Ch. 13 Kinetics of Particles: Energy and Momentum Methods

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13.0 Introduction

Motion of Particles : through $\vec{F} = m\vec{a}$

given $\vec{F}$, solve for $\vec{a}$, then use kinematics,

➢ velocity and position

Alternative : $\vec{F} = m\vec{a}$ + kinematics

➢ two methods
- work and energy : relates force, mass, $\vec{v}$, and $\vec{x}$
- impulse and momentum : relate force, velocity, time

- advantage : no need to determine $\vec{a}$

i) Method of work and energy

- Work of a force
- Kinetic energy of a particle
- Concept of power efficiency

13.0 Introduction

ii) Principle of impulse and momentum

- particularly effective in impulsive motion

  (very large forces are applied for a very short time)

- central impact of two bodies
  - certain relation exists between the relative velocity of the two colliding bodies.
- three fundamental methods
  - best suited for the given problem
- combination of { principle of conservation of energy / method of impulse and momentum }
  - only conservative forces or short impact phase during which impulsive forces must be taken into account

13.1A Work of a Force

$d\vec{r}$: displacement (infinitesimal)

Work of a force $dU = \vec{F} \cdot d\vec{r}$

$dU = F\,ds\cos\alpha$ (α : angle formed by $\vec{F}$ and $d\vec{r}$, ds : magnitude of $d\vec{r}$)

$dU = F_x\,dx + F_y\,dy + F_z\,dz$: rectangular component

- scalar quantity : no direction
  unit : $N \cdot m = J$ (but, moment of a force : $N \cdot m$, not J)

Sign : positive if α is acute
negative if α is obtuse

i) $\vec{F}$ same direction as $d\vec{r}$: $dU = F\,ds$
ii) $\vec{F}$ opposite direction as $d\vec{r}$: $dU = -F\,ds$
iii) $\vec{F}$ perpendicular to $d\vec{r}$: $dU = 0$

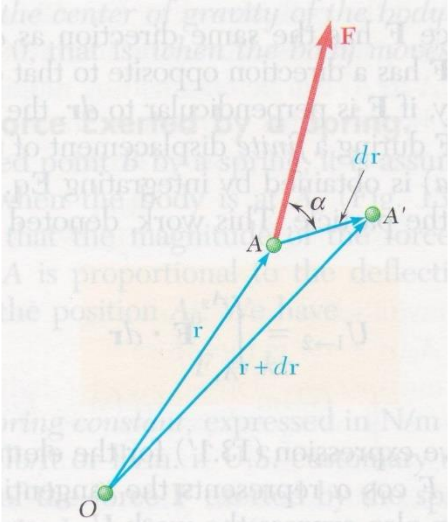

Fig. 13.1

13.1A Work of a Force

Work during a finite displacement ($A_1 \rightarrow A_2$)

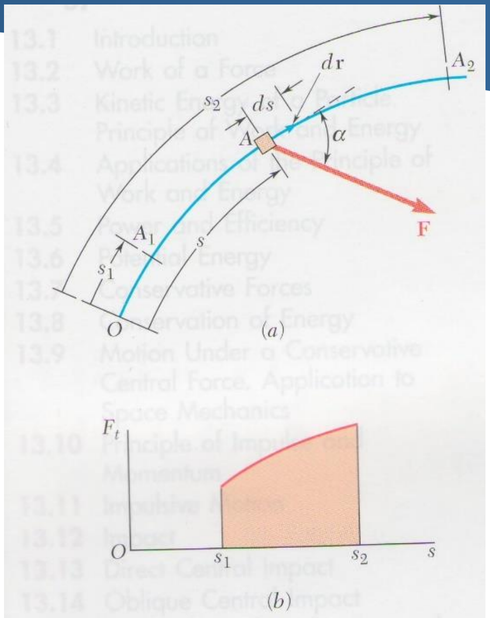

Fig. 13.2

$$U_{1\rightarrow 2} = \int_{A_1}^{A_2} \vec{F} \cdot d\vec{r}$$

$F \cos\alpha = F_t$ (tangential component)

$$U_{1\rightarrow 2} = \int_{s_1}^{s_2} (F \cos\alpha)\, ds = \int_{s_1}^{s_2} F_t \, ds$$

