It should be note:

This system is *not* conservative, since work must be done in order to maintain the constant speed of the cart.

Hence the total mechanical energy is not conserved.

The integral of motion (2.79) represents conservation of the energy as computed by an observer riding on the cart.

The Jacobi energy integral is one type of invariant of

motion associated with conservative systems. Certain forms of the Lagrangian admit other integ-rals of motion. These results when the Lagrangian does not contain some of the gene-ralized coordinates.

IGNORABLE COORDINATES

Lagrangian system (*n* dof) and generalized coordinates

$$q_1, q_2, \ldots, q_n .$$

Suppose : There are *m* coordinates $q_{n-m+1},\cdots,q_n$ **, do *not* appear in the Lagrangian, but the corresponding generalized velocities do.**

$$L = L(q_1, q_2, \ldots, q_{n-m}; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$$

Eqns of motion for the first $n-m$ **coordinates are**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, n-m$$

and the eqns for the remaining *m* coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad i = n-m+1, \ldots, n \tag{2.80}$$

Eqn (2.80) : Last *m* coordinates $q_{n-m+1}, \cdots, q_n$ **do not appear in the Lagrangian.**

Define it as ignorable coordinates or cyclic coordinates.

Or inactive coordinates.

Anyway, for $i = n - m + 1, \dots, n$**, eqns (2.80) can be as**

$$\frac{\partial L}{\partial \dot{q}_i} = C_i \qquad \textbf{(2.81)}$$

~ Generalized coordinates and velocities : ***conserved*****,**

➔ Eqns (2.81) are also referred to as conservation eqns.

Potential function ***V*** **does depend on generalized velocities,**

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

then,

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

Thus the integrals of motion (2.81) can be

$$p_i = C_i ..(2.82)$$

: Generalized momenta conjugate to the ignorable

coordinates are conser-ved. ~ The individual conservation

eqns may be physically interpreted based on the physical significance of each ignorable coordinate.

The striking result :

Eqns of motion corresponding to the ignorable coordinates have been partially integrated.

→ $n-m$ **eqns remain to be analyzed.**

Moreover, Eqns (2.81) do not contain any ignorable coordinates. So (2.81) or (2.82) can be solved for the generalized velocities of the ignorable coordinates

$\dot{q}_{n-m+1}, \ldots, \dot{q}_n$ **with remaining coordinates.**

: For only $n-m$ **eqns of motion in the non-ignorable generalized coordinates** $q_1, q_2, \ldots, q_{n-m}$ **.**

Remaining eqns of motion contain the *constants* c_i**, but**

these are determined from initial conditions.

~ Analysis of the system reduces to the analysis of only

$n-m$ **degrees of freedom.**

A more systematic approach for the elimination of

ignorable coordinates is to eliminate the ignorable

variables *before* the eqns of motion are formulated.

Introduce a new function of the generalized coordinates

and velocities.

As above, the *m* conservation eqns associated with each of the ignorable coordinates,

$$\frac{\partial L}{\partial \dot{q}_i} = C_i, \qquad i = n-m+1, \ldots, n \qquad \textbf{(2.83)}$$

are solved for $\dot{q}_{n-m+1}, \ldots, \dot{q}_n$ in terms of the remaining coordinates and the constants C_i.

Routhian function is defined as,

$$R = \sum_{i=n-m+1}^{n} C_i \dot{q}_i - L$$

: Generalized velocities $\dot{q}_i$ are replaced by the expressions obtained by solving Eqns (2.83) for $\dot{q}_i$.

The result is a function in the non-ignorable coordinates

:Partial derivatives of the Routhian function w.r.t the

Non-ignorable coordinates and velocities, then

$$\frac{\partial R}{\partial q_k} = -\frac{\partial L}{\partial q_k}, \qquad k = 1, 2, \dots, n-m$$

$$\frac{\partial R}{\partial \dot{q}_k} = -\frac{\partial L}{\partial \dot{q}_k}, \qquad k = 1, 2, \dots, n-m \qquad \textbf{(2.85)}$$

Substitution eqn (2.85) into Lagrange's eqns for non-ignorable coordinates results in the $n-m$ **eqns of motion**

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right) - \frac{\partial R}{\partial q_k} = 0, \qquad k = 1, 2, \dots, n-m \qquad \textbf{(2.86)}$$

Once again, ignorable coordinates have been effectively eliminated to reduce the problem to a mere $n-m$ d.o.f

- Reduced system of $n-m$ eqns contains the *m* constants of motion $C_{n-m+1},\ldots,C_n$.

