SEVERAL INDEPENDENT VARIABLES

- **Remember ~ t:x,y…; q, u etc**

Multi-dimensional problems

An integral functional may have an input consisting of a function $u(x,y)$ that depends on several independent variables. In such a case, the integral is an area integral with the domain of integration defined over a region of the plane. Local stationary behavior at an extremum leads to the necessary condition

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \qquad \textbf{(3.19)}$$

Integrating by part in this case is replaced by its higher dimensional analogue, known as gradient theorem. Note that the necessary condition for the extremum (3.19) is now a partial differential equation. Subscripts on the function $u(x, y)$ **in (3.19) refer to partial derivatives with respect to the independent variables** x **and** y

A similar derivation may be done for functions $u(x,y,z)$ of the independent variables. The Euler-Lagrange equation in this case is

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)=0$$

Practice: **examine the functional with input function $u(x,y)$ given by**

$$I[u(x,y)]=\iint_D\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+2\,\mathrm{uf(x,y)}\right]\mathrm{dxdy}=0$$

where the integral is taken over some domain D **in the plane.** <u>**Derivation**</u> ~~> !!

VARIATIONAL PROBLEMS WITH CONSTRAINTS

There are two types of extremum problems involving constraints.

<u>**Two Types of Constraint! :**</u> **Subsidiary conditions that must be satisfied during the variational process may be stated as integral constraints or as equation constraints.**

Find an extremal $y(x)$ of the functional :

$$I(y) = \int_{x_1}^{x_2} F(x, y, y')dx$$

satisfying the boundary conditions

$$y(x_0) = y_0, y(x_1) = y_1$$

and subject to the constraint that the functional

$$J(y) = \int_{x_1}^{x_2} G(x, y, y')dx = C \qquad \textbf{(3.20)}$$

maintains a specified value C

: Isoperimetric problems : To find the simple closed curve of constant length maximizing the enclosed area.

For $y(x)$ **to be an extremal,**

$$\delta I(y) = \int_{x_1}^{x_2} \delta F(x, y, y')dx = 0 \qquad \textbf{(3.21)}$$

Since the integral constraint (3.20) must be maintained at a constant value.

$$\int_{x_1}^{x_2} \delta G(x, y, y')dx = 0 \qquad \textbf{(3.22)}$$

For all admissible functions. Multiplying Eq.(3.22) by a constant λ **and adding to Eqn.(3.21) result in**

$$\int_{x_1}^{x_2} [\delta F(x,y,y') + \lambda \delta G(x,y,y')]dx = 0$$

or equivalently in

$$\delta \int_{x_0}^{x_1} [F(x,y,y') + \lambda G(x,y,y')]dx = 0$$

Hence the problem reduces to finding an extremal of the auxiliary functional :

$$I[y] + \lambda J[y] = \int_{x_0}^{x_1} [F(x,y,y') + \lambda G(x,y,y')]dx$$

- **Thus the Euler-Lagrange equation for the auxiliary functional is**

$$\frac{\partial(F+\lambda G)}{\partial y}-\frac{d}{dx}\left(\frac{\partial(F+\lambda G)}{\partial y'}\right)=0$$

Or

$$F,_{y}+\lambda G,_{y}-(F,_{y'}+\lambda G,_{y'})_{,x}=0 \qquad \textbf{(3.23)}$$

Eqn (3.23) is a differential eqn with parameter λ.

The value of the constant λ is determined by substituting the solution of (3.23) into the integral constraint (3.20)

A more general isoperimetric problem consists of finding the extremals of the functionals

$$I[\vec{y}] = \int_{x_0}^{x_1} F(x, y_1 \ldots y_n, y', \ldots y_n')dx$$

under the conditions that the m **functionals**

$$J_k[\vec{y}] = \int_{x_0}^{x_1} G(x, y_1 \ldots y_n, y', \ldots y_n')dx = C_k \qquad \textbf{(3.24)}$$

Take on the specified values C_k **for** $k = 1, \ldots., m.$ **It is also assumed that each of the input functions are assigned prescribed values at the endpoints. As a direct extension to the above, we construct the auxiliary functionl**

$$I_\lambda[\vec{y}] = I[\vec{y}] + \sum_{k=1}^{m} \lambda_k J_k[\vec{y}]$$

with the augmented Lagrangian

$$F_\lambda(x,\vec{y},\vec{y}\,') = F\,(x,\vec{y},\vec{y}\,') + \sum_{k=1}^{m} \lambda_k G_k(x,\vec{y},\vec{y}\,')$$

Based on the Euler-Lagrange equations for several input functions (3.18), the associated equations for the extremals are

$$\frac{\partial F_\lambda}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F_\lambda}{\partial y_i'}\right) = 0 : i = 1,\ldots,n$$

The *m* Lagrange multipliers λ_k are determined by substituting the solutions into the integral constraints (3.24).

