Cyclotrons and Synchrotrons

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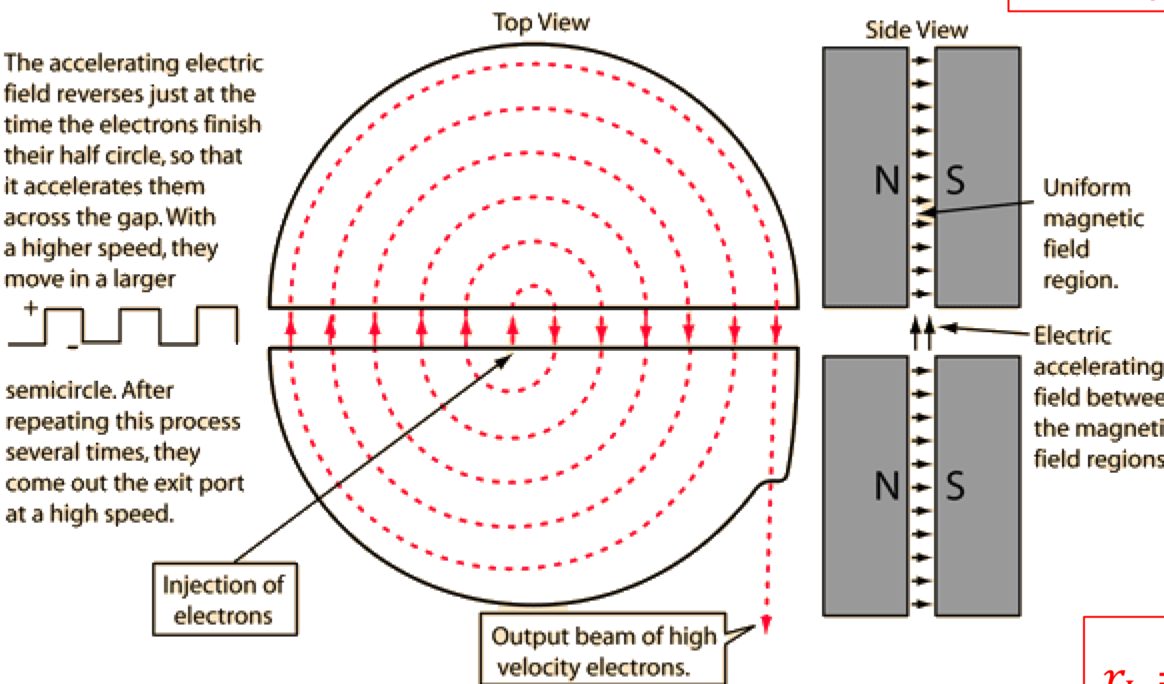
Introduction

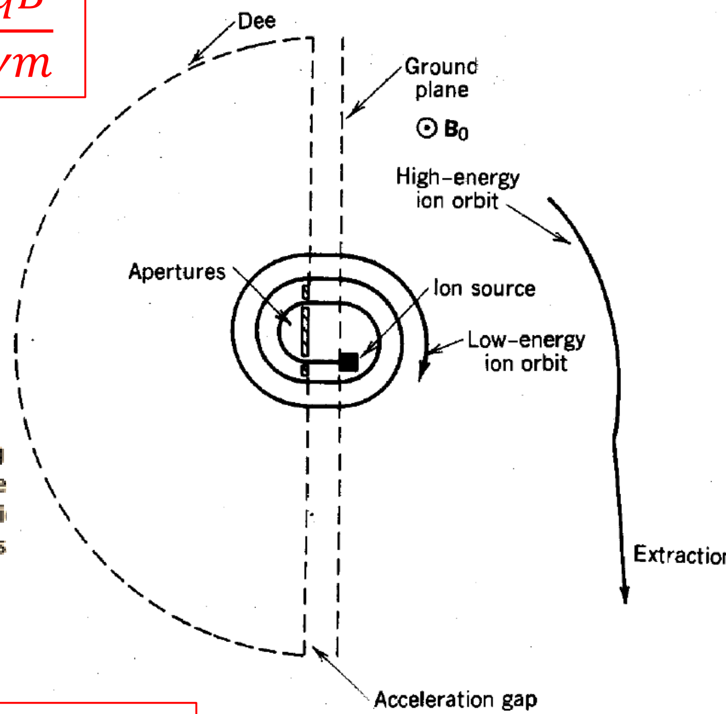
- The term circular accelerator refers to any machine in which beams describe a closed orbit.
- All circular accelerators have a vertical magnetic field to bend particle trajectories and one or more gaps coupled to inductively isolated cavities to accelerate particles.
- Beam orbits are often not true circles; for instance, large synchrotrons are composed of alternating straight and circular sections.
- The main characteristic of resonant circular accelerators is synchronization between oscillating acceleration fields and the revolution frequency of particles.
- Particle recirculation is a major advantage of resonant circular accelerators over rf linacs. In a circular machine, particles pass through the same acceleration gap many times (10^2 to greater than 10^8). High kinetic energy can be achieved with relatively low gap voltage. The effective gradient exceeds 50 MV/m.
- Most resonant circular accelerators can be classed as either cyclotrons or synchrotrons.

