Plasma and Sheath

Fall, 2018

Kyoung-Jae Chung

Department of Nuclear Engineering

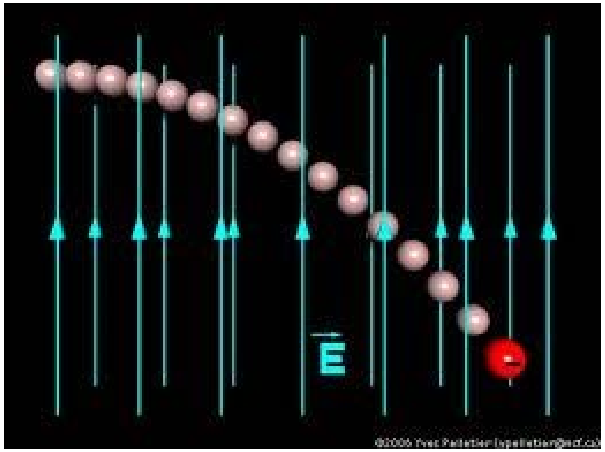
Seoul National University

Motions in uniform electric field

- Equation of motion of a charged particle in fields

$$m\frac{d\boldsymbol{v}}{dt} = q[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r},t)], \qquad \frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}(t)$$

- Motion in constant electric field



✓ For a constant electric field $\boldsymbol{E} = \boldsymbol{E_0}$ with $\boldsymbol{B} = 0$,

$$\boldsymbol{r}(t) = \boldsymbol{r_0} + \boldsymbol{v_0}t + \frac{q\boldsymbol{E_0}}{\boldsymbol{2}m}t^2$$

✓ Electrons are easily accelerated by electric field due to their smaller mass than ions.

✓ Electrons (Ions) move against (along) the electric field direction.

✓ The charged particles get kinetic energies.

Motions in uniform magnetic field

- Motion in constant magnetic field

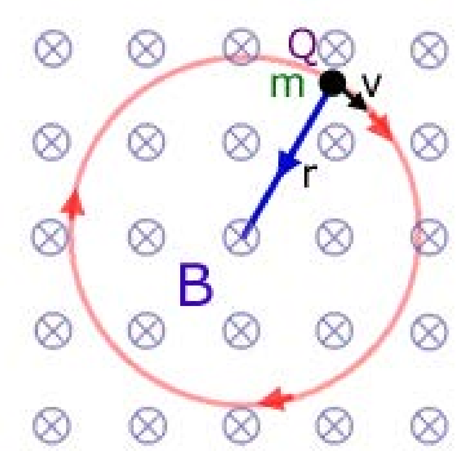
$$m\frac{d\boldsymbol{v}}{dt} = q\boldsymbol{v} \times \boldsymbol{B}$$

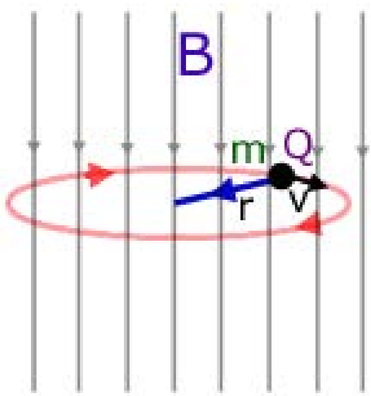
- For a constant magnetic field $\boldsymbol{B} = B_0\boldsymbol{z}$ with $\boldsymbol{E} = 0$,

$$m\frac{dv_x}{dt} = qB_0v_y$$

$$m\frac{dv_y}{dt} = -qB_0v_x$$

$$m\frac{dv_z}{dt} = 0$$





- Cyclotron (gyration) frequency

$$\frac{d^2v_x}{dt^2} = -\omega_c^2 v_x$$

$$\omega_c = \frac{|q|B_0}{m}$$

Motions in uniform magnetic field

- Particle velocity

$$v_x = v_\perp \cos(\omega_c t + \phi_0)$$
$$v_y = -v_\perp \sin(\omega_c t + \phi_0)$$
$$v_z = 0$$

- Particle position

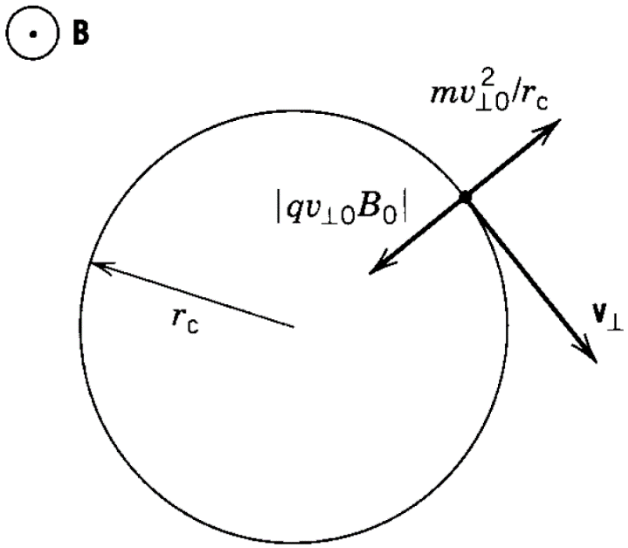
$$x = r_c \sin(\omega_c t + \phi_0) + (x_0 - r_c \sin\phi_0)$$
$$y = r_c \cos(\omega_c t + \phi_0) + (y_0 - r_c \cos\phi_0)$$
$$z = z_0 + v_{z0} t$$

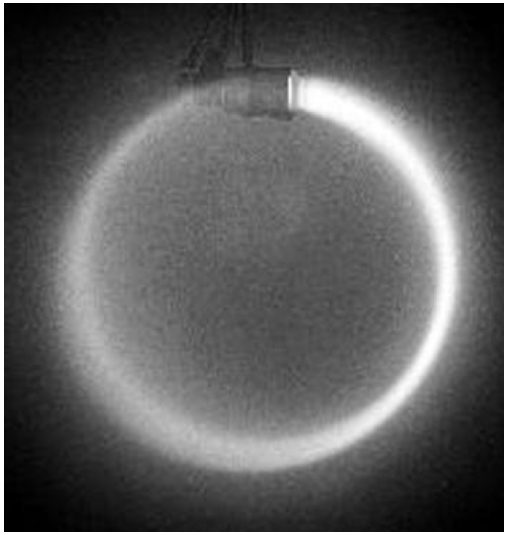
- Guiding center

$$(x_0, y_0, z_0 + v_{z0} t\ )$$

- Larmor (gyration) radius

$$r_c = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q| B_0}$$





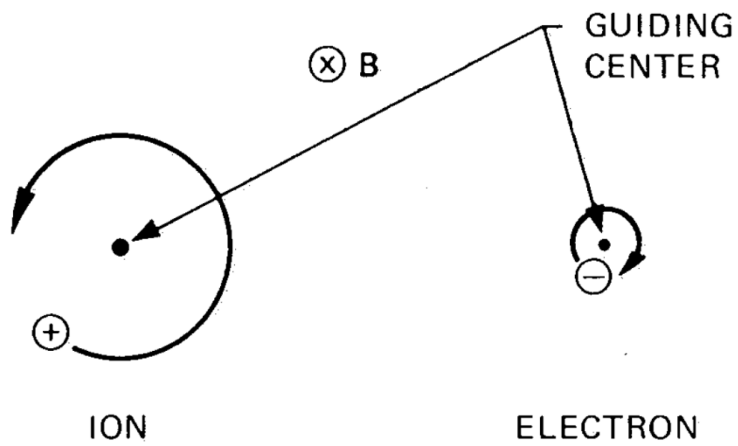
Gyro-frequency and radius

- The direction of gyration is always such that the magnetic field generated by the charged particle is opposite to the externally imposed field. → diamagnetic

- For electrons

$$f_{ce} = 2.80 \times 10^6\, B_0 \text{ [Hz]} \quad (B_0 \text{ in gauss})$$

$$r_{ce} = \frac{3.37\sqrt{E}}{B_0} \text{ [cm]} \quad (E \text{ in volts})$$



- For singly charged ions

$$f_{ci} = 1.52 \times 10^3\, B_0/M_A \text{ [Hz]} \quad (B_0 \text{ in gauss})$$

$$r_{ci} = \frac{144\sqrt{E M_A}}{B_0} \text{ [cm]} \quad (E \text{ in volts}, M_A \text{ in amu})$$

- Energy gain?