(s : the distance traveled by the particle along the path)

→ area under the curve obtained by plotting $F_t = F\cos\alpha$ vs. s

Rectangular components

$$U_{1\rightarrow 2} = \int_{A_1}^{A_2} (F_x\, dx + F_y\, dy + F_z\, dz)$$

13.1A Work of a Force

i) Constant force in Rectilinear Motion

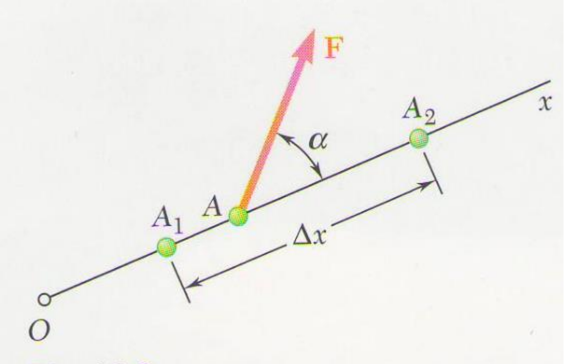

Fig. 13.3

$$U_{1\to 2} = (F\cos\alpha)\Delta x$$

α : angle that the force forms with direction of motion

Δx : displacement from A_1 to A_2

ii) Force of Gravity

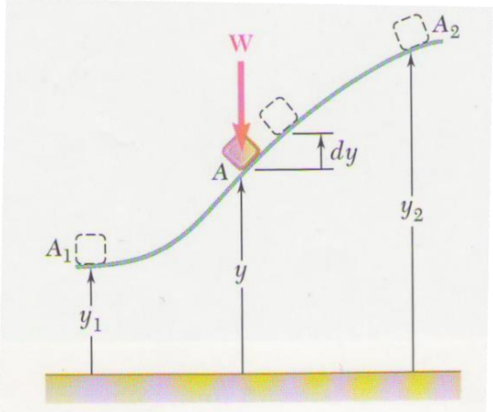

Fig. 13.4

$$F_x = 0,\ F_y = -W,\ F_z = 0$$

$$dU = -W\,dy$$

$$U_{1\to 2} = -\int_{y_1}^{y_2} W\,dy = W\,y_1 - W\,y_2$$

$$= -W(y_2 - y_1) = -W\Delta y$$

Δy : vertical displacement from A_1 to A_2

- Product of W and the vertical displacement of center of gravity

  → (+) when $\Delta y < 0$, when the body moves down

13.1A Work of a Force

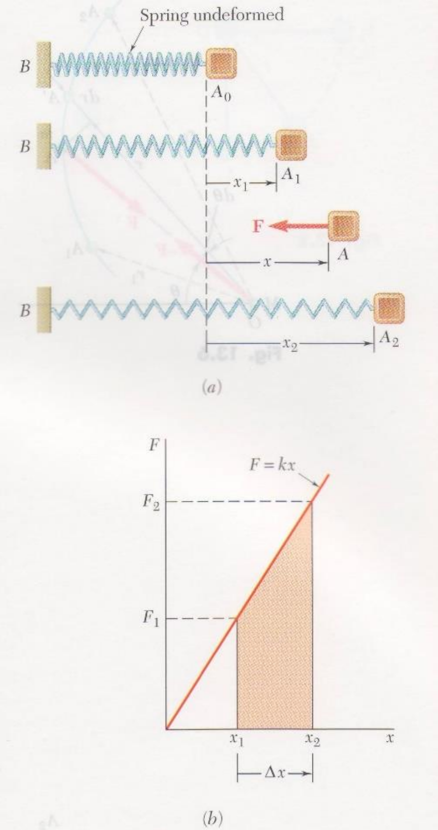

Fig. 13.5

iii) Spring

- $\vec{F}$ is proportional to the deflection x measured from to

$$F = kx$$

k : spring constant $(N/m, kN/m)$

under static condition only.

under dynamic condition, spring inertia should be accounted.