Finally, the ignorable coordinates of Routhian

$$\dot{q}_i = \frac{\partial R}{\partial C_i} \quad ...(2.87)$$

: Constant C_i in (2.87) is considered arbitrary until the initial conditions are invoked. ~ Eqn (2.87) can be integrated as

$$q_i(t) = \int_{t_0}^{t} \frac{\partial R}{\partial C_i} d\tau, \qquad i = n-m+1, \ldots, n,,,, (2.88)$$

Routhian Function :

A particle moving in a plane under to a central force

derivable from a potential function $V(r)$.

~ Conservative ! and a Lagrangian expression

in polar coordinates as

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- θ is ignorable ; conjugate momentum is constant,

$$p_\theta = mr^2\dot{\theta} = C_\theta$$

>> **Angular momentum** **of the particle is** **conserved.**

Furthermore, the Routhian function:

$$R(r,\dot{r},C_\theta)=\dot{\theta}C_\theta-L=\frac{C_\theta}{mr^2}\times C_\theta-\{\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2(\frac{C_\theta}{mr^2})^2-V(r)\}$$

>>

$$R(r,\dot{r},C_\theta)=-\frac{1}{2}m\dot{r}^2+\frac{C_\theta^2}{2mr^2}+V(r)$$

: **Cyclic Coordinate is Removed !: Single DOF !!**

Thus, Eqn becomes (2.86) for k=1…n-m as

$$m\ddot{r} - \frac{C_\theta^2}{mr^3} + V'(r) = 0....(2.89)$$

Note : Eqn (2.89) denotes an entire family! of des parameterized by the constant C_θ .

C_θ , : Conserved angular momentum >> Eqn(2.89) for

$r(t)$: Non-linear >> Numerical solution!

*** Jacobi energy function : Additional integral of motion.**

- **Energy integral from the Routhian function *R***

Eq.(2.89) * dr ~

$$\left(m\ddot{r}-\frac{C_{\theta}^{2}}{mr^{3}}+V'(r)=0\right)\bullet dr \ldots.(2.89)'$$

>>

$$\frac{1}{2}m\dot{r}^{2}+\frac{C_{\theta}^{2}}{2mr^{2}}+V(r)=E_{0} \qquad \textbf{(2.90)}$$

In this case, denote conservation of total mechanical energy.

Furthermore, since the ignorable coordinate θ has been suppressed, the KE associated with θ can be combined with the actual PE, $V(r)$, to define an effective potential:

$$V_{\text{eff}} = \frac{C_\theta^2}{2mr^2} + V(r)$$

Hence, construct the phase curves based on Eqn (2.90)

$$\frac{1}{2}m\dot{r}^2 + V_{eff}(r) = E_0 -> (\dot{r} = \frac{dr}{dt} = \sqrt{2(E_0 - V_{eff}(r))} / \sqrt{m} :$$

$$> dt = ...!$$

and then the solution

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{E_0 - V_{\text{eff}}}}$$

Also, $\dot{\theta} = \frac{C_\theta}{mr^2}$

thus

$$\theta(t) = \int_{t_0}^{t} \frac{C_\theta}{mr^2} d\tau + \theta_0$$

: Motion of the system has been entirely solved !

STEADY MOTION

An important and interesting class of motion :

Ignorable coordinate related to steady motion.

This type of motion : when the generalized velocities and

conjugate momenta of the *non-ignorable* coordinates are

zero.

That is, $\dot{q}_k = \dot{p}_k = 0 \ldots .(2.91)$

>>

for the $k = 1, 2, \ldots, n - m$ **of non-ignorable coordinates.**
This means that each of the non-ignorable coordinates

has a *constant value*.

>> Routhian becomes only a function of the constants of motion $C_{n-m+1},\ldots,C_n$ and does not depend on time *t*.

Hence, generalized velocities of the ignorable coordinates are constant. : solution for the ignorable coordinate results in

$$q_i(t) = v_i t + \text{const}, \qquad i = n-m+1,\ldots,n$$

Hence the characterization of steady motion. These

constant values of the non-ignorable coordinates are not completely arbitrary. Conditions on the non-ignorable coordinates q_k are obtained from the equations of motion (2.86). Conditions for steady motion are obtained by substituting

$$\dot{q}_k = 0 \quad \text{and} \quad \ddot{q}_k = 0$$

into the eqns of motion (2.86). It is actually more convenient to first insert the conditions (2.91) into the Routhian. Eqns of motion, and hence the conditions for steady motion, become

$$\frac{\partial R}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, n - m \ldots.(2.92)$$

Eqns (2.92) are solved for the constant values q_{k0} corresponding to steady motion.

One way to consider the situation is that

Non-ignorable coordinates are effectively in equilibrium, while the motion is maintained in a steady manner by conservation of momenta of the ignorable coordinates.

