# LAGRANGE'S EQUATIONS FOR IMPULSIVE FORCES

**Principle of Impulse and Momentum >>**

**Generalized in the Lagrangian formalism.**

**During impact : Very large forces are generated**

**over a very small time interval. ~ Not a practical matter**

**to record these forces over the very small time**

**>>> Instantaneous form of Newton's Second Law is of**

**little use in impact problems.**

**>>> Eqns of motion are integrated over the time**

**interval of impact.**

$$\hat{F}=\int_{t_0}^{t_0+\Delta t}\sum F(t)dt$$

**By the Principle of Impulse and Momentum,**

**velocities change by a finite amount over the time**

**interval** $\Delta t$ **. As long as the time interval is taken**

**infinitesimally small, the displacements do not change**

**and hence remain continuous.**

**Therefore, Impulsive force ~**

**Finding velocity change immediately after the impact.. without displacement change**

**Integrating Lagrange's eqns of motion for holomic systems over the time interval between $t_1 = t_0$ and $t_2 = t_0 + \Delta t$, we have**

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) dt - \int_{t_1}^{t_2} \frac{\partial T}{\partial q_k} dt = \int_{t_1}^{t_2} Q_k dt, \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.97)}$$

**Now letting** $\Delta t \to 0$**,**

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_2 - \left.\frac{\partial T}{\partial \dot{q}_k}\right|_1 = \hat{Q}_k \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.98)}$$

**Second term on the left-hand side of Eqn (2.97) vanishes, since the generalized coordinates are continuous and the generalized velocities remain bounded during the impact.**

**The integral on the right-hand side of Eqn (2.97) is the generalized impulse $\hat{Q}_k$.**

**The impulsive form of Lagrange's eqns (2.98) can also be**

$$\Delta p_k = \hat{Q}_k, \qquad k = 1, 2, \ldots, n \qquad \textbf{(2.99)}$$

**relating the change in generalized momentum $p_k$ to the applied generalized impulse $\hat{Q}_k$. Since the generalized momenta are polynomials in the generalized velocities,**

**there is no need to solve any differential equations to obtain the velocities immediately after impact. Computation of the generalized impulses is formally identical to finding generalized forces. At any instant, the virtual impulsive energy acquired by the system under virtual displacements compatible with the constraints is**

$$\delta \hat{W} = \sum_{j=1}^{n} \hat{Q}_j \delta q_j$$

**As with generalized forces, the independent degrees of freedom are incremented one at a time to determine the individual contributions to $\delta \hat{W}$.**

**Ex: A four-bar linkage constrained to slide smoothly along the the *x*-direction**

**(Fig. 2.17). The system has two degrees of freedom and as generalized coordinates we can take the location of the center of mass, $x_1$, and the angle $\theta$. We assume that the mechanism is at rest when an impulse $\hat{F}$ is suddenly applied, at point A, in the *x*-direction.**

**Solving this problem by vector methods involves**

**calculation of the linear and angular momenta of the system and invoking the momenta are easily derived from the kinetic energy of the system.**

**The generalized impulses are formally computed as if they were generalized forces.**

**The kinetic energy of the system is**

$$T = 2m\dot{x}_1^2 + \frac{8}{3}mb^2\dot{\theta}^2$$

**The generalized momenta conjugate to $x_1$ and $\theta$, respectively, are**

$$p_{x1} = 4m\dot{x}_1, \qquad p_\theta = \frac{16}{3} mb^2\dot{\theta}$$

**Similar to computing virtual work, we consider the independent virtual displacements**

$$x_1 \to x_1 + \delta x_1,\ \delta\theta = 0 \quad \text{and} \quad \delta x_1 = 0, \theta \to \theta + \delta\theta$$

**The virtual impulsive energy becomes**

$$\delta\hat{W} = \hat{Q}_{x1}\delta_{x1} + \hat{Q}_{\theta}\delta\theta$$

**in which the generalized impulses are**

$$\hat{Q}_{x1} = \hat{F}\ , \qquad \hat{Q}_{\theta} = 2b\sin\theta\hat{F}$$

**Since the system starts from rest, substitution of the above into Lagrange's equations for implusive systems (2.99) results in the acquired generalized velocities**

$$\dot{x}_1 = \frac{\hat{F}}{4m}, \qquad \dot{\theta} = \frac{3\sin\theta}{8mb}\hat{F}$$

## Practice 1

**A horizontal rod of mass m and length 2L falls under gravity and strakes a knife edge loaded one half of the way from the center to end of the rod. It's velocity just before impact is v . Coefficient of restitution between rod and knife edge is e.**

a. **Velocity of the center of mass**
b. **Angular velocity immediately after the rod strikes the ground.**

**Sol:** **Assume the impulse is applied at the impact.**

**Total enegy at any instant :** $T = \frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2}I\dot{\theta}^2$

**Virtual work of impulse :** $\delta W = \hat{F}(\delta y_c + \frac{1}{2}L\delta\theta) = \hat{Q}_{x_c}\delta x_c + \hat{Q}_{y_c}\delta y_c + \hat{Q}_\theta \delta\theta$

$$\sim \quad \hat{Q}_{x_c} = 0, \hat{Q}_{y_c} = \hat{F}, \hat{Q}_\theta = \frac{1}{2}L\hat{F}$$