Principles of uniform-field cyclotron

- The operation of the uniform-field cyclotron [E. O. Lawrence, Science 72, 376 (1930)] is based on the fact that the gyro-frequency for non-relativistic ions is independent of kinetic energy. Resonance between the orbital motion and an accelerating electric field can be achieved for ion kinetic energy that is small compared to the rest energy.

$$\boxed{\omega_g = \frac{qB}{\gamma m}}$$





$$\boxed{r_L = \frac{\gamma m v_\perp}{qB}}$$

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Typical parameters

- The rf frequency in cyclotrons is relatively low (~10 MHz).

$$f_0 = \frac{qB_0}{2\pi m_i} = 1.52 \times 10^7 \frac{B_0}{A} \text{ [Hz]} \quad (B_0 \text{ in tesla}, A \text{ in amu})$$

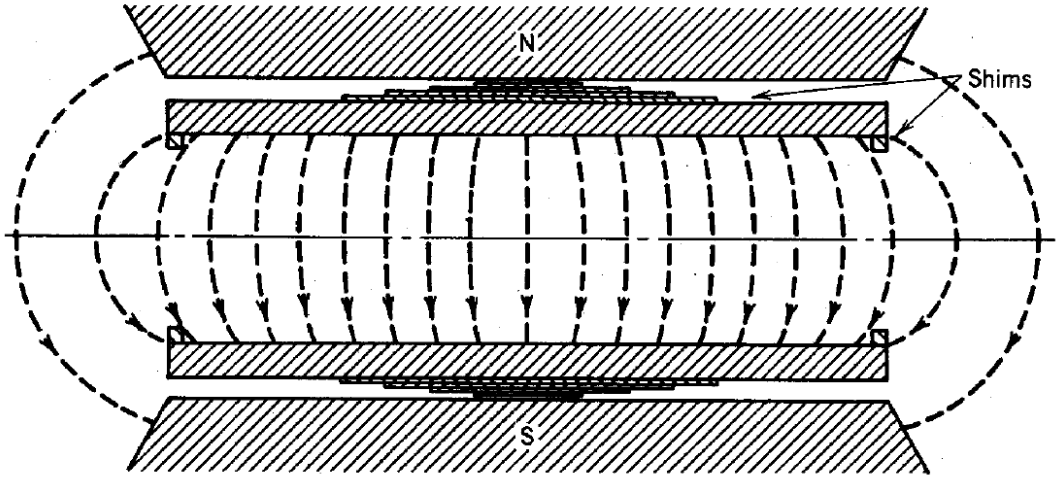
- The maximum energy of ions in a cyclotron is limited by relativistic detuning and radial variations of the magnetic field magnitude. The kinetic energy and orbit radius of non-relativistic ions are related by

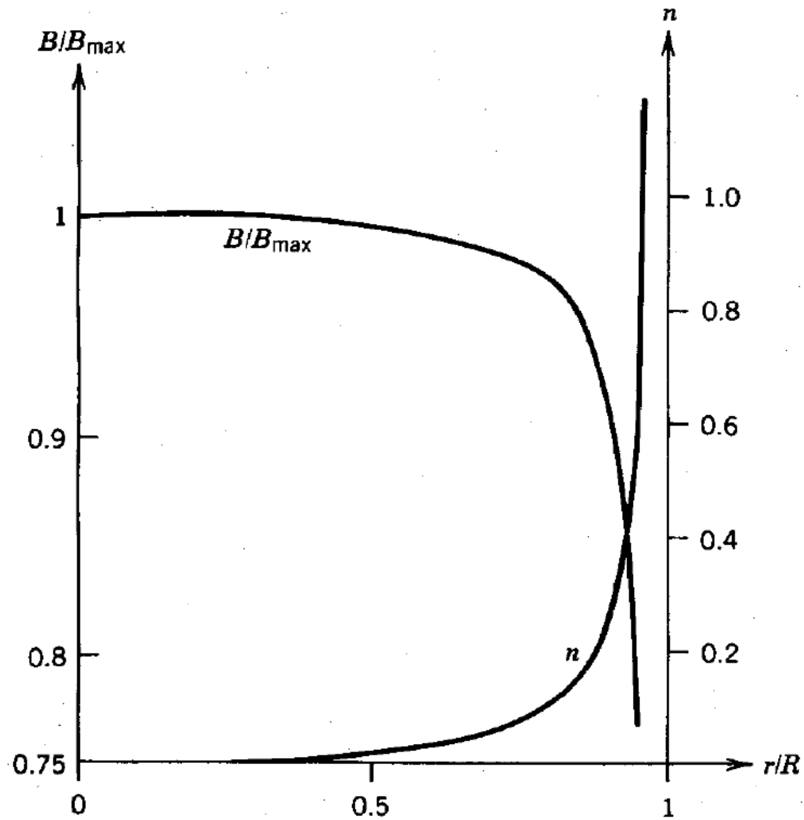
$$T_{max} = 48 \frac{(Z^* RB)^2}{A} \text{ [MeV]} \quad (R \text{ in meter}, B \text{ in tesla})$$