Motions in uniform E and B fields

- Equation of motion

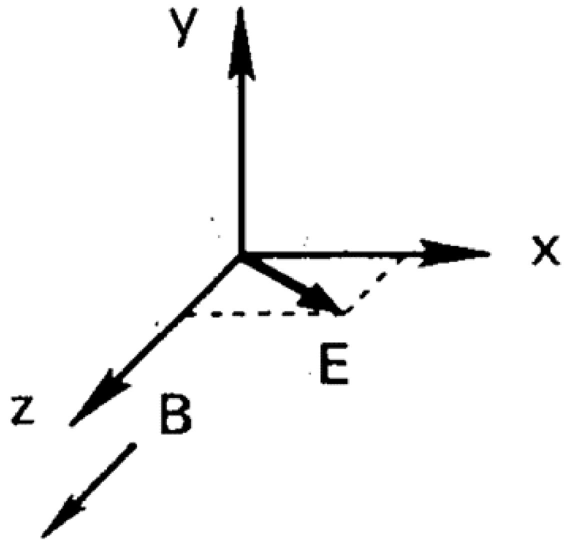
$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

- Parallel motion: $\boldsymbol{B} = B_0\boldsymbol{z}$ and $\boldsymbol{E} = E_0\boldsymbol{z}$,

$$m\frac{dv_z}{dt} = qE_z$$

$$v_z = \frac{qE_z}{m}t + v_{z0}$$

→ Straightforward acceleration along B



$\boldsymbol{E}\times\boldsymbol{B}$ drift

- Transverse motion: $\boldsymbol{B} = B_0\boldsymbol{z}$ and $\boldsymbol{E} = E_0\boldsymbol{x}$,

$$m\frac{dv_x}{dt} = qE_0 + qB_0v_y$$
$$m\frac{dv_y}{dt} = -qB_0v_x$$

- Differentiating,

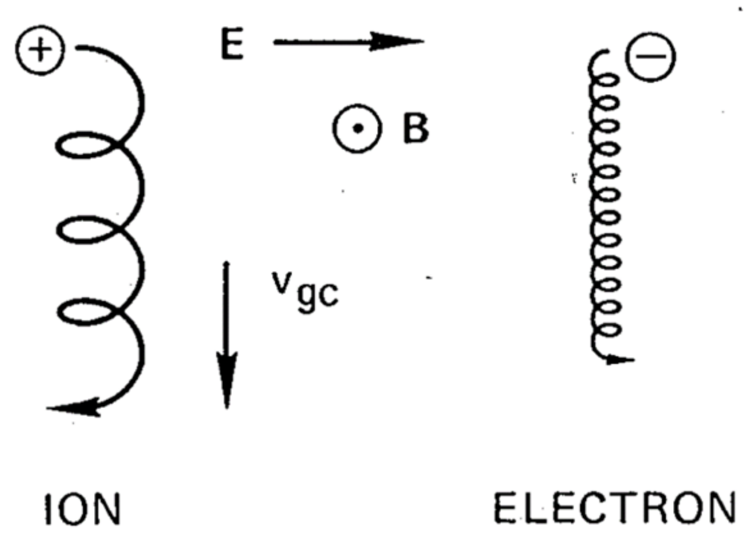
$$\frac{d^2v_x}{dt^2} = -\omega_c^2 v_x$$
$$\frac{d^2v_y}{dt^2} = -\omega_c^2\left(\frac{E_0}{B_0} + v_y\right)$$

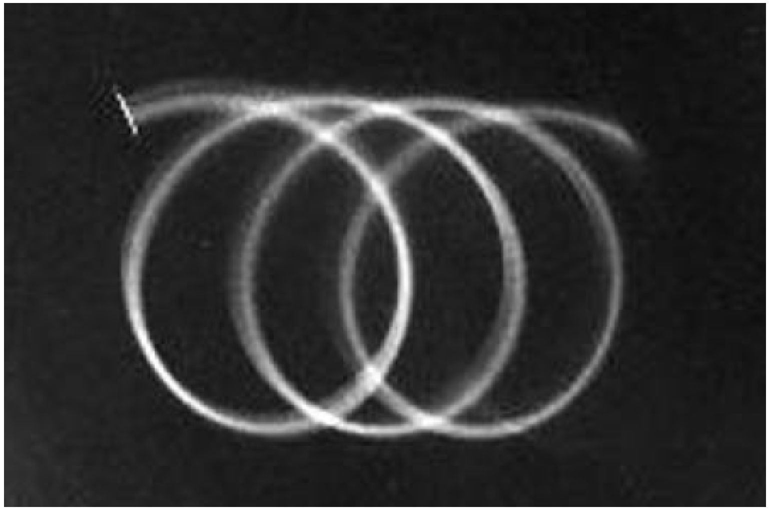
- Particle velocity

$$v_x = v_\perp \cos(\omega_c t + \phi_0)$$
$$v_y = -v_\perp \sin(\omega_c t + \phi_0) - \frac{E_0}{B_0}$$

v_{gc}

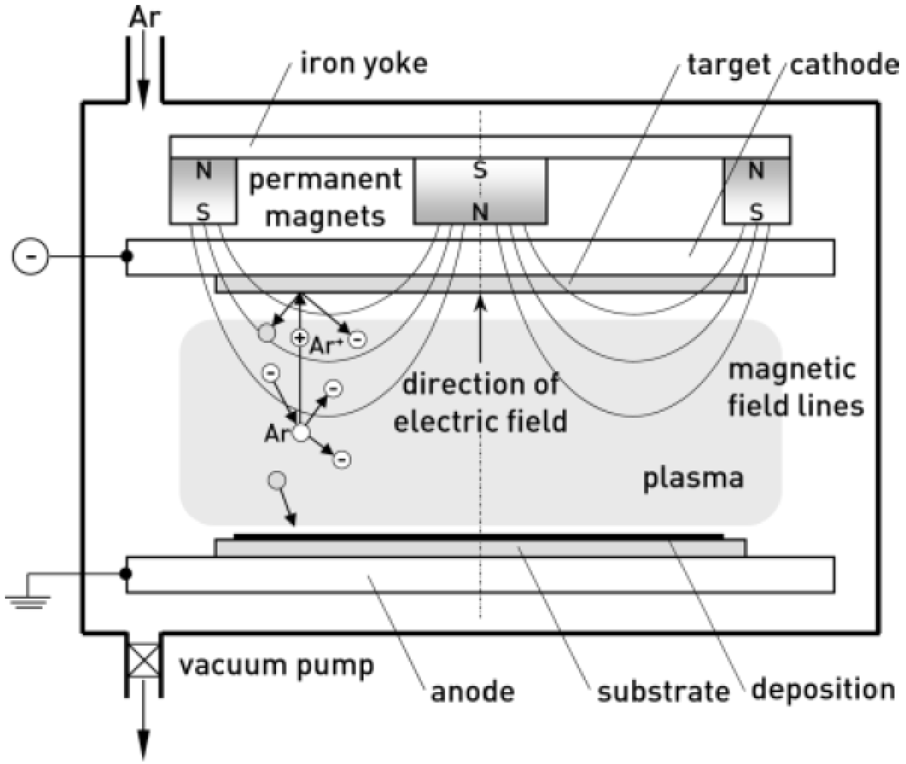
$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$$





DC magnetron

- A magnetron which is widely used in the sputtering system uses the $\boldsymbol{E}\times\boldsymbol{B}$ drift motion for plasma confinement.



- What is the direction of $\boldsymbol{E}\times\boldsymbol{B}$ drift motion?

Motions in gravitational field

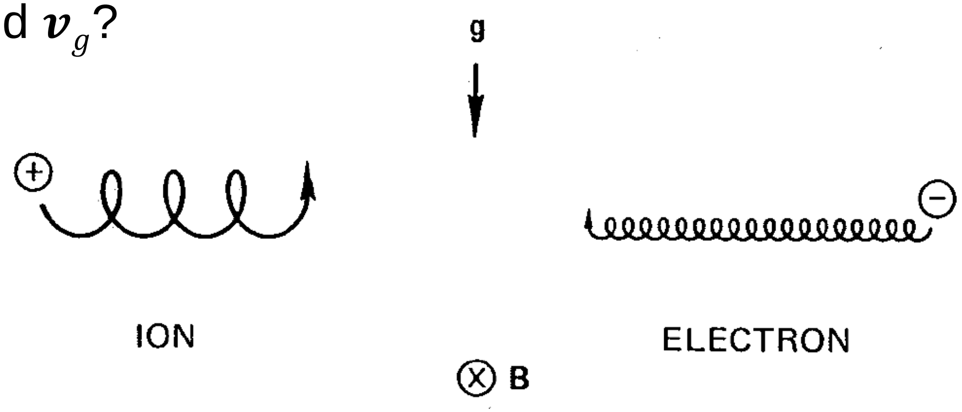
- Generally, the guiding center drift caused by general force $\boldsymbol{F}$

$$\boldsymbol{v}_f = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}$$

- If $\boldsymbol{F}$ is the force of gravity $m\boldsymbol{g}$,

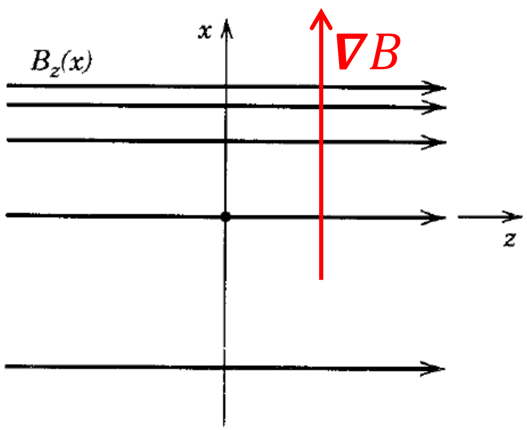
$$\boldsymbol{v}_g = \frac{m}{q}\frac{\boldsymbol{g}\times\boldsymbol{B}}{B^2}$$

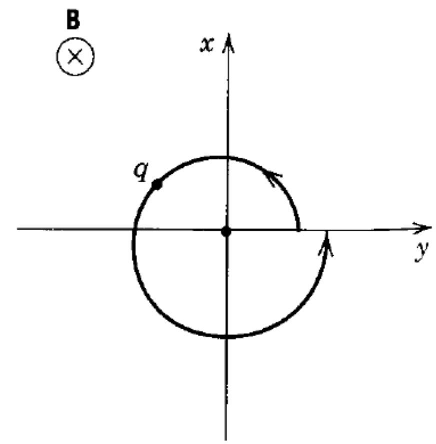
- What is the difference between $\boldsymbol{v}_E$ and $\boldsymbol{v}_g$?



$\nabla B \perp \boldsymbol{B}$: Grad-B drift

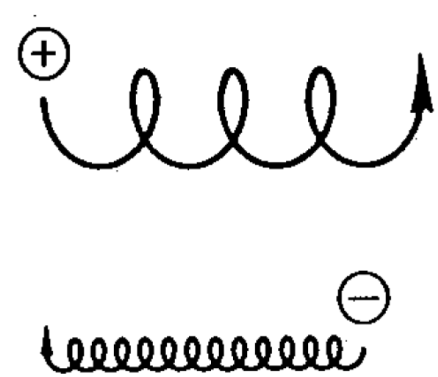
- The gradient in $|\boldsymbol{B}|$ causes the Larmor radius to be larger at the bottom of the orbit than at the top, and this should lead to a drift, in opposite directions for ions and electrons, perpendicular to both $\boldsymbol{B}$ and ∇B.





