CHAPTER THREE:

CALCULUS OF VARIATIONS

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INTRODUCTION

Lagrangian formulation of eqns of motion:

~ Energy of a system and work done by external forces.

: Dynamical system with inertial properties –

: Kinetic energy

Also, conservative generalized forces ~ Derivable from

: Potential energy

*** Non-conservative part ~**

Applying virtual work principle !

Most systems can be synthesized by the construction of a Lagrangian ~

Theoretical & Experimentally Verified !

o Primary feature of Lagrangian dynamics ~

Independent of any coordinate system

Physical coordinates ~> Generalized coordinates

: More intrinsic to the constraints

~ Associated analytical approach : Possible to use a system approach in deriving the governing eqns.

Concept of a configuration space using generalized coordinates.

Evolution of a dynamic system : A single point in the configuration space

As in Fig. 3.1, a system of *N* particles moving freely in space is described by *3N* generalized coordinates $q_1 \ldots q_{3N}$. : Configuration space is thus a *3N*-dimensional space.

As the system undergoes its motion between fixed times t_1 and t_2, the evolution is traced out by a unique path in the configuration space.

For a conservative holonomic system, the action is defined as

$$I[q] = \int_{to}^{t1} (T - V)dt......(3.1)$$

Many(?) dynamical systems evolve to extermize the value of the action integral (3.1).

~ Of all possible ones, <u>an</u> extremum relative to the values for the other paths. Therefore, the action integral assigns a number to each possible path in configuration space.

Evidently, we can follow another way of analyzing the motion of a dynamical system. ~ Advantage to this new point of view : Formulation is also independent of the particular generalized coordinates used.

Also, the ideas will directly extend to systems of infinite degrees of freedom, such as bodies composed of a continuum of points.

- **This is in contrast to the Lagrangian formulation, which by derivation is restricted to systems with only a finite number of degrees of freedom !**

~ Focus on the mathematics involved with finding extrema of integrals that depend in specific ways on functions as *inputs*. This objective is the essence of the area of analysis known as the calculus of variations.

:Differential calculus is full of standard tools available to analyze the extreme values of ordinary functions.

Fortunately, there are many parallels between the calculus of variations and the ordinary calculus of functions.

Define some concepts and terminology

: A function is usually taken as an assignment of real values. A function of one variable $f(\cdot)$ assigns to each x a given value $f(x) \in R$. A function of several variables assigns a value to a point given by real-valued coordinates

$$(x_1,\ldots\ldots x_N)\mapsto f(x_1,\ldots x_N)\in R.$$

Graph of a function of one variable is a curve

~ Graph of a function of two variables is a surface.

Generalization of a function is called a functional.

An assignment of a real value to a point, to a vector, or to an entire function.

Now concern here with functionals that are defined on some suitable space or set of functions.

: Functionals ~ Integral functionals.

- **Functionals defined by the integration of some expression involving an input function:**

$$f(x) \mapsto I[f(x)]$$

Integral functional $I[..]$ **may be of the form (3.1).**

Ex: integral functionals include *area under the graph* of a function and *arclength of a curve* between two points.

Ex might look like …

where $p(x)$ **and** $q(x)$ **are specified.**

Integral functionals may be defined as integrals over

some interval or as integrals over some region in space.

Integration interval is the domain of the input function.

Argument or input of an integral functional may be a single function or several functions.

It all depends on the context of the problem. It is important to distinguish between the domain of the input function and the domain of the functional itself, which is comprised of some class of admissible functions.

Consider a functionals of the form

$$I[y(x)] = \int_{x0}^{x1} F(x, y(x), y'(x))dx \qquad \textbf{(3.2)}$$

and its natural generalizations.

Integrand in (3.2) is called the *Lagrangian* of the integral functional ~ Fundamental objective of the calculus of variations is to establish conditions under which an integral functional attains an extreme value.

These conditions evidently depend on the form of the Lagrangian of the functional ~ will lead to the

conditions on the particular input functions that make the integral a maximum or minimum.

An input function that renders the value of the integral functional a maximum or minimum is called an *extramal*.

Now many interesting problems can be formulated in terms of integral functionals.

Geometry of curves and surfaces.

Most of the physical applications are based on mechanics. We will begin with several motivating examples.

Fig.3.2: **Given two fixed but arbitrary points P_1 and P_2 in a plane, we can connect these two points with a curve that is the graph of a continuous function $y(x)$.**

Now if we consider the collection of all continuously differentiable functions passing through the points P_1 and P_2 , we can consider the associated arclength of each curve. The arclength of each curve is given by the integral functional

$$L[y]=\int_{x1}^{x2}\sqrt{(dx)^2+(dy)^2}=\int_{x1}^{x2}\sqrt{1+y'^2}\,dx....(3.3)$$

Ex : Of all the continuously differentiable curves passing through point P_1 and P_2, find the function

whose arclength is minimal. That is, of all smooth functions passing through P_1 and P_2, find the one that has the smallest length. We intuitively know the correct answer, but it would be nice to have a formal way to decisively solve this problem.

Brachistochrone problem **(Fig. 3.3).**

A particle is free to slide down a frictionless wire with fixed endpoints at P_1 and P_2 .

Assuming wire has finite length, then determine the time t* it takes for the particle to slide along the wire. Given that the particle starts from rests, it would be

interesting to find the shape of the wire for which the time of travel between the two endpoints is as small as possible.