Vertical focusing in a uniform-field cyclotron

- Cyclotron magnets are designed for small field index n over most of the field area to minimize desynchronization of particle orbits. Therefore, vertical focusing in a uniform-field cyclotron is weak.
- The electric field pattern between the dees of a cyclotron act as the one-dimensional equivalent of the electrostatic immersion lens.

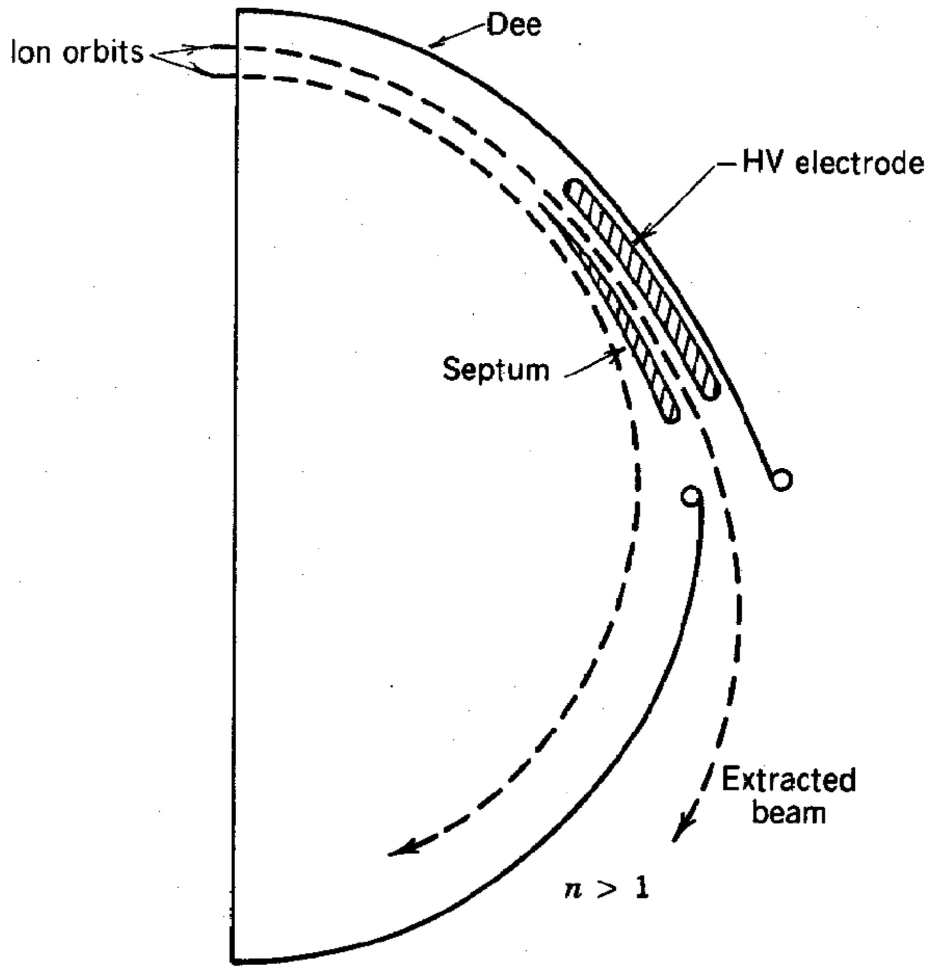
$$n(r) = -\frac{r}{B_z(r,0)}\frac{\partial B_z(r,0)}{\partial r}$$





Extraction of ions

- In order to extract ions from the machine at a specific location, deflection fields must be applied.