The other class of constraints involve finite equations or differential equations. Finite equations of constraint are also called *holonomic*, whereas differential constraints are called *nonholonomic*

A holonomic equation of constraint takes the form

$$G(x, y_1 \ldots y_n) = 0 \qquad \textbf{(3.25)}$$

The constraint (3.25) is required to be satisfied at every point in the domain,including the prescribed boundary conditions. Taking the variation of Equation(3.25), we have

$$\sum_{i=1}^{n} \frac{\partial G}{\partial y_i} \delta y_i = 0$$

Multiplying this constraint equation by an unknown function $\lambda(x)$ **and adding it to the variation (3.17) does not change the value:**

$$\int_{x_0}^{x_1}\{\sum_{i=1}^{n}[\frac{\partial F}{\partial y_i}-\frac{d}{dx}(\frac{\partial F}{\partial y_i'})]\delta y_i+\lambda(x)\sum_{i=1}^{n}\frac{\partial G}{\partial y_i}\delta y_i\}dx=0$$

Rearranging the terms results in

$$\int_{x_0}^{x_1}\{\sum_{i=1}^{n}[\frac{\partial F}{\partial y_i}-\frac{d}{dx}(\frac{\partial F}{\partial y_i'})+\lambda(x)\frac{\partial G}{\partial y_i}]\delta y_i\}dx=0$$

Now based on the constraint (3.25), only *n-1* of the variation δy_i

are independent. This means that not all of the variations may be arbitrarily varied. However, making the appropriate choice of λ , namely, choosing $\lambda(x)$ such that

$$\frac{\partial F}{\partial y_n} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'_n}\right) + \lambda(x)\frac{\partial G}{\partial y_n} = 0$$

results in

$$\int_{x_0}^{x_1} \{\sum_{i=1}^{n-1} [\frac{\partial F}{\partial y_i} - \frac{d}{dx}(\frac{\partial F}{\partial y'_i}) + \lambda(x)\frac{\partial G}{\partial y_i}]\delta y_i\} dx = 0$$

Now only *n-1* variations appear in the integral, so they can be arbitrarily varied. Hence we can obtain the necessary conditions

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) + \lambda(x)\frac{\partial G}{\partial y_i} = 0 : i = 1, \ldots n \quad \textbf{(3.26)}$$

Furthermore, since the constraint(3.25) does not contain derivatives, Eqn(3.26) are also the Euler-Lagrange Eqns for the arbitrary functional

$$I_\lambda[\vec{y}] = \int_{x_0}^{x_1} F(x, \vec{y}, \vec{y}\,') + \lambda(\mathrm{x})\mathrm{G}(\mathrm{x}, \vec{\mathrm{y}})]dx$$

Eq.(3.26) together with the constraint (3.25) represent *n+1*

Eqns for the extremals $y_1,\ldots y_n$ and the unknown multiplier function $\lambda(x)$. In the case of multiple finite-constraint equations, the modified functional becomes

$$I_\lambda[\vec{y}] = \int_{x_0}^{x_1} \left[F(x,\vec{y},\vec{y}\,') + \sum_{k=1}^{m} \lambda_k(\mathrm{x})\,\mathrm{G}(\mathrm{x},\vec{\mathrm{y}}) \right] dx$$

Differential or nonholonomic constraints have the form

$$G(x, y_1 \ldots y_n, y_1' \ldots y_n') = 0$$

which relate not only the values of the admissible functions but also their derivatives. The conditions for

local stationary behavior in the case of nonholonomic subsidiary conditions become identical with those for holonomic constraints. That is, we introduce multiplier functions $\lambda_k(x)$ **and construct the augmented functional**

$$I_\lambda[\vec{y}] = \int_{x_0}^{x_1} \left[F(x, \vec{y}, \vec{y}') + \sum_{k=1}^{m} \lambda_k(\mathrm{x}) \mathrm{G}(x, \vec{y}, \vec{y}') \right] dx$$