Still valid when spring mass is relatively small.

$$dU = -F\,dx = -kx\,dx$$

$$U_{1\to2} = -\int_{x_1}^{x_2} kx\,dx = \frac{1}{2}kx_1^{\,2} - \frac{1}{2}kx_2^{\,2}$$

: (+) when $x_2 < x_1$, when the spring is returning to its undeformed position.

- area of the trapezoid of slope k passing through the origin from A_1 to A_2 .

$$U_{1\to2} = -\frac{1}{2}(F_1 + F_2)\Delta x$$: more convenient

13.1A Work of a Force

iv) Gravitational force

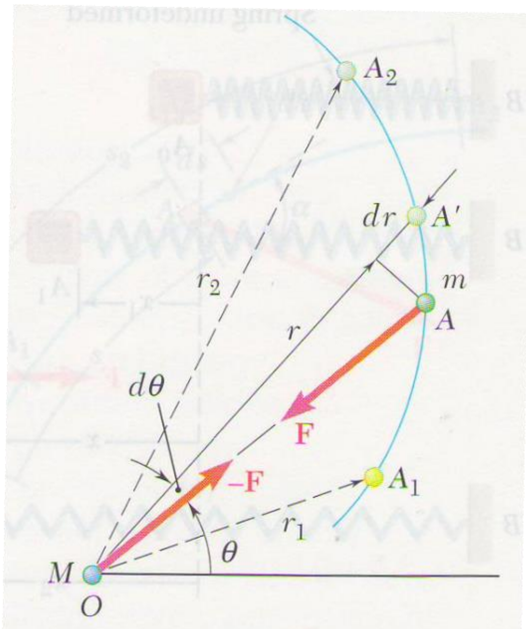

Fig. 13.6

$\vec{F}$ and $-\vec{F}$, $F = \dfrac{GMm}{r^2}$

Fig. 13.6 M fixed at O, m moves in the path (infinitesimal dr)

$$dU = -F\,dr = -G\frac{Mm}{r^2}dr$$

$$U_{1\to 2} = -\int_{r_1}^{r_2}\frac{GMm}{r^2}dr = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

- work of the force exerted by the earth on a body of mass m at a distance r from the center of the earth, where $r > R$ (R : radius of the earth) can replace GMm by WR^2.

13.1A Work of a Force

v) No work

- applied to fixed points ($ds = 0$)
- acting perpendicular to the motion ($\cos\alpha = 0$)

e.g.

- reaction of a frictionless pin when the body rotates
- reaction of a friction surface when the body moves
- reaction at a roller moving along a track
- weight of a body when its center of gravity moves horizontally

13.1B Principle of Work and Energy

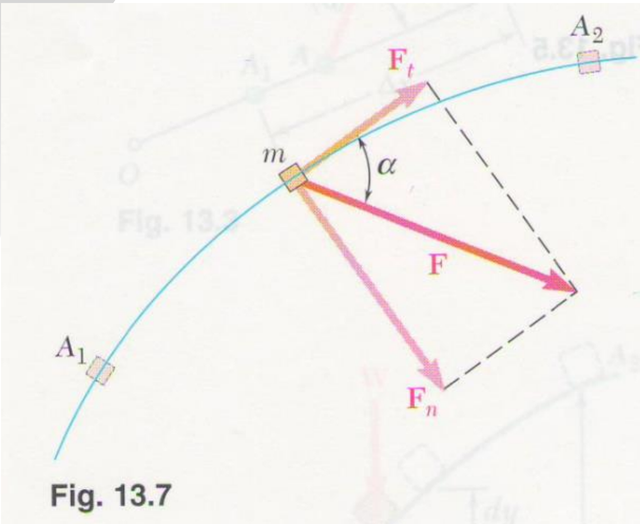

Fig. 13.7

Fig. 13.7 mass m, force $\vec{F}$ rectilinear or curved path

Newton's 2nd law in terms of the tangential component

$$F_t = ma_t \text{, or } F_t = m\frac{dv}{dt}$$

(v : speed of the particle)