Longitudinal dynamics

- In the uniform-field cyclotron, the oscillation frequency of gap voltage remains constant while the ion gyro-frequency continually decreases. The reduction in ω_g with energy arises from two causes: (1) the relativistic increase in ion mass and (2) the reduction of magnetic field magnitude at large radius.
- Assume that the voltage of dee1 relative to dee2 is given by

$$V(t) = V_0 \sin \omega t$$

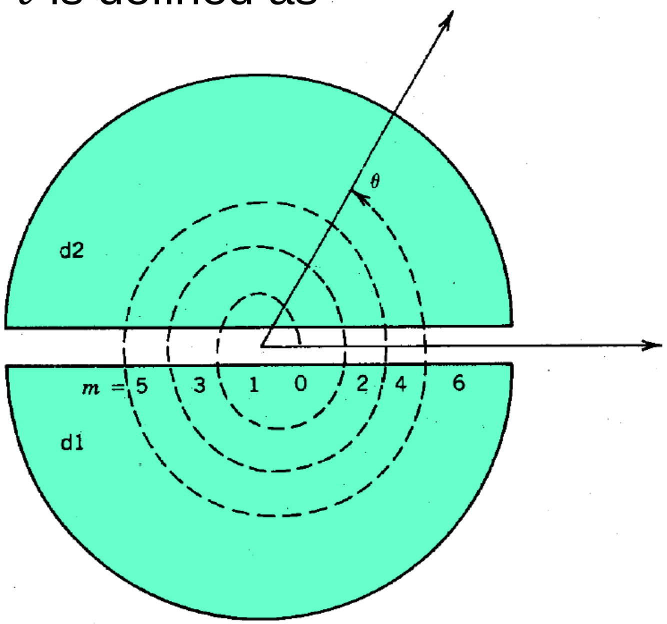
- The phase of an ion at azimuthal position θ and time t is defined as

$$\phi = \omega t - \theta(t)$$

$$\frac{d\phi}{dt} = \omega - \frac{d\theta}{dt} = \omega - \omega_g$$

$$\omega_g = \frac{qB_0}{\gamma m_0} = \frac{qc^2 B_0}{E}$$

$$E = T + m_0 c^2$$



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Longitudinal dynamics

- With the assumption of small gap width, particles making their mth transit of the gap with phase ϕ_m gain an energy

$$\Delta E_m = qV_0 \sin\phi_m$$

- The change of phase for a particle during the transit through a dee is

$$\Delta\phi = \frac{d\phi}{dt}\frac{\pi}{\omega_g} = (\omega - \omega_g)\frac{\pi}{\omega_g} = \pi\left[\frac{\omega E}{c^2 q B_0} - 1\right]$$

- We obtain an approximate equation for $\phi(E)$

$$\frac{\Delta\phi}{\Delta E} \cong \frac{d\phi}{dE} = \frac{\pi}{qV_0 \sin\phi}\left[\frac{\omega E}{c^2 q B_0} - 1\right]$$

- The cyclotron phase equation is usually expressed in terms of the kinetic energy T $(T = E - m_0 c^2)$

$$\cos\phi = \cos\phi_0 + \frac{\pi}{qV_0}\left(1 - \frac{\omega}{\omega_{g0}}\right)T - \frac{\pi}{2qV_0 m_0 c^2}\frac{\omega}{\omega_{g0}}T^2$$