Since the constraint now involve derivatives, the Euler-Lagrange eqns are

$$\frac{\partial F}{\partial y_i} + \sum_{k=1}^{m} \lambda_k(x) \frac{\partial G_k}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'} + \sum_{k=1}^{m} \lambda_k(x) \frac{\partial G_k}{\partial y_i}\right) = 0 : i = 1, \ldots n$$

These eqns are solved simultaneously, together with the m **nonholonomic eqns of constraints, for the extremals**

$y_1(x), \ldots, y_n(x).$

HAMILTON'S PRINCIPLE

We now make a connection between the Calculus of Variations and Lagrangian formulation

For each particle, $F_i + R_i = m_i \ddot{x}_i$

~> Total virtual work **(3.28)**

$$\delta W : (F_i + R_i)\delta x_i = m_i \ddot{x}_i \delta x_i ..(i = 1,,,3N)$$

Remember !

$$\frac{d}{dt}(m_i \dot{x}_i \delta x_i) = m_i \ddot{x}_i \delta x_i + m_i \dot{x}_i \delta \dot{x}_i = \delta W + \delta[\frac{1}{2} m_i x_i^{\ 2}]..(i = 1,,3N)$$

Eqn(3.28): $\delta T + \delta W = (m_i \dot{x}_i \delta x_i),_t$

Integrating over the time domain and applying BC in time

$$\delta x_i(t_0) = \delta x_i(t_1) = 0$$

For the *actual* motion:

$$\int_{x_0}^{x_1} (\delta T + \delta W) dt = 0 \qquad (3.29)$$

: **Hamilton Principle ~ interpretation !**

For a conservative system :

$$\delta \int_{x_0}^{x_1} (T - V) dt = 0$$

For $L = T - V$ **:**

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‘Hamilton’s Principle’

Advantage of variational point of view:

Hamilton’s principle may be extended to continuous systems with infinite number of DOF !

Wave eqn : $\rho u,_{tt} = \mu u,_{xx} + f(x,t)$

Euler beam vibration : $\rho u,_{tt} = EIu,_{xxxx} + f(x,t)$

Merits ? Disadvantages ?

Concept of Energy ; T : $\frac{1}{2}\iiint \rho \dot{u}_i \bullet \dot{u}_i \, dV$,

V: $\frac{1}{2}\iiint \sigma_{ij}\varepsilon_{ij} dV$

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$$

Geometrically and Materially Nonlinear problem

-> Geometrically Nonlinear problem :

von Karman theory of Plate

2-11

Sol)

$$I_z = \frac{1}{12}M[(2a)^2 + (2a)^2] = \frac{2}{3}Ma^2$$

Total energy : $$T = \frac{1}{2}M(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2}\cdot\frac{2}{3}Ma^2\dot{\theta}^2$$

Generalized moments :

$$\Delta(M\dot{x}_c) = M\dot{x}_c = \hat{Q}_x$$

$$\Delta(M\dot{y}_c) = M\dot{y}_c = \hat{Q}_y$$

$$\Delta(I\dot{\theta}) = \frac{2}{3}Ma^2\dot{\theta} = \hat{Q}_\theta$$

Virtual work of applied impulsive force *J*

$$\delta W = \hat{J}(\delta y_c + \sqrt{2}a\delta\theta) \equiv \hat{Q}_x\delta x_c + \hat{Q}_y\delta y_c + \hat{Q}_\theta\delta\theta$$

$$\hat{Q}_x = 0, \hat{Q}_y = \hat{J}, \hat{Q}_\theta = \sqrt{2}a\hat{J}$$

Angular velocity : $\dot{\theta} = \dfrac{3\hat{Q}_\theta}{2Ma^2} = \dfrac{3\sqrt{2}aJ}{2Ma^2} = \dfrac{3J}{\sqrt{2}Ma}$

4-10

<u>A cable of fixed length</u> ℓ suspended between points (-*a*,*b*)

and (*a*,*b*). The cable is of uniform mass/length μ .