Recalling $v = \dfrac{ds}{dt}$

$$F_t = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$F_t\, ds = mv\, dv$$

Integration from $A_1(s=s_1, v=v_1)$ to $A_2(s=s_2, v=v_2)$

$$\int_{s_1}^{s_2} F_t\, ds = m\int_{v_1}^{v_2} v\, dv = \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_1^{\,2} \qquad (13.8)$$

The work of the force $\vec{F}$ is equal to the change in kinetic energy

- Principle of work and energy

$$T_1 + U_{1\to 2} = T_2$$

13.1B Principle of Work and Energy

: Applies only with respect to a Newtonian frame of reference.

T should be measured with respect to a Newtonian frame of reference

- Several forces acting

  : $U_{1\to 2}$ obtained by adding algebraically the work of the various forces.

- Kinetic energy

  : $T = \frac{1}{2}mv^2$, always positive regardless of the motion direction

when $v_1 = 0, v_2 = v_1$ work done by the forces = T

: kinetic energy = work which must be done to bring the particle from rest to the speed v.

substituting $T_1 = T$ and $T_2 = 0$ (a particle with a speed v is brought to rest), the work done by the forces = $-T$

: assuming no energy dissipated into heat, the work done by the forces exerted by the particle on the bodies which cause it to come to rest is equal to T.

(kinetic energy = the capacity to do work associated with the speed of the particle)

- Unit : $kg\,(m/s)^2 = (kg \cdot m/s^2)\,m = N \cdot m = J$

13.1C Applications of the principle of work and energy

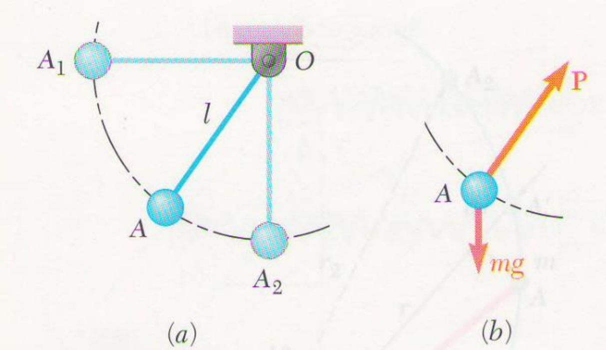

Fig. 13.8

Pendulum example

: bob A of weight W, cord length l (Fig. 13.8a)

- released with no velocity from a horizontal position OA
- wish to determine the speed of the bob at A_x

I. Determine the work from A_1 to A_2

- free body diagram → weight $\vec{W}_1$, cord force $\vec{P}$ (Fig. 13.8 b)

(inertia vector is not an actual force, should not be included)

- $\vec{P}$ does no work since it is normal to the path

$$\vec{W}: U_{1\to 2} = Wl$$

$$T_1 = 0,\ T_2 = \frac{1}{2}(W/g)v_2^{\ 2}$$

$$T_1 + U_{1\to 2} = T_2,\ 0 + Wl = \frac{1}{2}\frac{W}{g}v_2^{\ 2}$$

$v_2 = \sqrt{2gl}$ ⟶ speed of a body freely falling from a height l

13.1C Applications of the principle of work and energy

- Advantages
  - i) No need to determine the acceleration at an intermediate position A.
    no need to integrate the expressions from A_1 to A_2
  - ii) All quantities are scalars, can be added directly without using x, y component.
  - iii) Forces which do no work → eliminated from the solution.