Injection phase

$$\omega_{g0} = \frac{qB_0}{m_0}$$

Nonrelativistic gyro-frequency

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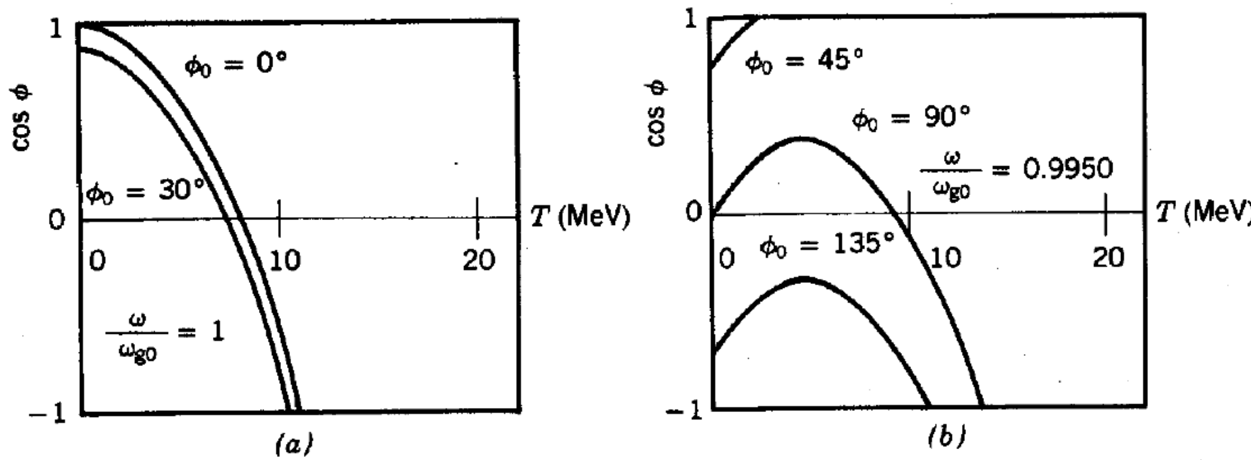
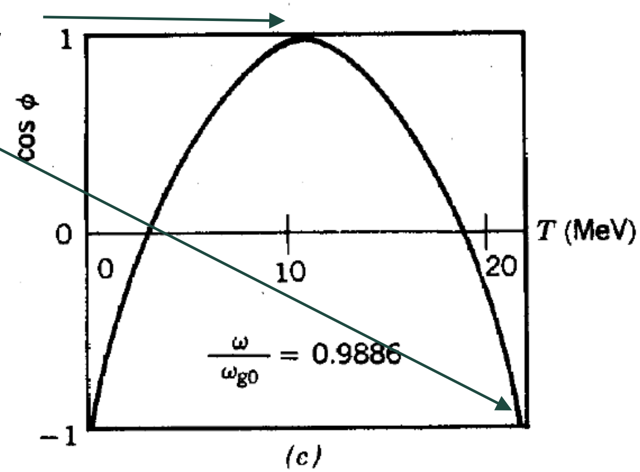
Longitudinal dynamics



$$\cos\phi = +1\ for\ T = \frac{1}{2}T_{max}$$

$$\cos\phi = -1\ for\ T = T_{max}$$



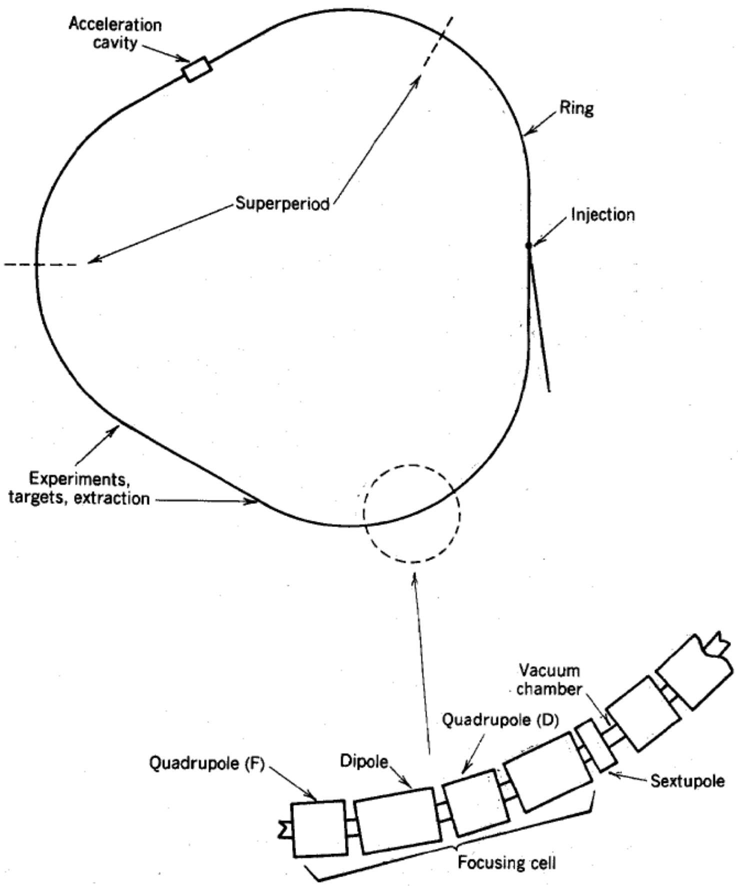
$$\frac{\omega}{\omega_{g0}} = \frac{1}{1 + \frac{T_{max}}{2m_0c^2}}$$