Once the conditions for steady motion are established, the next important consideration is the stability of these motions. That is, what happens to the steady solutions

under small disturbances? The nature of the motion near the steady solutions is analyzed by setting

$$q_k(t) = q_{k0} + s_k(t)$$

These expressions are substituted into the Routhian (2.84), which gives

$$\tilde{R} = \tilde{R}(s_1, s_2, \ldots, s_{n-m}, \dot{s}_1, \dot{s}_2, \ldots, \dot{s}_{n-m})$$

Localized eqns of motion *about* the steady motion are

$$\frac{d}{dt}\left(\frac{\partial \tilde{R}}{\partial \dot{s}_k}\right) - \frac{\partial \tilde{R}}{\partial s_k} = 0, \qquad k = 1, 2, \ldots, n-m$$

For small disturbances about steady motion, these

Eqn may be linearized, and then using standard methods

to characterize the stability of steady solution

Ex: A spherical pendulum (Fig. 2.16).

Using the spherical angles ϕ and θ, the Lagrangian :

$$L = \frac{1}{2} ml^2 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + mgl \cos \theta$$

>> Coordinate ϕ is ignorable ~ conjugate momentum

$$p_\phi = ml^2 \dot{\phi} \sin^2 \theta = C \text{....(2.93)}$$

: An integral of motion.

The analysis is reduced to a single degree of freedom

with the Routhian function

$$R(\theta,\dot{\theta},C)=\frac{C^2}{2ml^2\sin^2\theta}-\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta$$

and therefore the eqn of motion,

$$ml^2\ddot{\theta}-\frac{C^2\cos\theta}{ml^2\sin^3\theta}+mgl\sin\theta=0 \qquad \textbf{(2.94)}$$

Spherical pendulum is a conservative system, and then an effective potential:

$$V_{\text{eff}} = \frac{C^2}{2ml^2 \sin^2 \theta} - mgl \cos\theta \text{....(2.95)}$$

Eqn of motion (2.94) can be equivalently written as

$$ml^2\ddot{\theta} + \frac{dV_{\text{eff}}}{d\theta} = 0$$

Condition for steady motion :

$$\frac{C^2 \cos\theta}{ml^2 \sin^3 \theta} - mgl \sin\theta = 0$$

or

$$C^2 \cos\theta = m^2 g l^3 \sin^4\theta$$

From the conservation of angular momentum (2.93), the condition for steady motion reduces to

$$l\dot{\phi}_0^2 = g \sec\theta_0$$

(2.96)

So if the initial conditions ϕ_0 and θ_0 satisfy (2.96), the angle θ and the angular velocity $\dot{\phi}$ will remain constant

and the tip of the pendulum will execute uniform circular motion. To investigate the stability of perturbations from this steady motion, we set

$$\theta = \theta_0 + s(t)$$

and substitute into the eqn of motion (2.94). After linearization based on small values of $s(t)$**, we obtain the**

DE for the perturbation:

$$\ddot{s} + \frac{g}{l}(3\cos\theta_0 + \sec\theta_0)s = 0$$

The stability may also be determined by analyzing the effective potential V_{eff} (2.95) in the neighborhood of $\theta = \theta_0$.

LAGRANGE’S EQUATIONS FOR IMPULSIVE FORCES

Principle of Impulse and Momentum >>

Generalized in the Lagrangian formalism.

During impact : Very large forces are generated

over a very small time interval. ~ Not a practical matter

to record these forces over the very small time

>>> Instantaneous form of Newton’s Second Law is of

little use in impact problems.

>>> Eqns of motion are integrated over the time interval of impact.

$$\hat{F} = \int_{t_0}^{t_0+\Delta t} \sum F(t)\,dt$$

By the Principle of Impulse and Momentum, velocities change by a finite amount over the time interval Δt. As long as the time interval is taken

infinitesimally small, the displacements do not change

and hence remain continuous.

Therefore, Impulsive force ~ Finding velocity change immediately after the impact.. without displacement change

Integrating Lagrange's eqns of motion for holomic systems over the time interval between $t_1 = t_0$ **and**

$t_2 = t_0 + \Delta t$**, we have**

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) dt - \int_{t_1}^{t_2} \frac{\partial T}{\partial q_k} dt = \int_{t_1}^{t_2} Q_k dt, \qquad k = 1, 2, \ldots, n$$

(2.97)

Now letting $\Delta t \to 0$**,**

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_2 - \left.\frac{\partial T}{\partial \dot{q}_k}\right|_1 = \hat{Q}_k \qquad k = 1, 2, \ldots, n$$

(2.98)

Second term on the left-hand side of Eqn (2.97) vanishes,

since the generalized coordinates are continuous and the

generalized velocities remain bounded during the impact. The integral on the right-hand side of Eqn (2.97) is the

generalized impulse $\hat{Q}_k$.