- Disadvantages
  - i) Cannot directly determine the acceleration.
  - ii) To determine the force which is normal to the path, supplemented by Newton's 2nd law

[Example] To determine the tension in the cord, force-body diagram
Newton's 2nd low in terms of tangential and normal component.

$$\sum F_t = ma_t \,, \quad \sum F_n = ma_n$$

$$a_t = 0, \qquad P - W = ma_n = \frac{W}{g}\frac{v_2^{\,2}}{l}$$

since $v_2^{\,2} = 2gl,$

$$P = W + \frac{W}{g}\frac{2gl}{l} = 3W$$

13.1C Applications of the principle of work and energy

Several particles

: principle of work and energy applies to each particle

- Adding the kinetic energies, considering the work of all the forces

→ single equation for all the particles involved

$$T_1 + U_{1\to 2} = T_2$$

T : arithmetic sum of the kinetic energies of the particle involved (all (+))

$U_{1\to 2}$: work of all the forces, including the forces of action and reaction exerted by the particles on each other

Bodies connected by inextensible cords or links → work of the forces exerted by a given cord or link on the two bodies it connects cancels out.

(since the points of application moves the equal distance)

Friction forces

- direction opposite to the displacement

→ work is always (-)

- energy dissipated into heat and always results in a decrease in the kinetic energy

13.1D Power and Efficiency

- Power : Time rate at which work is done
  - selection of a motor or engine much more important criterion

$$\text{Average Power} = \frac{\Delta U}{\Delta t}$$

$$\text{Power} = \frac{dU}{dt}$$

$$= \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \qquad (13.12)$$

- Unit : $\text{watt} / W = 1\ J/s = 1\ N \times m/s \qquad 1\ hp = 746\ W$

- Mechanical efficiency $\eta = \dfrac{output\ \ work}{input\ \ work}$ (13.15)

assumption : work is done at a constant rate

$$\eta = \frac{Power\ output}{Power\ input}$$

Because of energy loss due to friction $\eta < 1$

Machine transforming mechanical → electric energy

thermal → mechanical

→ overall efficiency < 1

13.2A Potential Energy

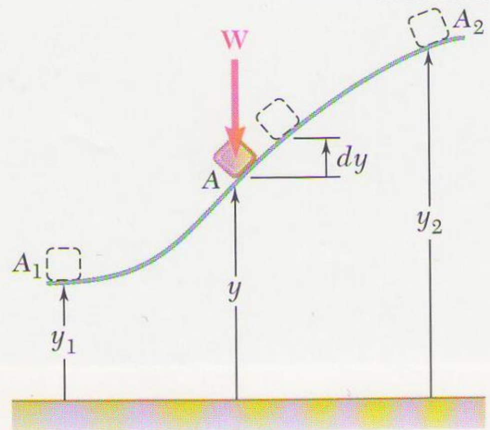

Fig. 13.4

A body of weight $\vec{W}$ which moves from A_1 to A_2

$$U_{1\to 2} = Wy_1 - Wy_2 \qquad (13.4)$$

- can be obtained by subtracting the value of the function Wy corresponding to y_2 from that of y_1

$\vec{W}$ is independent of the actual path followed

depends only on the initial and final values of the function

Wy : called potential energy w.r.t. the force of gravity $\vec{W}$

$$U_{1\to 2} = (V_g)_1 - (V_g)_2 \quad \text{with} \quad V_g = Wy$$

$(V_g)_2 > (V_g)_1$ potential energy increases (the case considered here)

$U_{1\to 2}$ (-)

work of $\vec{W}$ (+), potential energy decreases

- V_g is the measure of the work which can be done by $\vec{W}$

13.2A Potential Energy

- Only change in P.E. is involved in (13.6) an arbitrary constant can be added to the expression of
  - the level or datum for which y is measured can be chosen arbitrarily
- Unit : J

  only valid when $\vec{W}$ can be assumed to remain constant

  (→ the displacements are small compared with the radius of earth)
- Space vehicle : work of gravitational force

$$U_{1->2} = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

V_g expression when the variation in the force of gravity cannot be neglected

$$V_g = -\frac{GMm}{r} \quad \text{or} \quad = -\frac{WR^2}{r} \qquad (13.17)$$

valid when $r \geq R$ (body is above the earth surface)

always (-), approaches to zero for very large r.