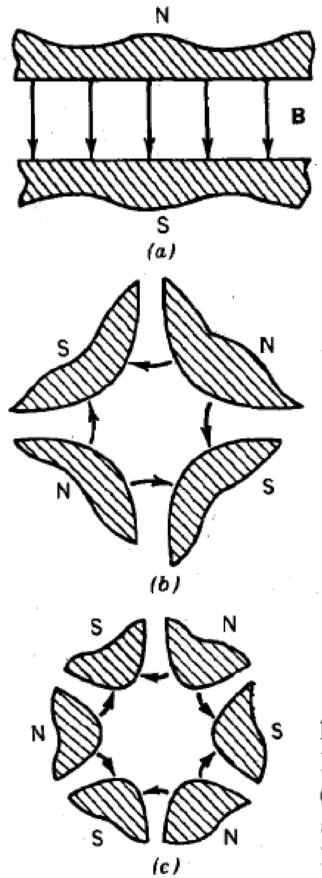
$$T_{max} \approx \sqrt{\frac{16qV_0m_0c^2}{\pi}}$$

Figure 15.5 Phase histories of protons in uniform-field cyclotron; cos ϕ versus kinetic energy (T) for different injection phase (ϕ_0). (a) $\omega/\omega_{g0} = 1$, where ω is rf angular frequency and ω_{g0} is nonrelativistic gyrofrequency. (b) $\omega/\omega_{g0} = 0.9950$. ($c$) $\omega/\omega_{g0} = 0.9896$, $\phi_0 = 180°$ (parameters for maximum kinetic energy).

Principles of synchrotron

- Synchrotrons are resonant circular particle accelerators in which both the magnitude of the bending magnetic field and the rf frequency are cycled. An additional feature of most modem synchrotrons is that focusing forces are adjustable independent of the bending field.





$$\boxed{\omega_g = \frac{qB}{\gamma m}}$$

$$\boxed{r_L = \frac{\gamma m v_\perp}{qB}}$$

Figure 15.15 Classification of synchrotron magnets. (*a*) Dipole magnet (bending particle orbits). (*b*) Quadrupole (transverse focusing). (*c*) Sextupole (chromatic correction; assuring that particles in a range of energy about the mean have the same ν).

Maximum energy is limited by synchrotron radiation

- Synchrotron radiation results from the continuous transverse acceleration of particles in a circular orbit. The total power emitted per particle is

$$P = \frac{2cE^4 r_0}{3R^2(m_0c^2)^3} \qquad r_0 = \frac{q^2}{4\pi\epsilon_0 m_0 c^2} \quad \text{(Classical radius of particle)}$$

- Synchrotron radiation has a negligible effect in ion accelerators: $\frac{P_i}{P_e} \sim \left(\frac{m_e}{m_i}\right)^4$

- To illustrate the significance of synchrotron radiation in electron accelerators, consider a synchrotron in which electrons gain an energy eV_0 per turn. The power input to electrons (in eV/s) is

$$P = \frac{cV_0}{2\pi R}$$

- The maximum allowed total energy is $E \leq \left(\frac{3V_0(m_0c^2)^3 R}{4\pi r_0}\right)^{1/4}$

- For example, with $R = 20\ \mathrm{m}$ and $V_0 = 100\ \mathrm{kV}$, the maximum energy is $E = 2.2\ \mathrm{GeV}$.

- Electron synchrotrons are a unique source of intense radiation over a wide spectral range via synchrotron radiation.

Longitudinal dynamics: general features

- Variations of longitudinal energy associated with stable phase confinement of particles in an rf bucket result in horizontal particle oscillations. The synchrotron oscillations sum with the usual betatron oscillations that arise from spreads in transverse velocity.

- The range of stable synchronous phase in a synchrotron depends on the energy of particles.
  - At energies comparable to or less than m_0c^2, particles are non-relativistic; therefore, their velocity depends on energy. In this regime, low-energy particles in a beam bunch take a longer time to complete a circuit of the accelerator and return to the acceleration cavity. Therefore, the accelerating voltage must rise with time at ϕ_s for phase stability ($0 < \phi_s < \pi/2$).
  - At relativistic energies, particle velocity is almost independent of energy; the particle orbit circumference is the main determinant of the revolution time. Low-energy particles have smaller orbit radii and therefore take less time to return to the acceleration gap. In this case, the range of stable phase is $\pi/2 < \phi_s < \pi$.
  - The energy that divides the regimes is called the transition energy. It is essential to shift the phase of the rf field before the bunched structure of the beam is lost. This effect is unimportant in electron synchrotrons because electrons are always injected above the transition energy.