- Guiding center motion

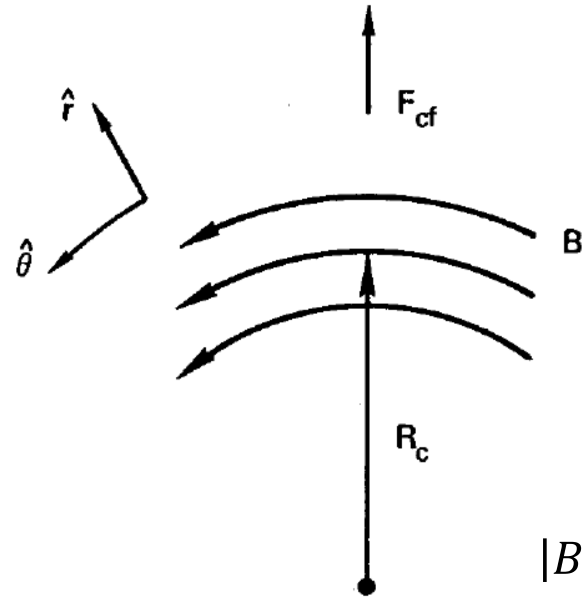
$$\boldsymbol{v}_{\nabla B} = \pm\frac{1}{2}v_{\perp}r_c\frac{\boldsymbol{B}\times\nabla B}{B^2}$$



Curved B: Curvature drift

- The average centrifugal force

$$\boldsymbol{F}_{cf} = \frac{m v_\parallel^2}{R_c}\hat{\boldsymbol{r}} = m v_\parallel^2 \frac{\boldsymbol{R}_c}{R_c^2}$$



- Curvature drift

$$\boldsymbol{v}_R = \frac{1}{q}\frac{\boldsymbol{F}_{cf} \times \boldsymbol{B}}{B^2} = \frac{m v_\parallel^2}{q B^2}\frac{\boldsymbol{R}_c \times \boldsymbol{B}}{R_c^2}$$

$$|B| \propto \frac{1}{R_c}$$

- Total drift in a curved vacuum field (curvature + grad-B)

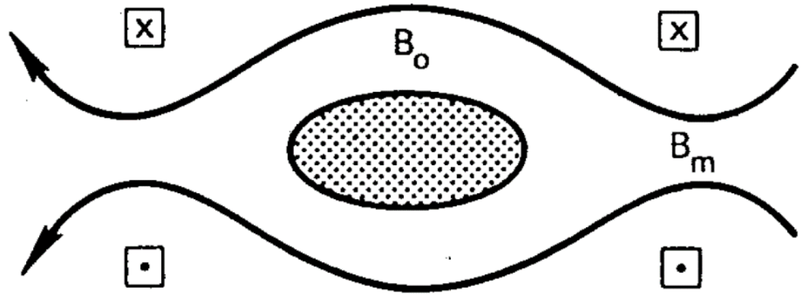
$$\boldsymbol{v}_R + \boldsymbol{v}_{\nabla B} = \frac{m}{q}\frac{\boldsymbol{R}_c \times \boldsymbol{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)$$

$$\frac{\boldsymbol{\nabla}|B|}{|B|} = -\frac{\boldsymbol{R}_c}{R_c^2}$$

$\nabla B \parallel \boldsymbol{B}$: Magnetic mirror

- Adiabatic invariant: Magnetic moment

$$\mu = IA = \frac{\frac{1}{2} m v_\perp{}^2}{B}$$



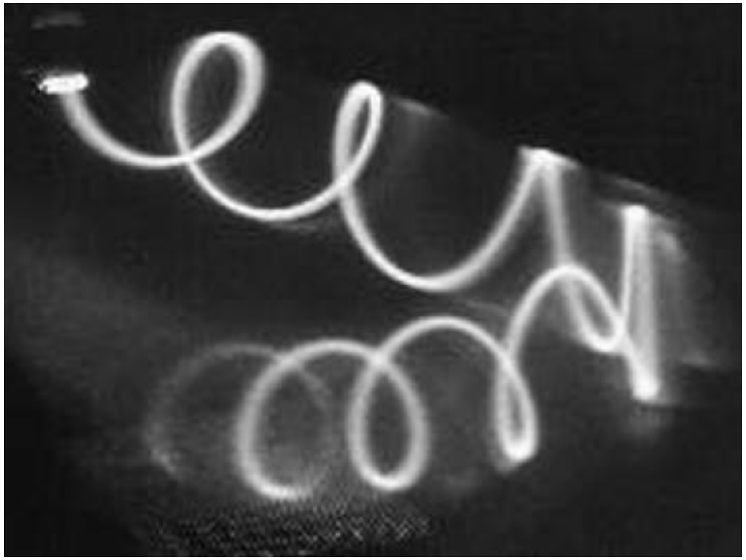
- As the particle moves into regions of stronger or weaker B, its Larmor radius changes, but μ remains invariant.

- Magnetic mirror

$$\frac{\frac{1}{2} m v_{\perp 0}{}^2}{B_0} = \frac{\frac{1}{2} m v_{\perp m}{}^2}{B_m}$$

$$v_{\perp m}{}^2 = v_{\perp 0}{}^2 + v_{\parallel 0}{}^2 \equiv v_0{}^2$$

$$\frac{B_0}{B_m} = \frac{v_{\perp 0}{}^2}{v_{\perp m}{}^2} = \frac{v_{\perp 0}{}^2}{v_0{}^2} = \sin^2\theta$$