The impulsive form of Lagrange's eqns (2.98) can also be

$$\Delta p_k = \hat{Q}_k, \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.99)}$$

relating the change in generalized momentum p_k to the

applied generalized impulse $\hat{Q}_k$. Since the generalized

momenta are polynomials in the generalized velocities, there is no need to solve any differential equations to obtain the velocities immediately after impact. Computation of the generalized impulses is formally identical to finding generalized forces. At any instant, the virtual impulsive energy acquired by the system under virtual displacements compatible with the constraints is

$$\delta\hat{W} = \sum_{j=1}^{n} \hat{Q}_j \delta q_j$$

As with generalized forces, the independent degrees of freedom are incremented one at a time to determine the individual contributions to $\delta\hat{W}$.

Ex: A four-bar linkage constrained to slide smoothly along the the *x*-direction

(Fig. 2.17). The system has two degrees of freedom and as generalized coordinates we can take the location of the center of mass, x_1, and the angle θ. We assume that the mechanism is at rest when an impulse $\hat{F}$ is suddenly applied, at point A, in the *x*-direction.

Solving this problem by vector methods involves calculation of the linear and angular momenta of the

system and invoking the momenta are easily derived from the kinetic energy of the system.

The generalized impulses are formally computed as if they were generalized forces.

The kinetic energy of the system is

$$T = 2m\dot{x}_1^2 + \frac{8}{3}mb^2\dot{\theta}^2$$

The generalized momenta conjugate to x_1 **and** θ**,**

respectively, are

$$p_{x1} = 4m\dot{x}_1, \qquad p_\theta = \frac{16}{3} mb^2 \dot{\theta}$$

Similar to computing virtual work, we consider the

independent virtual displacements

$$x_1 \to x_1 + \delta x_1,\ \delta\theta = 0 \quad \text{and} \quad \delta x_1 = 0, \theta \to \theta + \delta\theta$$

The virtual impulsive energy becomes

$$\delta\hat{W} = \hat{Q}_{x1}\delta_{x1} + \hat{Q}_{\theta}\delta\theta$$

in which the generalized impulses are

$$\hat{Q}_{x1} = \hat{F}\ , \qquad \hat{Q}_{\theta} = 2b\sin\theta\hat{F}$$

Since the system starts from rest, substitution of the above into Lagrange's equations for implusive systems (2.99) results in the acquired generalized velocities

$$\dot{x}_1 = \frac{\hat{F}}{4m}, \qquad \dot{\theta} = \frac{3\sin\theta}{8mb}\hat{F}$$

Practice !

A horizontal rod of mass m and length 2L falls under gravity and strakes a knife edge loaded one half of the way from the center to end of the rod. It's velocity just before impact is v . Coefficient of restitution between rod and knife edge is e.

a. **Velocity of the center of mass**
b. **Angular velocity immediately after the rod strikes the ground.**

Sol: Assume the impulse is applied at the impact.

Total enegy at any instant : $T = \frac{1}{2} m(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2} I \dot{\theta}^2$

Virtual work of impulse : $\delta W = \hat{F}(\delta y_c + \frac{1}{2} L \delta\theta) = \hat{Q}_{x_c} \delta x_c + \hat{Q}_{y_c} \delta y_c + \hat{Q}_\theta \delta\theta$

$\sim \quad \hat{Q}_{x_c} = 0, \hat{Q}_{y_c} = \hat{F}, \hat{Q}_\theta = \frac{1}{2} L \hat{F}$

Change of generalize Momentum:

$$\Delta(m\dot{x}_c) = m\dot{x}_c = 0 - - - (1) : \dot{x}_c = 0$$

$$\Delta(m\dot{y}_c) = m(\dot{y} + v) = \hat{F} - (2)$$

$$\Delta(I\dot{\theta}) = I\dot{\theta} = \frac{L}{2}\hat{F} - - - (3) : (I = \frac{1}{3}mL^2)$$

$$And..\dot{y}_c + \frac{L}{2}\dot{\theta} = ev >>>> \dot{y}_c = ev - \frac{L}{2}\dot{\theta} - -(4)$$

$$(4) : \dot{\theta} = \frac{2}{L}(ev - \dot{y}_c) - - - > (3)\hat{F} = ...$$

$$: \dot{y}_c = \frac{v}{7}(4e - v)..\dot{\theta} = \frac{6}{7L}(1 + e)v$$

ELECTROMECHANICAL ANALOGIES

The Lagrangian formalism is based on energy and therefore has applicability that goes far beyond simple mechanical systems (Fig. 2.18). A very practical extension of the theory is to electrical circuits and combined electromechanical systems. A direct application of Lagrangian's equations to electrical circuits is based on the parameters given in Table 2.1. Energy carried by an inductor coil is