13.2A Potential Energy

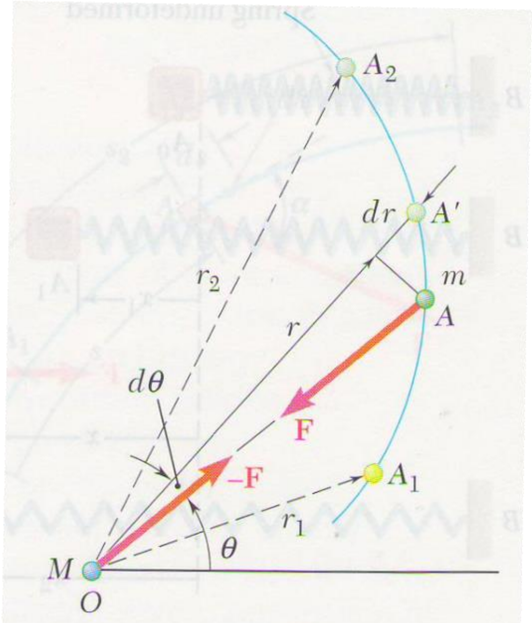

Fig. 13.6

13.2A Potential Energy

- Spring $U_{1->2} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$

  potential energy w.r.t. the elastic force $\vec{F}$

  $U_{1->2} = (V_e)_1 - (V_e)_2$ with $V_e = \frac{1}{2}kx^2$ (13.18)

  work of the force $\vec{F}$ exerted by the spring on the body (-), P.E. V_e increases.

  (13.18) can be used when the spring is rotated (Fig. 13. 10a)

  -> depends only on the initial and final deflections (Fig. 13. 10b)

- P.E. concept : Can be used to other than gravity and elastic forces

  As long as the work ,"conservative forces", is independent of the path followed

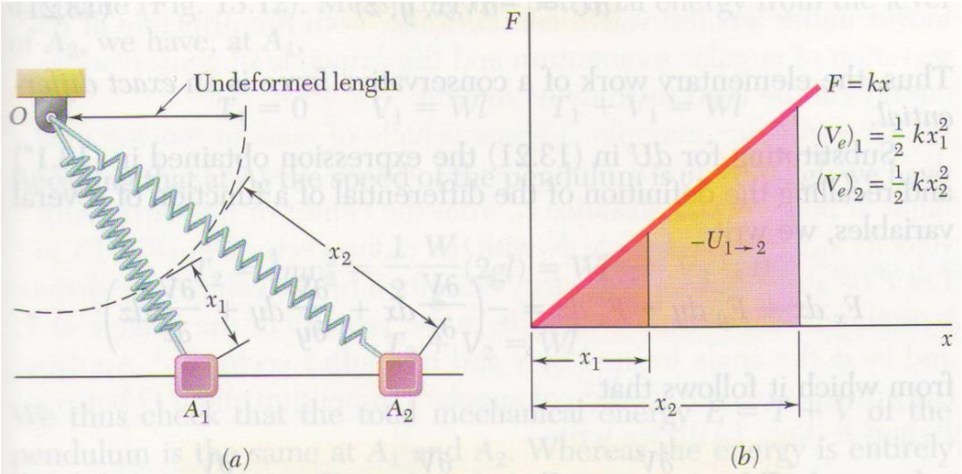

Fig. 13.10

13.2C Conservation of Energy

Work of conservative force : can be expressed as a change in P.E.