Determine $y(x)$

For the equilibrium state which has the minimum potential energy.

sol) $d\ell = \sqrt{dx^2 + dy^2} = \sqrt{1^2 + (\frac{dy}{dx})^2}\, dx = \sqrt{1^2 + (y')^2}\, dx$

$$F = \int_{-a}^{a} \mu g y \sqrt{1^2 + (y')^2}\, dx : Stationary..value$$

$$with..constra\,\mathrm{int}...G = \int_{-a}^{a} \sqrt{1^2 + (y')^2}\, dx = \ell$$

Stationary function of $F + \lambda G = J : (\lambda : Lagrange..multiplier)$

x **is absent ->** $J - y'\dfrac{\partial J}{\partial y'} = C$

~~ **Let C = C_1**

Integrate one more time,

$$\mu g y + \lambda = C_1 \cosh\left(\int_{-a}^{a} \frac{\mu g x}{C_1} + C_2\right)$$

$$where..C_1, C_2, \lambda :$$

Coefficients are determined using <u>Constraint and BCS</u>

y(-a)=y(a)=b

$$y = \frac{C_1}{\mu g}\cosh\left(\frac{\mu g x}{C_1}\right) - \frac{\lambda}{\mu g}$$

~~~
~~~

Practice

- Find the largest volume of rectangular solid containing the

ellipsoid given by

$$g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

The volume of a rectangular solid with sides (2*x*,2*y*,2*z*) is

$$f(x, y, z) = 2^3 xyz = 8xyz$$

Then

$$\overline{f}(x, y, z) = f(x, y, z) + \lambda g(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\frac{\partial \overline{f}(x,y,z)}{\partial x} = 0 : 8yz + \lambda \frac{2x}{a^2} = 0..or..8xyz + \lambda \frac{2x^2}{a^2} = 0....(1)$$

$$\frac{\partial \overline{f}(x,y,z)}{\partial y} = 0 : 8xz + \lambda \frac{2y}{b^2} = 0..or..8xyz + \lambda \frac{2y^2}{b^2} = 0....(2)$$

$$\frac{\partial \overline{f}(x,y,z)}{\partial z} = 0 : 8xy + \lambda \frac{2z}{c^2} = 0..or..8xyz + \lambda \frac{2z^2}{c^2} = 0....(3)$$

From (1)~(3)

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = -\frac{4xyz}{\lambda} \text{.....(4)}$$

Substitute it in

$$g(x,y,z) = 0$$

Then,

$$-\frac{4xyz}{\lambda} - \frac{4xyz}{\lambda} - \frac{4xyz}{\lambda} = 1$$

$$-> \lambda = -12xyz \qquad (5)$$

Insert (5) into (4)

$$\frac{x^2}{a^2} = \frac{4xyz}{12xyz} = \frac{1}{3}.....or...x = \pm\frac{a}{\sqrt{3}}$$

and $$y = \pm\frac{b}{\sqrt{3}},....z = \pm\frac{c}{\sqrt{3}}$$

*** Practice**

Consider the non-holonomic system in Text page ?

Find the equations of motion for the system and solve for x(t),y(t) with

initial conditions

$$x(0) = y(0) = 0$$
$$\dot{x}(0) = v_o, \dot{y}(0) = 0$$
$$\varphi(0) = 0, \dot{\varphi}(0) = \omega$$

Sol:

$$ds^2 = dx^2 + dy^2 :$$

$$dx = \cos\varphi ds : dy = \sin\varphi ds$$

$$-> \dot{x} = \cos\varphi\dot{s}, \dot{y} = \sin\varphi\dot{s}$$

Eliminate

$$\dot{s}$$

Then we get, nonholonomic constraint :

$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$$

$$or..dx\sin\varphi - dy\cos\varphi = 0$$

hence

$$a_{1x} = \sin\varphi, a_{1y} = \cos\varphi, a_{1\varphi} = 0$$

Also,

$$T = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_c \dot{\varphi}^2 ..with..I_c = \frac{1}{2} ml^2$$

$$V = 0$$

Lagrange Equations of Motion :

$$x : 2m\ddot{x} = \lambda \sin\varphi ....y : ??$$

$$\varphi : \frac{1}{2}ml^2\ddot{\varphi} = 0 - > \ddot{\varphi} = 0 - > \dot{\varphi} = \omega - > \varphi = \omega t$$

Then

$$\dot{x} = \frac{\lambda}{2m\omega}\cos\omega t + C_1 : \dot{y} = ??$$

Using Initial Conditions

$$C_1 = v_0 + \frac{\lambda}{2m\omega} \ldots\ldots\ldots C_2 = 0$$

$$=> C_1 = v_0 \; \& \; 2m\omega : const. => \lambda = const.$$

Also, T+V=Const = T_0 since V = 0