* set of initial conditions.

Aim of analytical mechanics ?

~Analysis of the eqns of motion themselves, without actually solving the system of eqns.

Such qualitative analysis was introduced in Chapter 1 with the energy analysis of conservative systems.

Conservative systems are disting-uished by conservation of total mechanical energy.

- Allowed the partial integration of eqn of motion.

This concept is readily extended to general systems.

Suppose that a certain combination of the generalized coordinates and velocities remains *invariant* during the evolution of the system. : If there exists some function

$$G(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$$

that remains constant over time, then

$$G(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) = C \qquad \textbf{(2.74)}$$

or equivalently

$$\frac{dG}{dt} = 0$$

The relation (2.74) is called an integral of the motion.

C **is called a *constant of motion*.**

~ An integral of motion represents a quantity that is conserved during the motion.

There are <u>only first derivatives</u> in an integral of motion, so each integral of motion

~ a partial integration of the original system : used as reduction of the order of the system.

Lagrange's eqns represent *n* second-

order (partial) diff. eqns. ~

Ideally, the solution of Lagrange's equations consists of finding 2*n* integral of motion (2.74), each containing *only* the generalized coordinates.(?)

This is typically not possible, but certain systems do admit some integrals of motion.

For example, in a *conservative system*, the total mechanical energy is an *invariant* of the system.

$$T(q,\dot{q},t) + V(q) = \text{const}$$

is an integral of the motion. Value of the constant of motion is determined by initial conditions.

A conservative system is a special case of a Lagrangian system.

~ Eqs of motion for a Lagrangian system :

$$\frac{d}{dt}\left[\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_k}\right]-\frac{\partial L(q,\dot{q},t)}{\partial q_k}=0 \qquad \textbf{(2.75)}$$

Total time derivative of $L(q,\dot{q},t)$ **is**

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$(\sim L,_{q_k} \bullet \dot{q}_k + L,_{\dot{q}_k} \bullet \ddot{q}_k + L,_t \,.\, for.k = 1..n)$$

*** Lagrange's Eqns (2.75) we have**

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

this means

$$\frac{dL}{dt} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_{k=1}^{n} \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial t}$$

Therefore

$$\frac{d}{dt}\left(\sum_{k=1}^{n}\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}-L\right)=-\frac{\partial L}{\partial t} \quad ..(2.76)$$

Dimension of *L* is energy, the quantity in the parentheses is known as the Jacobi energy function

$$h(\mathrm{q},\dot{\mathrm{q}},t)=\sum_{k=1}^{n}\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}-L$$

Eqn (2.76) can be written :

$$\frac{dh}{dt}=-\frac{\partial L}{\partial t}$$

~ If the Lagrangian does not contain time *t* explicitly,

then the Jacobi energy function is *invariant* during the motion. ~

Energy function = an integral of the motion

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = h = \text{cosnt}$$

Generally : Energy integral into a more familiar form by referring to the kinetic energy expression as

$$L = T_2 + T_1 + T_0 - V$$

If *V* depends only on the generalized

coordinates, then

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = 2T_2 + T_1$$

the Jacobi energy integral has

$$T_2 - T_0 + V = h \qquad \textbf{(2.77)}$$

It is important to note here that the Jacobi energy integral is not in general the *total energy*, since the term T_1 is missing. It is still a constant of motion.

Without moving coordinates,

$T_1 = T_0 = 0$, **energy integral *is* the total energy ~ Conservation of total mechanical energy:**

$$T + V = h$$

~Kinetic energy is purely quadratic in the generalized coordinates are called *natural systems*.

The cart-pendulum system is a natural system, ~

Jacobi energy integral is the total energy of the system.

As a modification of this example,

Suppose :

Motion of the cart ~

A constant speed $\dot{x} = v_0$ **for 1 DOF for the pendulum,** θ **.**

Kinetic energy :

$$T = \frac{1}{2}(m+M)v_0^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mv_0\dot{\theta}l\cos\theta$$

Potential energy :

$$V = -mgl\cos\theta \, .$$

Then, Lagrangian :

$$L = \frac{1}{2}(m+M)v_0^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mv_0\dot{\theta}l\cos\theta + mgl\cos\theta$$

- **Kinetic energy is not purely quadratic, but**

Eqn (2.77) still gives the Jacobi energy integral as

$$\frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}(m+M)v_0^2 + mgl\cos\theta = h : (2.78)$$

Constant *h* is specified with initial conditions !

Setting $t = t_0$ in Eqn (2.78), then

$$h = \frac{1}{2} m l^2 \dot{\theta}_o{}^2 - \frac{1}{2}(m+M) v_0^2 + mgl\cos\theta_o$$

Hence Eqn (2.78) may also be written as

$$\frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta = \frac{1}{2} m l^2 \dot{\theta}_0{}^2 - mgl\cos\theta_0$$

Eqns (2.78) and (2.79) : Equivalent forms of the energy

integral for the system.