Modified form of principle of work and energy

$$V_1 - V_2 = T_2 - T_1$$

$$T_1 + V_2 = T_2 + V_2 \qquad (13.24)$$

- Sum of kinetic E and P.E. remains constant under the action of conservative forces

$T + V$: total mechanical energy, E

Pendulum example : Fig. 13.12

$T_1 = 0$, $V_1 = Wl$, $T_1 + V_1 = Wl$

$$T_2 = \frac{1}{2} m v_2^2 = \frac{1}{2}\frac{W}{g}(2gl) = Wl, \quad V_2 = 0, \quad T_2 + V_2 = Wl$$

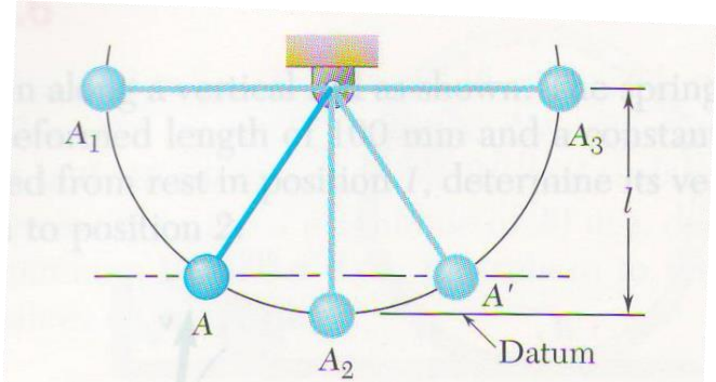

Fig. 13.12

$E = T + V$ is the same at A_1 and A_2

entirely potential at A_1, entirely kinetic at A_2

At A_3, $T_3 = 0$ and $V_3 = Wl$

- kinetic energy will have the same value at any two points located to the same level

13.2C Conservation of Energy

➢ Particle moving along any given path (Fig. 13. 13)
→ frictionless track the same speed at A, A′ and A′′

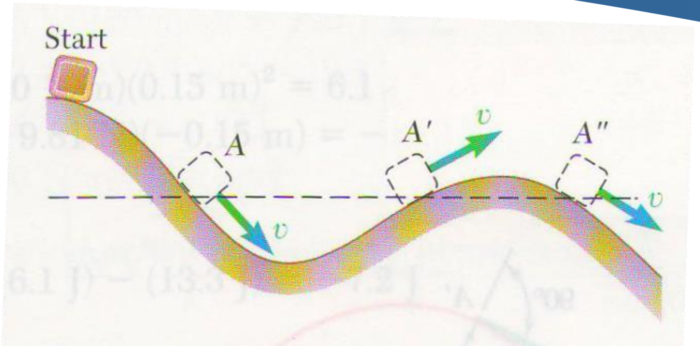

Fig. 13.13

➢ Friction force → non conservative force

➡ cannot be expressed as a change in P.E.

depends on the path followed by its point of application

work of friction force is always (-)

→ when friction is involved, total mechanical energy does not remain constant decreased.

But, energy of the system is NOT lost

transformed into heat

➡ sum of mechanical energy and thermal energy still remains constant

13.2C Conservation of Energy

- Other forms of energy converter

  generator - mechanical energy → electric energy

  gasoline engine - chemical energy → mechanical energy

  nuclear reactor – mass energy → thermal energy

  ➡ Energy can be considered constant ("principle of conservation of energy")

- When non-conservative force is involved

$$T_1 + V_{g1} + V_{e1} + U_{1\to 2}^{NC} = T_2 + V_{g2} + V_{e2} \qquad (13.24')$$

13.2D Conservative Central Forces

Under conservative central force,

both principle of conservation of angular momentum

principle of conservation of energy

Fig. 13. 14: space vehicle of mass m

conservation of angular momentum → $r_o m v_o \sin\phi_o = r m v \sin\phi$ (13.25)