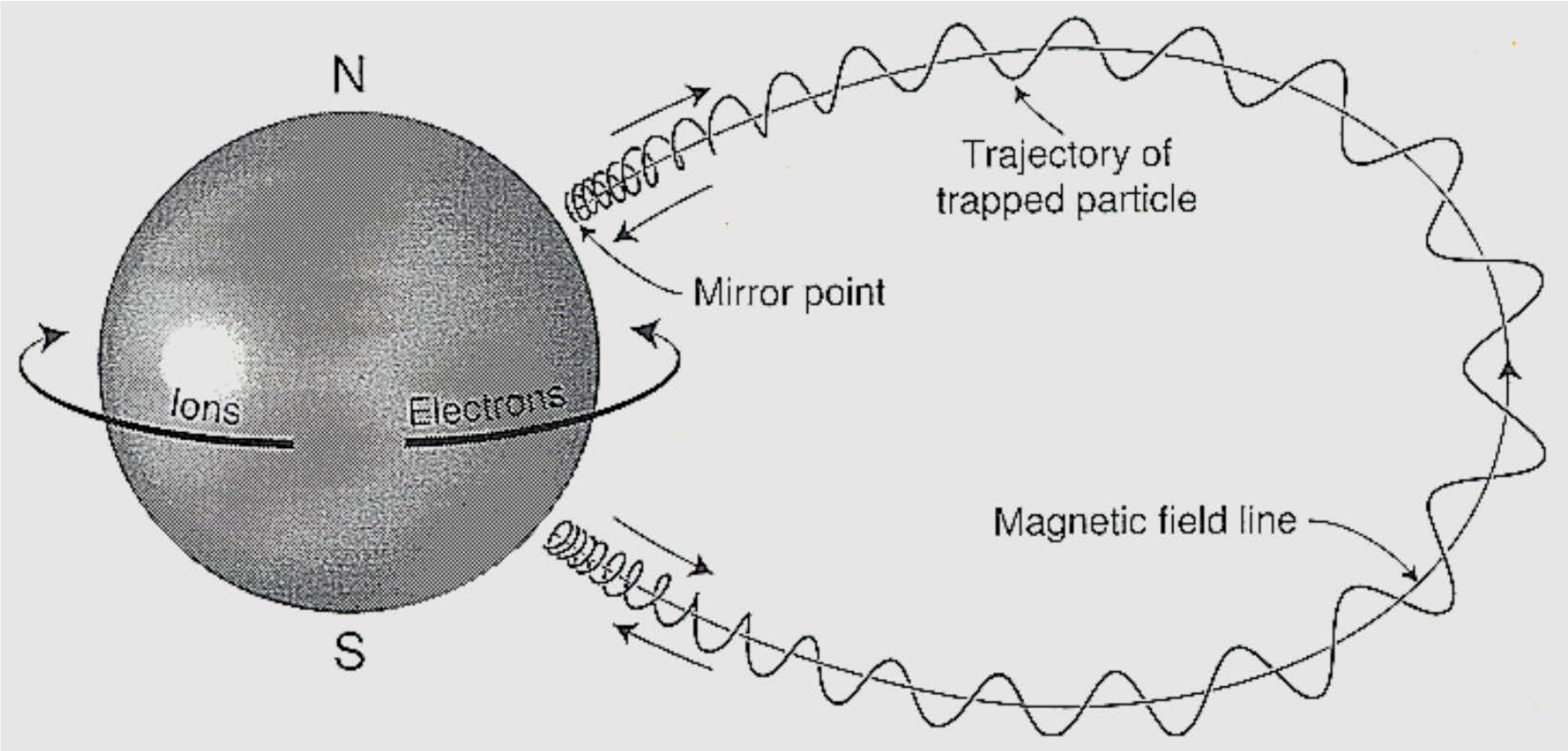


SEOUL NATIONAL UNIVERSITY

Motions in a dipole magnetic field

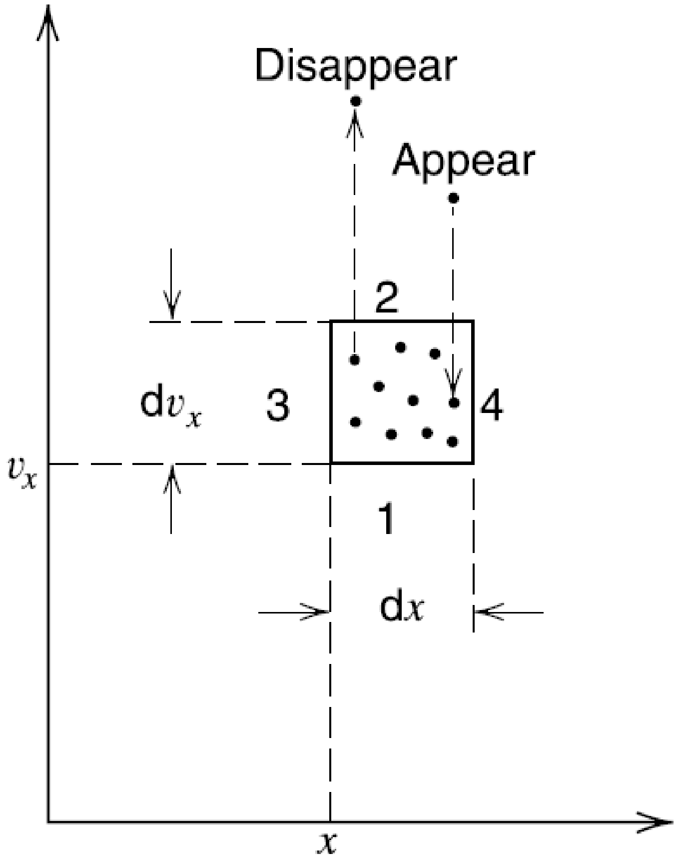
- Trajectories of particles confined in a dipole field

  → Particles experience gyro-, bounce- and drift- motions



Boltzmann's equation & macroscopic quantities

$f(\mathbf{r}, \mathbf{v}, t)\mathrm{d}^3r\,\mathrm{d}^3v =$ number of particles inside a six-dimensional phase space volume $\mathrm{d}^3r\,\mathrm{d}^3v$ at $(\mathbf{r}, \mathbf{v})$ at time t



$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = \left.\frac{\partial f}{\partial t}\right|_{\mathrm{c}}$$

- Particle density

$$n(\mathbf{r}, t) = \int f\,\mathrm{d}^3v$$

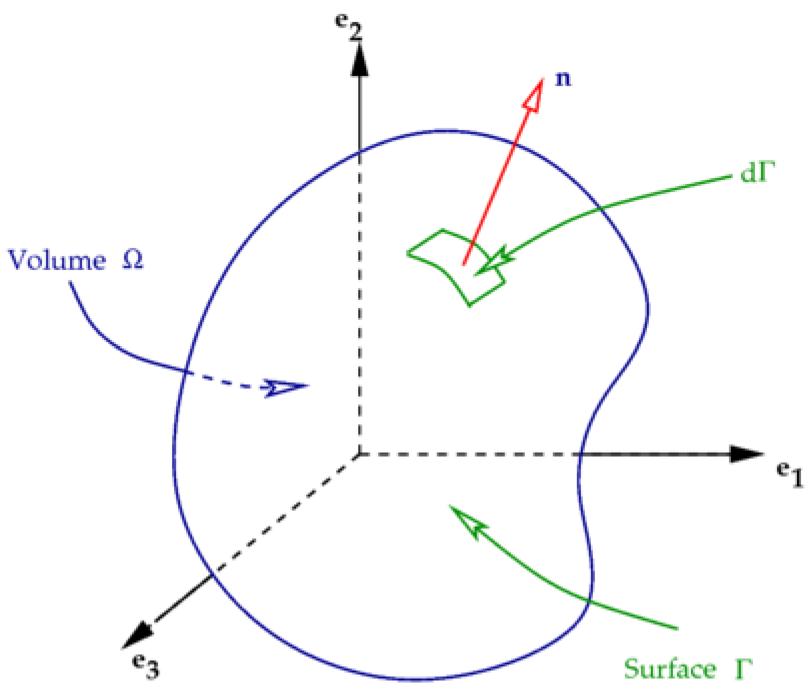
- Particle flux

$$\Gamma(\mathbf{r}, t) = n\mathbf{u} = \int \mathbf{v}f\,\mathrm{d}^3v$$

- Particle energy density

$$w = \frac{3}{2}p + \frac{1}{2}mu^2n = \frac{1}{2}m\int v^2 f\,\mathrm{d}^3v$$

Particle conservation



$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = G - L$$

The net number of particles per second generated within Ω either flows across the surface Γ or increases the number of particles within Ω.

For common low-pressure discharges in steady-state:

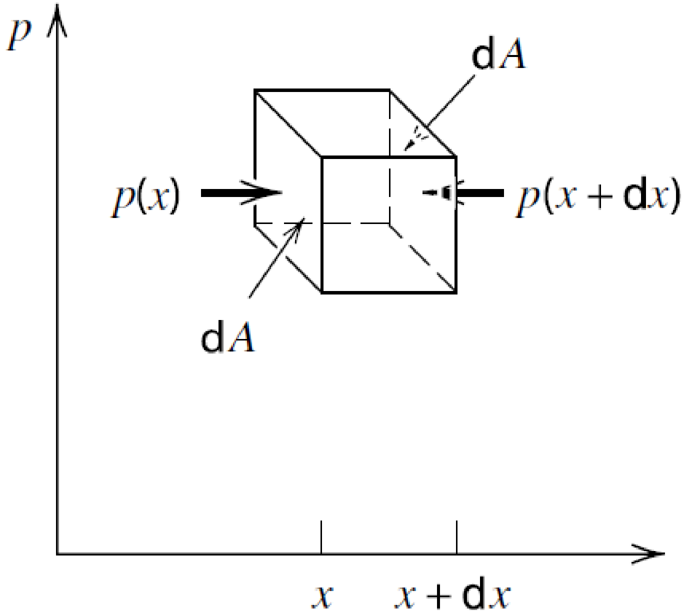
$$G = \nu_{iz} n_e, \qquad L \approx 0$$

Hence, $\boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = \nu_{iz} n_e$

- The continuity equation is clearly not sufficient to give the evolution of the density n, since it involves another quantity, the mean particle velocity $\boldsymbol{u}$.

Momentum conservation

$$\underbrace{mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right]}_{\text{Convective derivative}} = qn(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla\cdot\mathbf{\Pi} + \mathbf{f}\Big|_{\mathrm{c}}$$



- Pressure tensor
  → isotropic for weakly ionized plasmas

$$\Pi_{ij} = mn\langle (v_i - u)(v_j - u)\rangle_{\mathbf{v}}$$

$$\nabla\cdot\mathbf{\Pi} = \nabla p$$

- The time rate of momentum transfer per unit volume due to collisions with other species

$$\mathbf{f}|_{\mathrm{c}} = -\sum_{\beta} mn\nu_{\mathrm{m}\beta}(\mathbf{u} - \mathbf{u}_{\beta}). - m(\mathbf{u} - \mathbf{u}_{\mathrm{G}})G + m(\mathbf{u} - \mathbf{u}_{\mathrm{L}})L$$

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = qn\boldsymbol{E} - \boldsymbol{\nabla}p - mn\nu_m\boldsymbol{u}$$

Equation of state (EOS)

- An equation of state is a thermodynamic equation describing the state of matter under a given set of physical conditions.

$$p = p(n, T), \qquad \varepsilon = \varepsilon(n, T)$$

- Isothermal EOS for slow time variations, where temperatures are allowed to equilibrate. In this case, the fluid can exchange energy with its surroundings.

$$\boxed{p = nkT, \qquad \nabla p = kT\nabla n} \qquad n_{\mathrm{g}}\,(\mathrm{cm}^{-3}) \approx 3.250 \times 10^{16}\, p\,(\mathrm{Torr})$$

→ The energy conservation equation needs to be solved to determine *p* and *T*.

- Adiabatic EOS for fast time variations, such as in waves, when the fluid does not exchange energy with its surroundings

$$\boxed{p = Cn^{\gamma}, \qquad \frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}} \qquad \gamma = \frac{C_p}{C_v} \quad \text{(specific heat ratio)}$$

→ The energy conservation equation is not required.

- Specific heat ratio vs degree of freedom (f) $\qquad \gamma = 1 + \dfrac{2}{f}$

Equilibrium: Maxwell-Boltzmann distribution

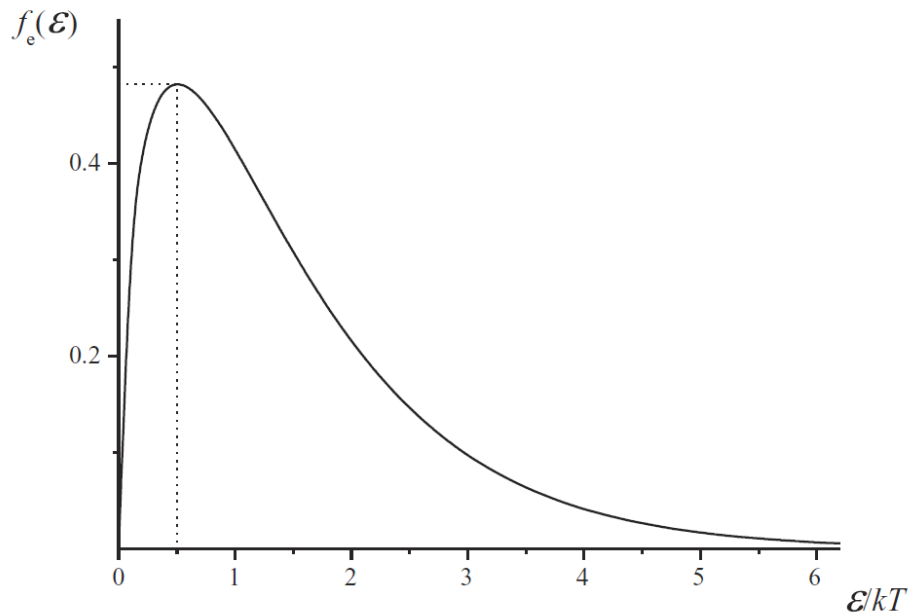
- For a single species in thermal equilibrium with itself (e.g., electrons), in the absence of time variation, spatial gradients, and accelerations, the Boltzmann equation reduces to

$$\left.\frac{\partial f}{\partial t}\right|_c = 0$$

- Then, we obtain the Maxwell-Boltzmann velocity distribution

$$f(v) = \sqrt{\frac{2}{\pi}\left(\frac{m}{kT}\right)^3}\, v^2 exp\left(-\frac{mv^2}{2kT}\right)$$

$$f(\varepsilon) = \frac{2}{\sqrt{\pi}}\left(\frac{\varepsilon}{kT}\right)^{1/2} exp\left(-\frac{\varepsilon}{kT}\right)$$