Longitudinal dynamics: synchronous particles

- Both the magnetic field and frequency of accelerating electric fields must vary in a synchrotron to maintain a synchronous particle with constant radius R. There are a variety of possible acceleration histories corresponding to different combinations of synchronous phase, cavity voltage amplitude, magnetic field strength, and rf frequency.
- Assume the acceleration gap has narrow width δ so that transit-time effects can be neglected. The electric force acting on the synchronous particle in a gap with peak voltage V_0 is $qE = qV_0 \sin\varphi_s/\delta$. The momentum change passing through the gap is the electric force times the transit time, or

$$\Delta p_s = \left(\frac{qV_0 \sin\varphi_s}{\delta}\right)\left(\frac{\delta}{v_s}\right)$$

- Acceleration occurs over a large number of revolutions; it is sufficient to approximate p_s as a continuous function of time.

$$\frac{dp_s}{dt} \approx \frac{\Delta p_s}{\tau_0} = \frac{qV_0 \sin\varphi_s/v_s}{2\pi R/v_s} = \frac{qV_0 \sin\varphi_s}{2\pi R} \qquad p_s(t) = p_{s0} + \frac{qV_0 \sin\varphi_s}{2\pi R} t$$

$$\Rightarrow \quad \gamma_s(t) = \sqrt{1 + (p_s(t)/(m_0c^2))^2} \qquad \omega_{g0}(t) = \frac{v_s}{R} = \frac{c}{R}\sqrt{1 - (\gamma_s(t))^{-2}}$$

Longitudinal dynamics: synchronous particles

- The rf frequency must be an integer multiple of the revolution frequency, $\omega = M\omega_{g0}$.
- In small synchrotrons, M may equal 1 to minimize the rf frequency. In larger machines, M is usually greater than unity. In this case, there are M circulating beam bunches contained in the ring.
- The rf frequency is related to the particle energy by

$$\omega(t) = M\omega_{g0}(t) = \frac{Mc}{R}\sqrt{1-(\gamma_s(t))^{-2}}$$

- Similarly, the magnetic field magnitude is related to the particle energy by

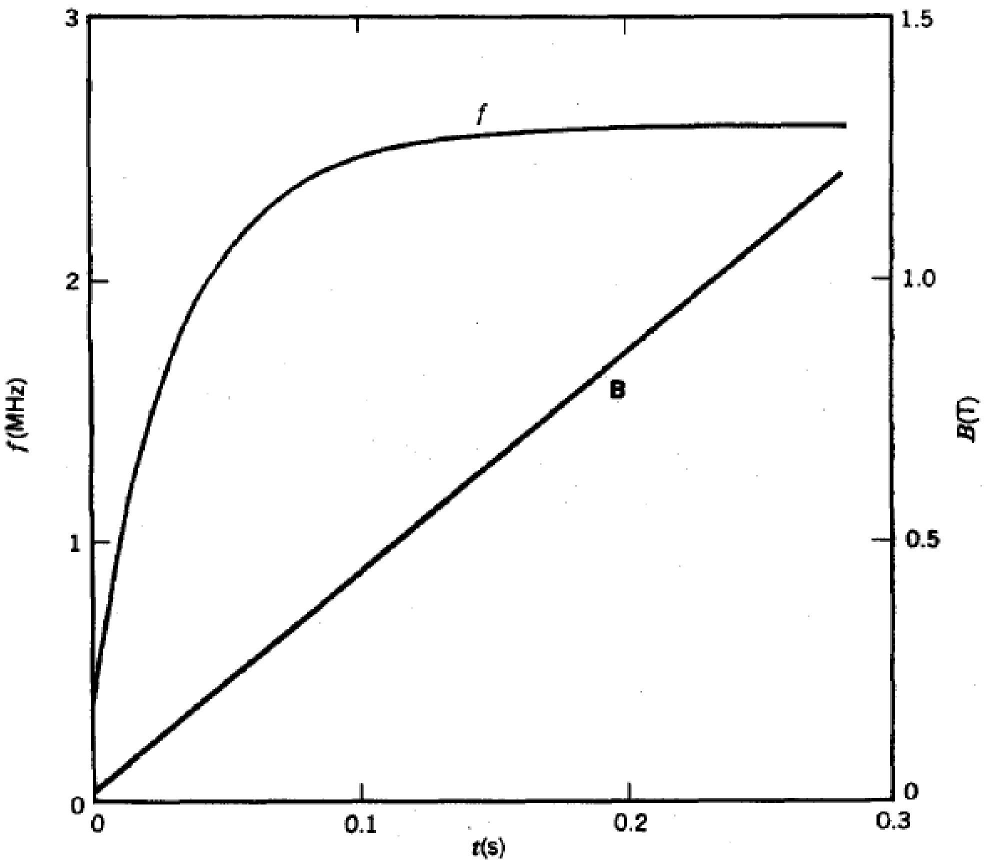
$$B_0(t) = \frac{\gamma_s m_0 v_s}{qR} = \frac{m_0 c}{qR}\sqrt{(\gamma_s(t))^2 - 1}$$