We can formulate this problem as follows: Applying the Principle of Work and Energy,

$$W_{1\to 2} = T_2 - T_1$$

at any point $(x, y(x))$ **along the curve,**

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2$$

Starting from rest, the speed of the particle is given as

$$v = \sqrt{2gy}$$

Now since

$$v = \frac{ds}{dt}$$

We have the differential relation

$$dt = \frac{ds}{\sqrt{2gy}} \qquad \textbf{(3.4)}$$

Integrating both sides of Eqn (3.4),

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{ds}{\sqrt{2gy}}$$

Thus the total time of travel along the curve is obtained

$$t^* = \int_{x_1}^{x_2} \frac{\sqrt{1+[y'(x)]^2}}{\sqrt{2gy(x)}} dx$$

Notice that in each of these problems the quantity to be minimized was formulated in terms of an integral.

Given an appropriate function $y(x)$ that satisfies certain conditions, the function is entered as an argument and the resultant integral evaluated. This assignment of a scalar value to an entire function is the operation assigned to an integral functional. Symbolically, the value of a functional associated with a specified input function $y(x)$ is expressed as $I[y(x)]$. Integral functionals are a special case of such evaluations.

The domain of a functional, that is, the collection of functions satisfying certain conditions, is defined as the associated set of *admissible functions*. In the case of integral functionals, admissibility typically requires certain differentiablility conditions and specified boundary conditions. These conditions are usually specified with the problem.

The problem of finding a function $y(x)$ out a set of admissible functions that minimized (maximizes) a given functional $I[y]$ is called a variational problem. The actual value of the functional, or even if it is a maximum or a minimum, is of little concern. The

important thing is that the value of the extremal is *stationary*.

A representative integral functional, as in the examples, has the form

$$I[y(x)] = \int_{x0}^{x1} F(x, y(x), y'(x))dx \quad \textbf{...(3.2)}$$

in which the Lagrangian $F(x, y, y')$ **is a smooth function of three variables. The Lagrangian is the integrand of the functional** $I[y]$**. The function of three variables**

$F(\alpha, \beta, \gamma)$ **specifies the relationship of all the variables in the integrand of the functional.**

For the minimal length problem :

For the branchistochrone problem:

EXTREMA OF FUNCTIONS

An extremum problem consists of finding the largest or smallest value of a quantity. For functions of one or two variables, the function can be graphed and we immediately see where the function attains its extreme values. These may be inside of a domain, or the extrema may be located at points on the boundary. For functions of more than two variables, graphing is

not possible, so we must report to performing a comparison of the values at a point with neighboring values. That is, we examine the local rate of change of a function. These ideas carry over directly to finding extrema of integral functionals.

To find the extrema of a function inside an interval, we look for *local stationary behavior*. At a point x where a function attains a local extremum, given an infinitesimal change, the value of the function should remain the same; otherwise we do not have an extremum. This is the same as examing the *local linearization* of the function. At an extremum, the

function should be flat. The rate of change in every possible direction must be zero. Since there are only two directions, this is easy. Analytically, this means that the differential of of the function is equal to zero.

Hence, a necessary condition that the function $f(x)$ be stationary at a point x_0 is that the derivative $f'(x_0)$ is equal to zero. The location where this happens is called *a critical point*. This condition is only necessary, since the condition implies that at the critical point the function can have a local maximum, a local minimum, or an inflection point. Further examination is required, namely checking out the

local curvature at the critical point. This involves the second-derivative test. For functions defined on finite intervals, the values at the endpoints must be checked separately, since the derivates are not defined there.

For functions of two variables, say $f(x, y)$**, let us assume that a point** (x_0, y_0) **is a critical point. In order for the function to be stationary at this point, we must examine the variation of the function as we move an infinitesimal amount in any possible direction. So let r =** Δx **i+**Δy**j be any vector that will denote some fixed but arbitrary direction. We can use a small parameter,**

ε, to test the variation of the function under a infinitesimal displacement:

Now the function

$$f(\mathbf{r},\varepsilon) = f(x_0 + \varepsilon\Delta x, y_0 + \varepsilon\Delta y)$$

is a function of a single variable ε. This can be thought of as cutting a slice through the surface defined by $f(x,y)$ along the direction of the vector r. This curve is parameterized by ε. Note that at ε=0, $f(\mathbf{r},0) = f(x_0, y_0)$. Thus we have reduced the analysis to searching condition that $f(\mathbf{r},\varepsilon)$ is stationary at ε=0 is

that $f'($r ,0)=0. The rate of change with respect to ε is

$$\frac{df}{d\varepsilon} = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \quad \textbf{(3.5)}$$

Setting ε =0 in Eqn (3.5), we find that

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = 0 \quad \textbf{(3.6)}$$

in which the partial derivatives are evaluated *at the point* (x_0, y_0).The left-hand side of Eqn (3.6) is equal to zero for an arbitrary direction specified by Δx and Δy if and only if the partial derivatives are equal to zero at (x_0, y_0). That is,

**(3.7)

The condition (3.7) give a necessary and sufficient condition for local stationary behavior of ** at the point **.

These conditions extend to higher dimensions for functions of *n* variables **, as

**

We now generalize these concepts to establish criteria for stationary values of scalar quantities that depend on entire functions as arguments. Before formally

developing the theory, let us consider a simple motivational example.