- The mean speed

$$\bar{v} = \left(\frac{8kT}{\pi m}\right)^{1/2}$$

Particle and energy flux

- The directed particle flux: the number of particles per square meter per second crossing the $z = 0$ surface in the positive direction

$$\Gamma_z = n\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \sin\theta \,\mathrm{d}\theta \int_0^{\infty} v\cos\theta \exp\left(-\frac{v^2}{2v_{\mathrm{th}}^2}\right) v^2 \,\mathrm{d}v \qquad v_{th} = \left(\frac{kT}{m}\right)^{1/2}$$

$$\Gamma_z = \frac{1}{4} n\bar{v}$$

- The average energy flux: the amount of energy per square meter per second in the +z direction

$$S_z = n\left\langle \frac{1}{2} m v^2 v_z \right\rangle_v = 2kT\Gamma_z$$

- The average kinetic energy W per particle crossing $z = 0$ in the positive direction

$$W = 2kT$$

Diffusion and mobility

- The fluid equation of motion including collisions

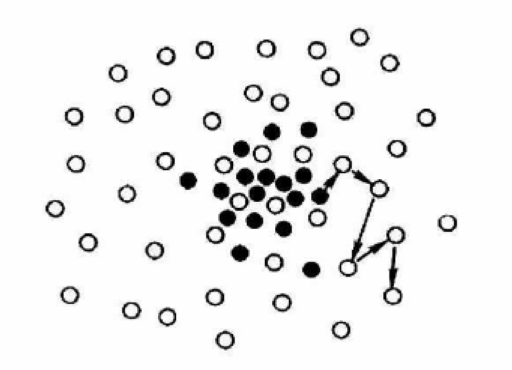
$$mn\frac{d\boldsymbol{u}}{dt} = mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = qn\boldsymbol{E} - \boldsymbol{\nabla}p - mn\nu_m\boldsymbol{u}$$

- In steady-state, for isothermal plasmas

$$\boldsymbol{u} = \frac{1}{mn\nu_m}(qn\boldsymbol{E} - \boldsymbol{\nabla}p) = \frac{1}{mn\nu_m}(qn\boldsymbol{E} - kT\boldsymbol{\nabla}n)$$

$$= \frac{q}{m\nu_m}\boldsymbol{E} - \frac{kT}{m\nu_m}\frac{\boldsymbol{\nabla}n}{n} = \pm\mu\boldsymbol{E} - D\frac{\boldsymbol{\nabla}n}{n}$$

Drift Diffusion



Diffusion is a random walk process.

- In terms of particle flux

$$\boldsymbol{\Gamma} = n\boldsymbol{u} = \pm n\mu\boldsymbol{E} - D\boldsymbol{\nabla}n$$

$\mu = \frac{|q|D}{kT}$: Einstein relation

$\mu = \frac{|q|}{m\nu_m}$: Mobility

$D = \frac{kT}{m\nu_m}$: Diffusion coefficient

Ambipolar diffusion

- The flux of electrons and ions out of any region must be equal such that charge does not build up. Since the electrons are lighter, and would tend to flow out faster in an unmagnetized plasma, an electric field must spring up to maintain the local flux balance.

$$\Gamma_i = +n\mu_i E - D_i \nabla n \qquad \Gamma_e = -n\mu_e E - D_e \nabla n$$

- Ambipolar electric field for $\Gamma_i = \Gamma_e$

$$E = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}$$

- The common particle flux

$$\Gamma = -\frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e}\nabla n = -D_a \nabla n$$

- The ambipolar diffusion coefficient for weakly ionized plasmas

$$D_a = \frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e} \approx D_i + \frac{\mu_i}{\mu_e} D_e \approx D_i\left(1 + \frac{\mathrm{T_e}}{\mathrm{T_i}}\right) \sim \mu_i \mathrm{T_e}$$

Steady-state plane-parallel solutions

- Diffusion equation

volume source and sink

$$\frac{\partial n}{\partial t} - D\nabla^2 n = G - L$$

- For a plane-parallel geometry with no volume source or sink

$$-D\frac{d^2 n}{dx^2} = 0 \Longrightarrow n = Ax + B$$

$$\Longrightarrow n = \frac{\Gamma_0}{D}\left(\frac{l}{2} - x\right)$$

B.C. $\Gamma(x = 0) = \Gamma_0$

$n(x = l/2) = 0$

- For a uniform specified source of diffusing particles

n_0 is determined

$$-D\frac{d^2 n}{dx^2} = G_0 \Longrightarrow n = \frac{G_0 l^2}{8D}\left[1 - \left(\frac{2x}{l}\right)^2\right]$$

B.C. symmetric at $x = 0$

$n(x = \pm l/2) = 0$

- The most common case is for a plasma consisting of positive ions and an equal number of electrons which are the source of ionization

$$\nabla^2 n + \frac{\nu_{iz}}{D} n = 0$$

where, $D = D_a$ and ν_{iz} is the ionization frequency

Boltzmann's relation

- The density of electrons in thermal equilibrium can be obtained from the electron force balance in the absence of the inertial, magnetic, and frictional forces

$$mn_e\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right]=-en_e\boldsymbol{E}-\boldsymbol{\nabla}p_e-mn_e\nu_m\boldsymbol{u}$$

Setting $\boldsymbol{E}=-\boldsymbol{\nabla}\Phi$ and assuming $p_e=n_ekT_e$

$$en_e\boldsymbol{\nabla}\Phi-kT_e\boldsymbol{\nabla}n_e=n_e\boldsymbol{\nabla}(e\Phi-kT_e\ln n_e)=0$$

Integrating, we have

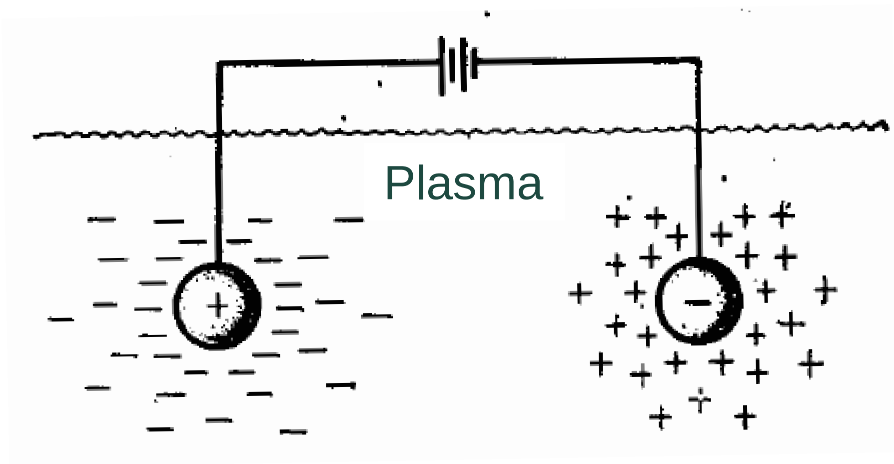
$$n_e(\boldsymbol{r})=n_0\exp\left(\frac{e\Phi(\boldsymbol{r})}{kT_e}\right)=n_0\exp\left(\frac{\Phi(\boldsymbol{r})}{\mathrm{T_e}}\right)$$

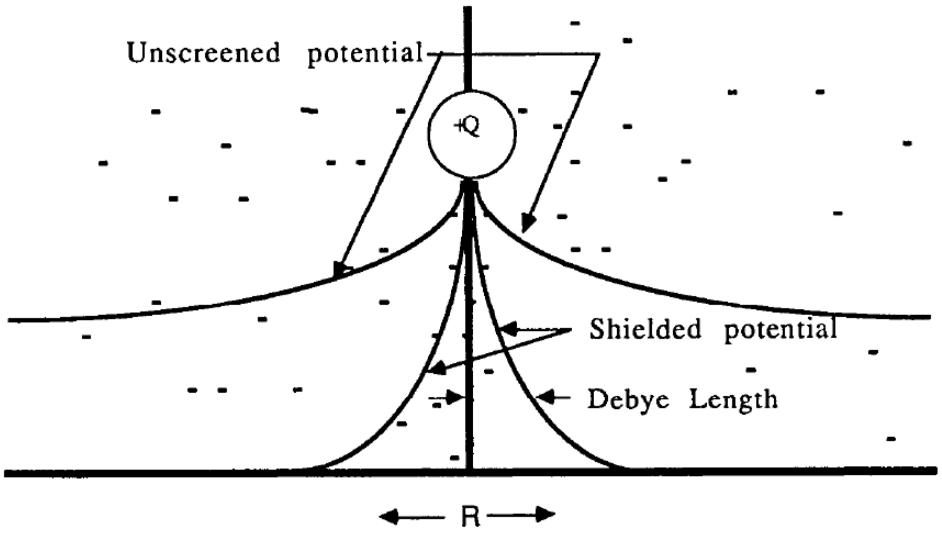
Boltzmann's relation for electrons

- For positive ions in thermal equilibrium at temperature $\mathrm{T_i}$ $\quad n_i(\boldsymbol{r})=n_0\exp\left(-\frac{\Phi(\boldsymbol{r})}{\mathrm{T_i}}\right)$

- However, positive ions are almost never in thermal equilibrium in low pressure discharges because the ion drift velocity u_i is large, leading to inertial or frictional forces comparable to the electric field or pressure gradient forces.