- The rf frequency and magnetic field are related to each other by

$$\omega = \frac{MqB_0/m_0}{\sqrt{1+(qB_0R/m_0c)^2}}$$

Longitudinal dynamics: synchronous particles

- As an example, consider the parameters of a moderate-energy synchrotron (the Bevatron). The injection and final energies for protons are 9.8 MeV and 6.4 GeV. The machine radius is 18.2 m and $M = 1$. The magnetic field rises from 0.025 to 1.34 T and the rf frequency increases from 0.37 to 2.6 MHz.



Longitudinal phase dynamics

- A momentum equation for a nonsynchronous particle:

$$\frac{d\delta p}{dt} \approx \frac{\Delta p - \Delta p_s}{\tau_0} = \frac{qV_0 \sin\phi - qV_0 \sin\phi_s}{2\pi R} = \frac{qV_0\omega_{g0}}{2\pi\beta_s c}(\sin\phi - \sin\phi_s)$$

- Changes of phase can be related to the difference between the orbital frequency of a nonsynchronous particle to the rf frequency:

$$\frac{d\phi}{dt} = \omega - \frac{d\theta}{dt} = \omega - M\omega_g$$

- The differential change in momentum ($p = \gamma m_0 \beta c$) is

$$\frac{\delta p}{p_s} = \frac{\delta\gamma}{\gamma_s} + \frac{\delta\beta}{\beta_s} = \frac{\delta\beta}{\beta_s(1-\beta_s^2)} \qquad \frac{\delta\beta}{\beta_s} = (1-\beta_s^2)\frac{\delta p}{p_s} = \frac{1}{\gamma_s^2}\frac{\delta p}{p_s}$$

- Differential changes in τ arise from variations in particle velocity and changes in orbit radius

$$\frac{\delta\tau}{\tau_0} = -\frac{\delta\omega_g}{\omega_{g0}} = \frac{\delta r}{R} - \frac{\delta v}{v_s} = \frac{\delta r}{R} - \frac{\delta\beta}{\beta_s} = \left[\frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}\right]\frac{\delta p}{p_s}$$

$$\gamma_t^2 = \frac{\delta p/p_s}{\delta R/R}$$

Transition gamma factor

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Longitudinal phase dynamics

- Changes of gyro-frequency:

$$\frac{d\omega_g}{dt} = -\frac{d\delta p}{dt}\frac{\omega_{g0}}{p_s}\left[\frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}\right]$$

- For the long acceleration cycle of synchrotrons (in the limit that the parameters of the synchronous particle and the rf frequency change slowly compared to the time scale of a phase oscillation), we obtain phase dynamics equation:

$$\frac{d^2\phi}{dt^2} \approx -M\frac{d\omega_g}{dt} = \frac{M\omega_{g0}^2}{\gamma_s m_0 \beta_s^2 c^2}\frac{qV_0}{2\pi}\left[\frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}\right](\sin\phi - \sin\phi_s)$$

- Phase oscillations lead to changes of momentum about p_s and hence to oscillation of particle orbit radii. These radial oscillations are called synchrotron oscillations.
- The coefficient of the sine terms may be either positive or negative, depending on the average particle energy.

Longitudinal phase dynamics

- In the limit of small phase excursion ($\Delta\phi \ll 1$), the phase dynamics equation reduces to

$$\frac{d^2\Delta\phi}{dt^2} \cong \frac{M\omega_{g0}^2}{\gamma_s m_0 \beta_s^2 c^2}\frac{qV_0}{2\pi}\left[\frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}\right]\cos\phi_s\,\Delta\phi \quad \Rightarrow \quad \Delta\phi \cong \Delta\phi_0 \cos\omega_s t$$

- The angular frequency for phase oscillations in a synchrotron is

$$\omega_s = \omega_{g0}\sqrt{-\frac{M\cos\phi_s}{2\pi\beta_s^2}\frac{qV_0}{\gamma_s m_0 c^2}\left[\frac{1}{\gamma_t^2} - \frac{1}{\gamma_s^2}\right]}$$

- For $\gamma_s < \gamma_t$, the sign of the square bracket is negative. In this case, the stability range is the same as in a linear accelerator, $0 < \phi_s < \pi/2$.

- For $\gamma_s > \gamma_t$, the sign of the square bracket is positive, and the stable phase regime becomes $\pi/2 < \phi_s < \pi$.