# Change of generalize Momentum:

$$\Delta(m\dot{x}_c) = m\dot{x}_c = 0 --- (1) : \dot{x}_c = 0$$

$$\Delta(m\dot{y}_c) = m(\dot{y}+v) = \hat{F} - (2)$$

$$\Delta(I\dot{\theta}) = I\dot{\theta} = \frac{L}{2}\hat{F} --- (3) : (I = \frac{1}{3}mL^2)$$

$$And..\dot{y}_c + \frac{L}{2}\dot{\theta} = ev >>>> \dot{y}_c = ev - \frac{L}{2}\dot{\theta} -- (4)$$

$$(4) : \dot{\theta} = \frac{2}{L}(ev - \dot{y}_c) --- > (3)\hat{F} = ...$$

$$: \dot{y}_c = \frac{v}{7}(4e - v)..\dot{\theta} = \frac{6}{7L}(1+e)v$$

## Practice 2

# Rinked pair of rode on a smooth horizontal plane

$m : Mass..of..each..bar$

$\vec{F} : Sharp..blow..at..the..right..end$

$x, y : Coordnates..of..the..link$

$I : Moment..of..inertia..wrt..the..center..of..mass..of..each..rod$

$$T = \frac{1}{2} m\dot{\vec{r}}_1 \bullet \dot{\vec{r}}_1 + \frac{1}{2} m\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2 + \frac{1}{2} I(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})$$

$$\text{where}..I = \frac{m}{3}a^2 = mr^2 (radus..of..gyration)$$

$$\vec{r}_1 = \vec{R} + \vec{\rho}_1 .. \dot{\vec{r}}_1 = \dot{\vec{R}} + \dot{\vec{\rho}}_1 = \dot{\vec{R}} + \vec{\omega}_1 \times \vec{\rho}_1 .. (\vec{\omega}_1 = \vec{k}\dot{\theta}_1)$$

$$\dot{\vec{\rho}}_1 = a(-\vec{i}\cos\theta_1, -\vec{j}\sin\theta_1), .. \dot{\vec{R}} = x\vec{i} + y\vec{j}$$

$$\vec{\omega}_1 \times \vec{\rho}_1 = a\dot{\theta}_1(-\vec{j}\cos\theta_1, \vec{i}\sin\theta_1)$$

**At the instance when impulse acts,** $\theta_1 -> 0$

$$(\vec{\omega}_1 \times \vec{\rho}_1)_{\theta_1 -> 0} = -a\dot{\theta}_1\vec{j}$$

**Hence**

$$\dot{\vec{r}}_1 = \dot{x}\vec{i} + (\dot{y} - a\dot{\theta}_1)\vec{j}$$

**So**

$$\dot{\vec{r}}_1 \bullet \dot{\vec{r}}_1 = \dot{x}^2 + (\dot{y} - a\dot{\theta}_1)^2$$

**Similarly,**

$$\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2 = \dot{x}^2 + (\dot{y} + a\dot{\theta}_2)^2$$

**Therefore**

$$T = \frac{1}{2}m[\dot{x}^2 + (\dot{y} - a\dot{\theta}_1)^2 + r^2 \dot{\theta}_1^2] + \frac{1}{2}m[\dot{x}^2 + (\dot{y} + a\dot{\theta}_2)^2 + r^2 \dot{\theta}_2^2]$$

**The virtual displacement of the right end point**

**( Hitted by** $\vec{F}$ **) is**

$$\delta y_c = \delta y + 2a\delta\theta_2$$

**The virtual work statement is**

$$\delta W = F\delta y_c = F\delta y + 2aF\delta\theta_2$$

$$= \hat{Q}_x\delta x + \hat{Q}_y\delta y + \hat{Q}_{\theta_1}\delta\theta_1 + \hat{Q}_{\theta_2}\delta\theta_2$$

$$\therefore \hat{Q}_x = 0, \hat{Q}_y = F, \hat{Q}_{\theta_1} = 0, \hat{Q}_{\theta_2} = 2aF$$

**Hence**

$$\Delta p_j = \hat{Q}_j ..(j = 1, \ldots.., n)$$

$$\Delta p_x = 0 \ldots 2m\dot{x} = 0 \rightarrow x = 0.$$

$$\Delta p_y = F \ldots m(\dot{y} - a\dot{\theta}_1) + m(\dot{y} + a\dot{\theta}_2) = F$$

$$\Delta p_{\theta_1} = 0 \ldots -ma(\dot{y} - a\dot{\theta}_1) + mr^2\dot{\theta}_1 = 0 \rightarrow \dot{\theta}_1 = \frac{3\dot{y}}{4a}$$

$$\Delta p_{\theta_2} = 2aF \ldots ma(\dot{y} + a\dot{\theta}_2) + mr^2\dot{\theta}_2 = 2aF \rightarrow \dot{\theta}_2 = \frac{3}{4}\frac{2F - m\dot{y}}{ma}$$

$$\therefore \dot{x} = 0, \dot{y} = -\frac{F}{m}, \dot{\theta}_1 = -\frac{3F}{4m}, \dot{\theta}_2 = \frac{9F}{4ma}$$

**There are the velocities resulting from the impact !**

## ELECTROMECHANICAL ANALOGIES

**The Lagrangian formalism is based on energy and therefore has applicability that goes far beyond simple mechanical systems (Fig. 2.18). A very practical extension of the theory is to electrical circuits and combined electromechanical systems. A direct application of Lagrangian's equations to electrical circuits is based on the parameters given in Table 2.1. Energy carried by an inductor coil is**