Debye shielding (screening)





- Coulomb potential

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

- Screened Coulomb potential

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}\exp\left(-\frac{r}{\lambda_D}\right)$$

- Poisson's equation

$$\nabla^2\Phi = -\frac{en_0}{\epsilon_0}\left(1 - e^{e\Phi/kT_e}\right) \approx \frac{e^2 n_0}{\epsilon_0 kT_e}\Phi = \frac{\Phi}{\lambda_D^2}$$

- Debye length

$$\lambda_D = \left(\frac{\epsilon_0 kT_e}{e^2 n_0}\right)^{1/2} = \left(\frac{\epsilon_0 \mathrm{T_e}}{en_0}\right)^{1/2}$$

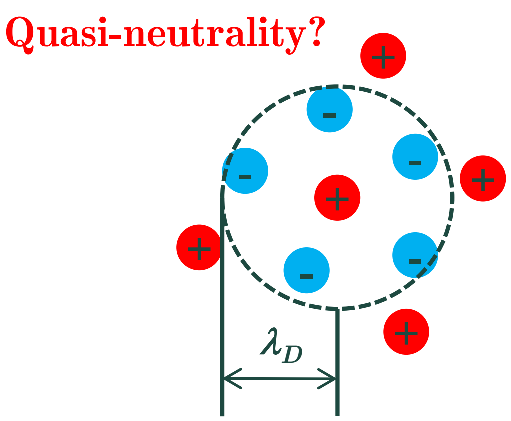
Debye length

- The electron Debye length λ_{De} is the characteristic length scale in a plasma.
- The Debye length is the distance scale over which significant charge densities can spontaneously exist. For example, low-voltage (undriven) sheaths are typically a few Debye lengths wide.

- Typical values for a processing plasma (n_e = 10^{10} cm^{-3}, T_e = 4 eV)

$$\lambda_D[\mathrm{cm}] = \left(\frac{\epsilon_0 \mathrm{T_e}}{e n_0}\right)^{1/2} = 743\sqrt{\frac{\mathrm{T_e[eV]}}{n_0[\mathrm{cm}^{-3}]}} = 743\sqrt{\frac{4}{10^{10}}} \approx 0.14\ \mathrm{mm}$$

- It is on space scales larger than a Debye length that the plasma will tend to remain neutral.
- The Debye length serves as a characteristic scale length to shield the Coulomb potentials of individual charged particles when they collide.



Quasi-neutrality

- The potential variation across a plasma of length $l \gg \lambda_{De}$ can be estimated from Poisson's equation

$$\nabla^2\Phi \sim \frac{\Phi}{l^2} \sim \left|\frac{e}{\epsilon_0}(Zn_i - n_e)\right|$$

We generally expect that

$$\Phi \lesssim \mathrm{T_e} = \frac{e}{\epsilon_0} n_e \lambda_{De}^2$$

Then, we obtain

$$\frac{|Zn_i - n_e|}{n_e} \lesssim \frac{\lambda_{De}^2}{l^2} \ll 1$$

Plasma approximation

$$Zn_i = n_e$$

- The plasma approximation is violated within a plasma sheath, in proximity to a material wall, either because the sheath thickness $s \approx \lambda_{De}$, or because $\Phi \gg \mathrm{T_e}$.

Plasma oscillations

- Electrons overshoot by inertia and oscillate around their equilibrium position with a characteristic frequency known as plasma frequency.

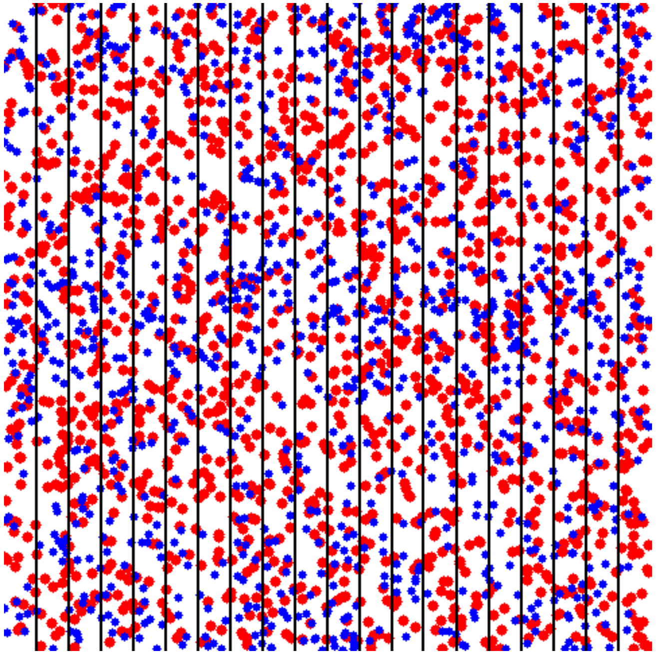
- Equation of motion (cold plasma)

$$m\frac{d^2\zeta_e}{dt^2} = -eE_x = -\frac{n_0 e^2}{\varepsilon_0}\zeta_e$$

$$\frac{d^2\zeta_e}{dt^2} + \frac{n_0 e^2}{m\varepsilon_0}\zeta_e = 0$$

- Electron plasma frequency

$$\omega_{pe} = \left(\frac{n_0 e^2}{m\epsilon_0}\right)^{1/2}$$



- If the assumption of infinite mass ions is not made, then the ions also move slightly and we obtain the natural frequency

$$\omega_p = \left(\omega_{pe}^2 + \omega_{pi}^2\right)^{1/2}$$

where, $\omega_{pi} = (n_0 e^2/M\epsilon_0)^{1/2}$ (ion plasma frequency)

Plasma frequency

- Plasma oscillation frequency for electrons and ions

$$f_{pe} = \frac{\omega_{pe}}{2\pi} = 8980\sqrt{n_0}\ [\mathrm{Hz}] \quad (n_0 \text{ in cm}^{-3})$$

$$f_{pi} = \frac{\omega_{pi}}{2\pi} = 210\sqrt{n_0/M_A}\ [\mathrm{Hz}] \quad (n_0 \text{ in cm}^{-3}, M_A \text{ in amu})$$

- Typical values for a processing plasma (Ar)

$$f_{pe} = \frac{\omega_{pe}}{2\pi} = 8980\sqrt{10^{10}}\ [\mathrm{Hz}] = 9 \times 10^{8}\ [\mathrm{Hz}]$$

$$f_{pi} = \frac{\omega_{pi}}{2\pi} = 210\sqrt{10^{10}/40}\ [\mathrm{Hz}] = 3.3 \times 10^{6}\ [\mathrm{Hz}]$$

- Collective behavior

$$\omega_{pe}\tau > 1$$

Plasma frequency > collision frequency

Criteria for plasmas

- The picture of Debye shielding is valid only if there are enough particles in the charge cloud. Clearly, if there are only one or two particles in the sheath region, Debye shielding would not be a statistically valid concept. We can compute the number of particles in a "Debye sphere":

$$N_D = n\frac{4}{3}\pi\lambda_D^3$$

Wigner-Seitz radius $a = \sqrt[3]{4\pi n_e/3}$

- Plasma parameter

$$\Lambda = 4\pi n\lambda_D^3 = 3N_D$$

- Coupling parameter

$$\Gamma = \frac{\text{Coulomb energy}}{\text{Thermal energy}} = \frac{q^2/(4\pi\epsilon_0 a)}{kT_e} \sim \Lambda^{-2/3}$$

- Criteria for plasmas:

$$\lambda_D \ll L$$

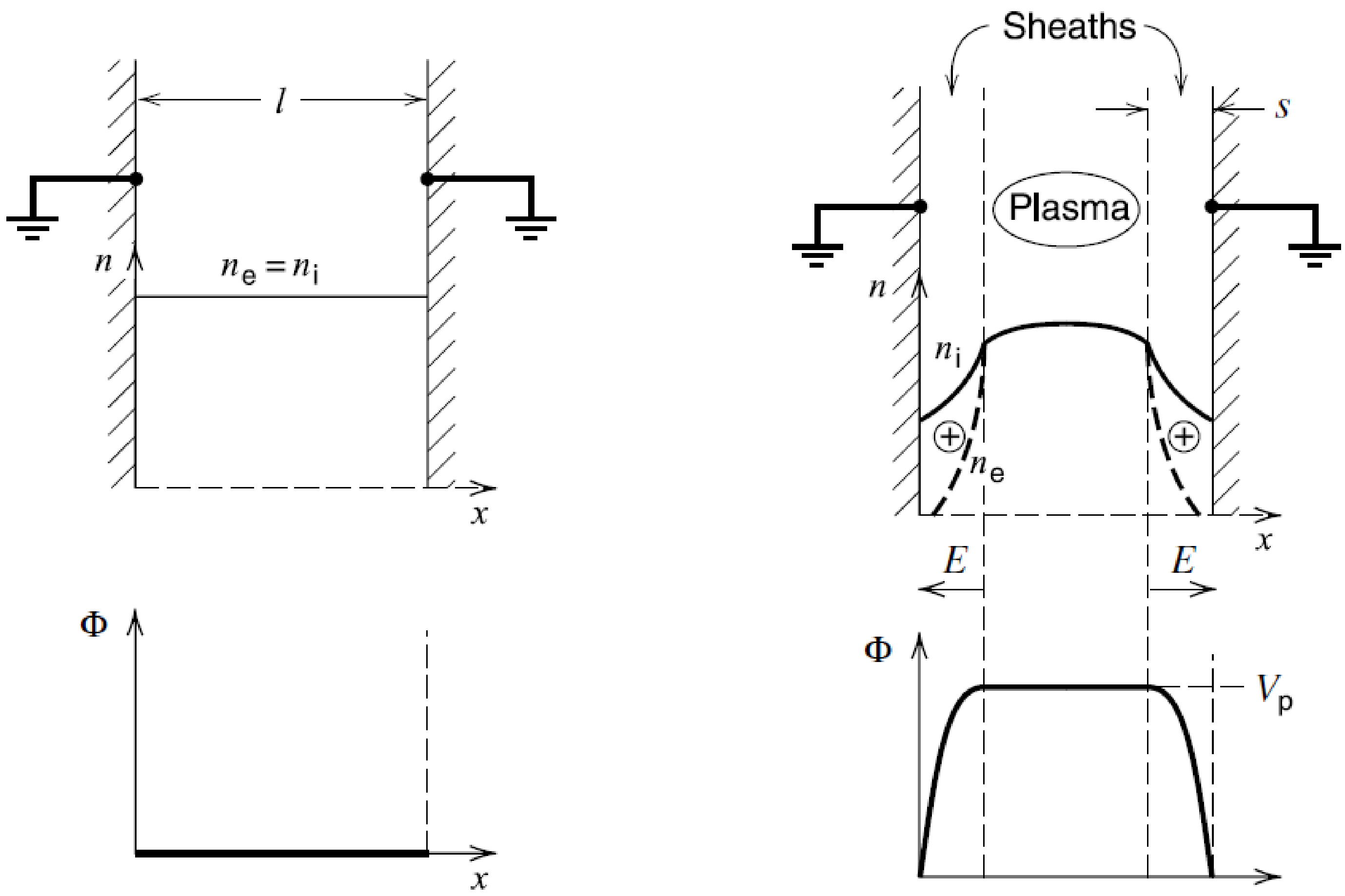
$$N_D \gg 1$$

$$\omega_{pe}\tau_c > 1$$

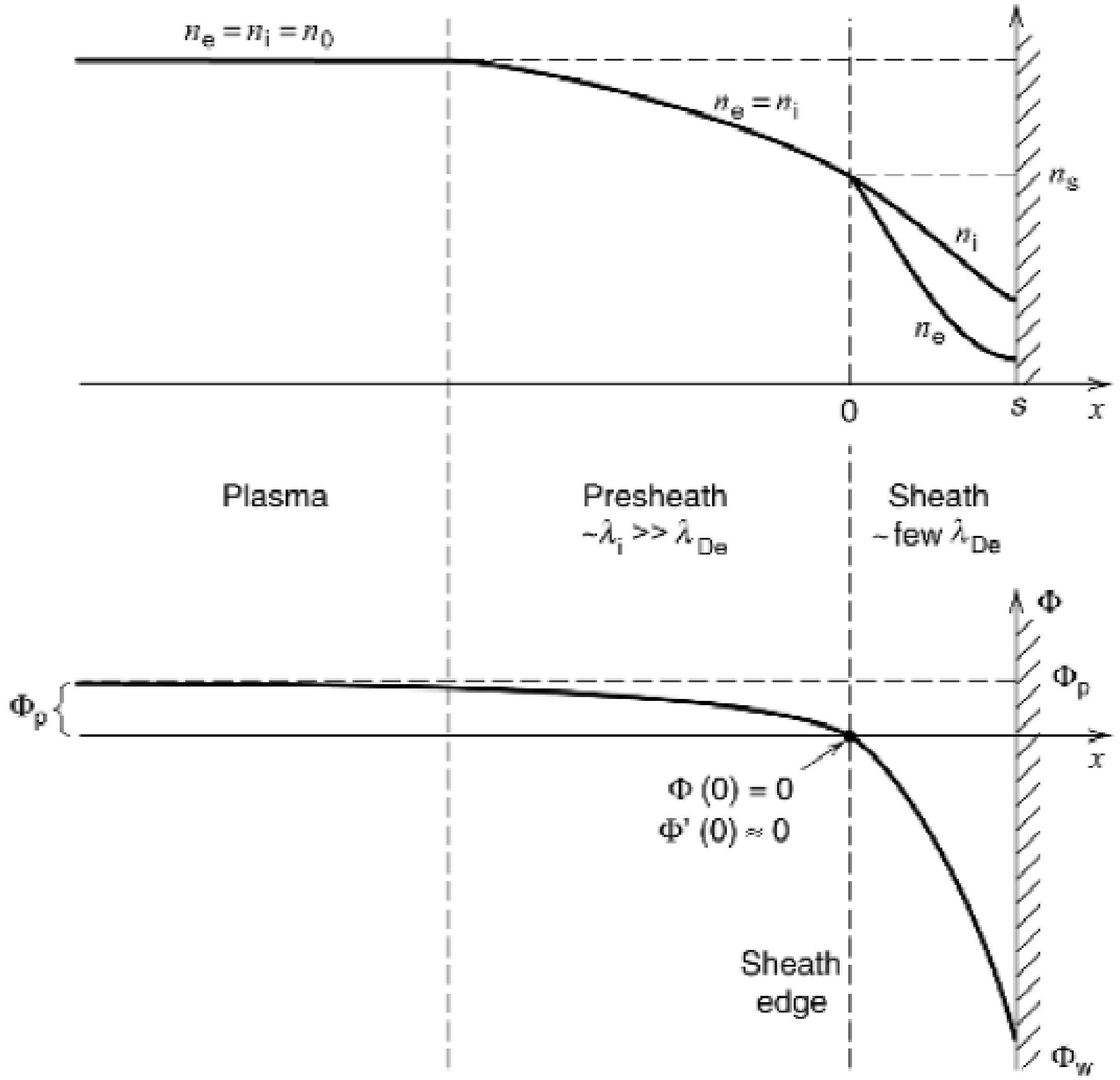
Description	Plasma parameter magnitude	
	$\Lambda \ll 1$ ($\Gamma \gg 1$)	$\Lambda \gg 1$ ($\Gamma \ll 1$)
Coupling	Strongly coupled plasma	Weakly coupled plasma
Debye sphere	Sparsely populated	Densely populated
Electrostatic influence	Almost continuously	Occasional
Typical characteristic	Cold and dense	Hot and diffuse
Examples	Solid-density laser ablation plasmas Very "cold" "high pressure" arc discharge Inertial fusion experiments White dwarfs / neutron stars atmospheres	Ionospheric physics Magnetic fusion devices Space plasma physics Plasma ball

Formation of plasma sheaths

- Plasma sheath: the non-neutral potential region between the plasma and the wall caused by the balanced flow of particles with different mobility such as electrons and ions.



Plasma-sheath structure



Collisionless sheath

- Ion energy & flux conservations (no collision)

$$\frac{1}{2}Mu(x)^2 + e\Phi(x) = \frac{1}{2}Mu_s^2 \qquad n_i(x)u(x) = n_{is}u_s$$

- Ion density profile

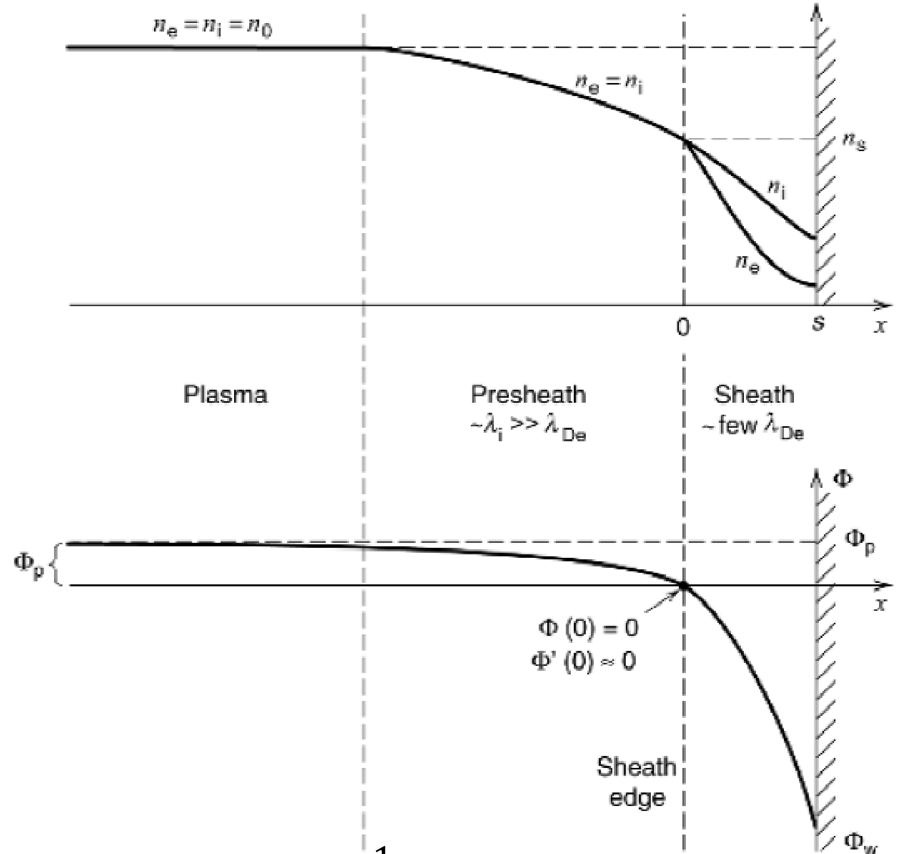
$$n_i(x) = n_{is}\left(1 - \frac{2e\Phi(x)}{Mu_s^2}\right)^{-1/2}$$