It should be note:

This system is *not* conservative, since work must be done

in order to maintain the constant speed of the cart.

Hence the total mechanical energy is not conserved. The integral of motion (2.79) represents conservation of the

energy as computed by an observer riding on the cart.

The Jacobi energy integral is one type of invariant of motion associated with conservative systems. Certain forms

of the Lagrangian admit other integrals of motion.

These results when the Lagrangian does not contain some of the generalized coordinates.

IGNORABLE COORDINATES

Lagrangian system (*n* dof) and generalized coordinates

$q_1, q_2, \ldots, q_n$ **.**

Suppose : There are *m* coordinates $q_{n-m+1}, \ldots, q_n$ **,**

do *not* appear in the Lagrangian, but the

corresponding generalized velocities do.

$$L = L(q_1, q_2, \ldots, q_{n-m}; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$$

Eqns of motion for the first $n-m$ **coordinates are**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, n-m$$

and the eqns for the remaining *m* coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad i = n - m + 1, \ldots, n$$

(2.80)

Eqn (2.80) : Last *m* coordinates $q_{n-m+1}, \ldots, q_n$ **do not appear in the Lagrangian.**

Define it as ignorable coordinates or cyclic coordinates.

Or inactive coordinates.

Anyway, for $i = n - m + 1, \ldots, n$**, eqns (2.80) can be as**

$$\frac{\partial L}{\partial \dot{q}_i} = C_i$$

(2.81)

~ Generalized coordinates and velocities : ***conserved,***

➔ Eqns (2.81) are also referred to as conservation eqns.

Potential function ***V*** **does depend on generalized**

velocities,

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

then,

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

Thus the integrals of motion (2.81) can be

$$p_i = C_i \quad ..(2.82)$$

: Generalized momenta conjugate to the ignorable

coordinates are conserved. ~ The

individual

conservation eqns may be physically interpreted based

on the physical significance of each ignorable

coordinate.

The striking result : Eqns of motion corresponding to the

ignorable coordinates have been partially integrated.

→ $n - m$ **eqns remain to be analyzed.**

Moreover, Eqns (2.81) do not contain any ignorable

coordinates. So (2.81) or (2.82) can be solved for

the generalized velocities of the ignorable coordinates

$\dot{q}_{n-m+1}, \ldots, \dot{q}_n$ **with remaining coordinates.**

: For only $n-m$ **eqns of motion in the non-ignorable**

generalized coordinates

$q_1, q_2, \ldots, q_{n-m}$ **.**

Remaining eqns of motion contain the *constants* c_i **, but**

these are determined from initial conditions.

~ Analysis of the system reduces to the analysis of only

$n-m$ **degrees of freedom.**

A more systematic approach for the elimination of

ignorable coordinates is to eliminate

the ignorable

variables *before* the eqns of motion are formulated.

Introduce a new function of the generalized coordinates

and velocities.

As above, the *m* conservation eqns associated with each

of the ignorable coordinates,

$$\frac{\partial L}{\partial \dot{q}_i} = C_i, \qquad i = n-m+1,\ldots,n$$

(2.83)

are solved for $\dot{q}_{n-m+1},\ldots,\dot{q}_n$ in terms of the remaining

coordinates and the constants C_i.

Routhian function is defined as,

$$R = \sum_{i=n-m+1}^{n} C_i \dot{q}_i - L$$

: Generalized velocities $\dot{q}_i$ are replaced by the

expressions obtained by solving Eqns (2.83) for $\dot{q}_i$.

The result is a function in the non-ignorable coordinates

??

:Partial derivatives of the Routhian function w.r.t the

Non-ignorable coordinates and velocities, then

$$\frac{\partial R}{\partial q_k} = -\frac{\partial L}{\partial q_k}, \qquad k = 1, 2, \dots, n-m$$

$$\frac{\partial R}{\partial \dot{q}_k} = -\frac{\partial L}{\partial \dot{q}_k}, \qquad k = 1, 2, \dots, n-m$$

(2.85)

Substitution eqn (2.85) into Lagrange's eqns for non-

ignorable coordinates results in the $n-m$ **eqns of motion**

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right)-\frac{\partial R}{\partial q_k}=0, \qquad k=1,2,\ldots,n-m$$

(2.86)

Once again, ignorable coordinates have been effectively

eliminated to reduce the problem to a mere $n-m$ **d.o.f**

- Reduced system of $n-m$ **eqns**

contains the *m* constants

of motion $c_{n-m+1}, \ldots, c_n$**.**

Finally, the ignorable coordinates of Routhian

$$\dot{q}_i = \frac{\partial R}{\partial C_i} \quad \ldots(2.87)$$

: Constant c_i **in (2.87) is considered arbitrary until**

the initial conditions are invoked. ~ Eqn (2.87) can be

integrated as

$$q_i(t) = \int_{t_0}^{t} \frac{\partial R}{\partial C_i} d\tau, \qquad i = n - m + 1, \ldots, n,,,, (2.88)$$