- Electron density profile

$$n_e(x) = n_{es}\exp\left(\frac{\Phi(x)}{\mathrm{T_e}}\right)$$

- Setting $n_{es} = n_{is} \equiv n_s$

$$\frac{d^2\Phi}{dx^2} = \frac{en_s}{\epsilon_0}\left[\exp\left(\frac{\Phi}{\mathrm{T_e}}\right) - \left(1 - \frac{\Phi}{\mathcal{E}_s}\right)^{-1/2}\right]$$



where, $e\mathcal{E}_s \equiv \frac{1}{2}Mu_s^2$

Bohm sheath criterion

- Multiplying the sheath equation by $d\Phi/dx$ and integrating over x

$$\int_0^{\Phi} \frac{d\Phi}{dx}\frac{d}{dx}\left(\frac{d\Phi}{dx}\right)dx = \frac{en_s}{\varepsilon_0}\int_0^{\Phi}\frac{d\Phi}{dx}\left[\exp\left(\frac{\Phi}{\mathrm{T_e}}\right)-\left(1-\frac{\Phi}{\mathcal{E}_s}\right)^{-1/2}\right]dx$$

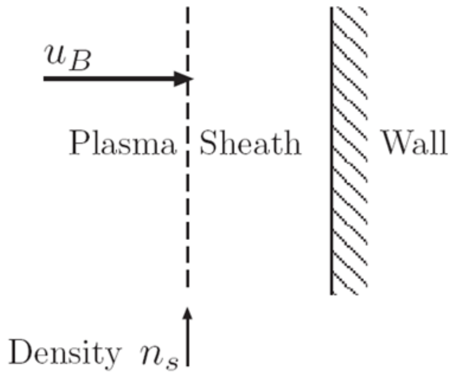
- Cancelling dx's and integrating with respect to Φ

$$\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 = \frac{en_s}{\varepsilon_0}\left[\mathrm{T_e}\exp\left(\frac{\Phi}{\mathrm{T_e}}\right)-\mathrm{T_e}+2\mathcal{E}_s\left(1-\frac{\Phi}{\mathcal{E}_s}\right)^{1/2}-2\mathcal{E}_s\right]\geq 0$$

$$\Rightarrow \quad \boxed{\mathcal{E}_s \equiv \frac{1}{2e}Mu_s^2 \geq \frac{\mathrm{T_e}}{2}}$$

- Bohm sheath criterion

$$\boxed{u_s \geq \left(\frac{e\mathrm{T_e}}{M}\right)^{1/2} \equiv \left(\frac{kT_e}{M}\right)^{1/2} \equiv u_B \quad \text{(Bohm speed)}}$$



Presheath

- To give the ions the directed velocity u_B, there must be a finite electric field in the plasma over some region, typically much wider than the sheath, called the presheath.
- At the sheath–presheath interface there is a transition from subsonic ($u_i < u_B$) to supersonic ($u_i > u_B$) ion flow, where the condition of charge neutrality must break down.
- The potential drop across a collisionless presheath, which accelerates the ions to the Bohm velocity, is given by

$$\frac{1}{2}Mu_B^2 = \frac{e\mathrm{T_e}}{2} = e\Phi_p$$

where, Φ_p is the plasma potential with respect to the potential at the sheath–presheath edge

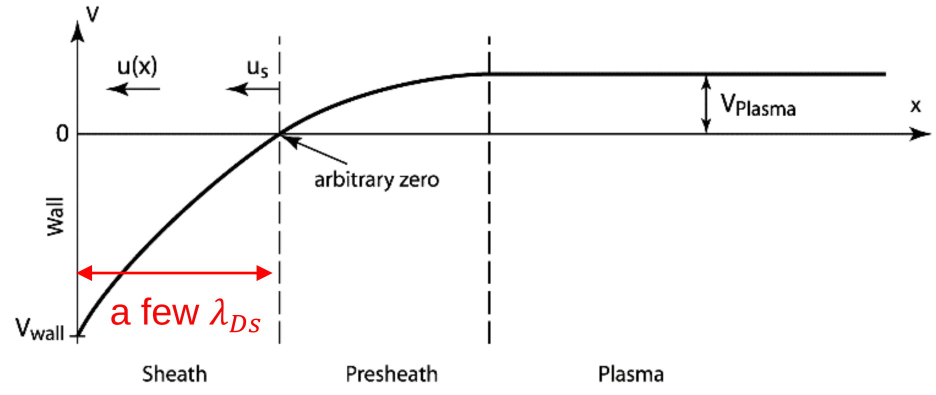
- The spatial variation of the potential $\Phi_p(x)$ in a collisional presheath (Riemann)

$$\frac{1}{2} - \frac{1}{2}\exp\left(\frac{2\Phi_p}{\mathrm{T_e}}\right) - \frac{\Phi_p}{\mathrm{T_e}} = \frac{x}{\lambda_i}$$

- The ratio of the density at the sheath edge to that in the plasma

$$n_s = n_b e^{-\Phi_p/\mathrm{T_e}} \approx 0.61 n_b$$

Sheath potential at a floating Wall



- Ion flux

$$\Gamma_i = n_s u_B$$

- Electron flux

$$\Gamma_e = \frac{1}{4} n_{ew} \bar{v}_e = \frac{1}{4} n_s \bar{v}_e \exp\left(\frac{\Phi_w}{\mathrm{T_e}}\right)$$

- Ion flux = electron flux for a floating wall

$$n_s \left(\frac{e\mathrm{T_e}}{M}\right)^{1/2} = \frac{1}{4} n_s \left(\frac{8e\mathrm{T_e}}{\pi m}\right)^{1/2} \exp\left(\frac{\Phi_w}{\mathrm{T_e}}\right)$$

- Wall potential

$$\Phi_w = -\frac{\mathrm{T_e}}{2} \ln\left(\frac{M}{2\pi m}\right)$$

$\Phi_w \approx -2.8\mathrm{T_e}$ for hydrogen and $\Phi_w \approx -4.7\mathrm{T_e}$ for argon

- Ion bombarding energy

$$\mathcal{E}_{ion} = \frac{e\mathrm{T_e}}{2} + e|\Phi_w| = \frac{e\mathrm{T_e}}{2}\left(1 + \ln\left(\frac{M}{2\pi m}\right)\right)$$

High-voltage sheath: matrix sheath

- The potential Φ in high-voltage sheaths is highly negative with respect to the plasma–sheath edge; hence $n_e \sim n_s e^{\Phi/\mathrm{T_e}} \rightarrow 0$ and only ions are present in the sheath.
- The simplest high-voltage sheath, with a uniform ion density, is known as a matrix sheath (not self-consistent in steady-state).
- Poisson's eq.

$$\frac{d^2\Phi}{dx^2} = -\frac{en_s}{\epsilon_0} \quad \Longrightarrow \quad \Phi = -\frac{en_s}{\epsilon_0}\frac{x^2}{2}$$

- Setting $\Phi = -V_0$ at $x = s$, we obtain the matrix sheath thickness

$$s = \left(\frac{2\epsilon_0 V_0}{en_s}\right)^{1/2}$$

- In terms of the electron Debye length at the sheath edge

$$s = \lambda_{Ds}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{1/2} \qquad \text{where, } \lambda_{Ds} = (\epsilon_0 \mathrm{T_e}/en_s)^{1/2}$$

High-voltage sheath: space-charge-limited current

- In the limit that the initial ion energy $\mathcal{E}_s$ is small compared to the potential, the ion energy and flux conservation equations reduce to

$$\frac{1}{2}Mu^2(x) = -e\Phi(x)$$

$$en(x)u(x) = J_0$$

$$\Rightarrow \quad n(x) = \frac{J_0}{e}\left(-\frac{2e\Phi}{M}\right)^{-1/2}$$

- Poisson's eq.

$$\frac{d^2\Phi}{dx^2} = -\frac{e}{\epsilon_0}(n_i - n_e) = -\frac{J_0}{\epsilon_0}\left(-\frac{2e\Phi}{M}\right)^{-1/2}$$

- Multiplying by $d\Phi/dx$ and integrating twice from 0 to x

$$-\Phi^{3/4} = \frac{3}{2}\left(\frac{J_0}{\epsilon_0}\right)^{1/2}\left(\frac{2e}{M}\right)^{-1/4}x$$

B.C. $\left.\dfrac{d\Phi}{dx}\right|_{x=0} = 0$

- Letting $\Phi = -V_0$ at $x = s$ and solving for J_0, we obtain

$$J_0 = \frac{4}{9}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{V_0^{3/2}}{s^2}$$

Child law:
Space-charge-limited current in a plane diode

High-voltage sheath: Child law sheath

- For a plasma $J_0 = en_s u_B$

$$en_s u_B = \frac{4}{9}\epsilon_0 \left(\frac{2e}{M}\right)^{1/2} \frac{V_0^{3/2}}{s^2}$$

- Child law sheath

$$s = \frac{\sqrt{2}}{3}\left(\frac{\epsilon_0 \mathrm{T_e}}{en_s}\right)^{1/2}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{3/4} = \frac{\sqrt{2}}{3}\lambda_{Ds}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{3/4}$$

- Potential, electric field and density within the sheath

$$\Phi = -V_0\left(\frac{x}{s}\right)^{4/3} \qquad E = \frac{4}{3}\frac{V_0}{s}\left(\frac{x}{s}\right)^{1/3} \qquad n = \frac{4}{9}\frac{\epsilon_0}{e}\frac{V_0}{s^2}\left(\frac{x}{s}\right)^{-2/3}$$

- Assuming that an ion enters the sheath with initial velocity $u(0) = 0$

$$\frac{dx}{dt} = v_0\left(\frac{x}{s}\right)^{2/3}$$

where, v_0 is the characteristic ion velocity in the sheath

$$v_0 = \left(\frac{2eV_0}{M}\right)^{1/2}$$

- Ion transit time across the sheath

$$\frac{x(t)}{s} = \left(\frac{v_0 t}{3s}\right)^3 \quad \Rightarrow \quad \tau_i = \frac{3s}{v_0}$$