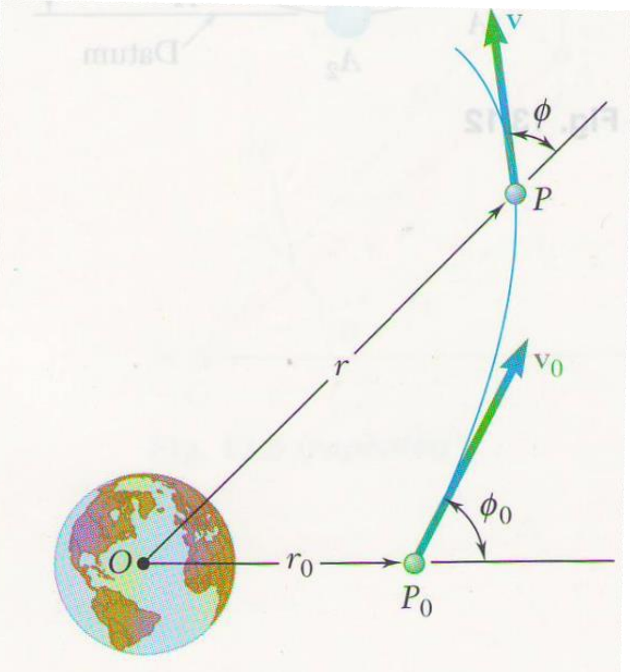

Fig. 13.14

$$T_o + V_o = T + V$$

$$\frac{1}{2} m v_o^2 - \frac{GMm}{r_o} = \frac{1}{2} v^2 - \frac{GMm}{r} \qquad (13.26)$$

(13.26) → solved for v when r is known

(13.25) → determine ϕ

Eqns (13.25) and (13.26) → to determine the maximum and minimum value of r for the satellite (Fig. 13. 15)

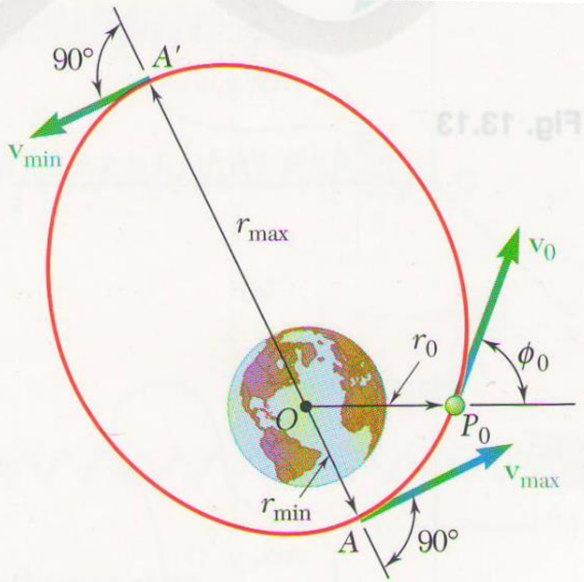

Fig. 13.15

→ by putting ϕ =90° , eliminate v between Eqs. (13.25) and (13.26)

$$r_o m v_o \sin\phi_o = rmv \qquad v = r_o m v_o \sin\phi_o \frac{1}{rm}$$

$$\frac{1}{2}mv_o^2 - \frac{GMm}{r_o} = \frac{1}{2}m\left(r_o m v_o \sin\phi_o \frac{1}{rm}\right)^2 - \frac{GMm}{r}$$

$$\left(\frac{1}{2}mv_o^2 - \frac{GMm}{r_o}\right)r^2 + (GMm)r - \frac{1}{2}m\left(r_o m v_o \sin\phi_o \frac{1}{m}\right)^2 = 0$$

13.3A Principle of Impulse and Momentum

→ 3rd method – to solve problems involving force, mass, velocity, time impulsive motion, involving impact

- Newton's 2nd law $\vec{F} = \frac{d}{dt}(m\vec{v})$ ← Linear momentum

$$\vec{F}dt = d(m\vec{v})$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\overrightarrow{v_2} - m\overrightarrow{v_1}$$

$$m\overrightarrow{v_1} + \int_{t_1}^{t_2} \vec{F}\,dt = m\overrightarrow{v_2} \qquad (13.28)$$

↑ Linear impulse, impulse

$$\overrightarrow{Imp}_{1\to 2} = \int_{t_1}^{t_2} \vec{F}\,dt = \vec{i}\int_{t_1}^{t_2} F_x\,dt + \vec{j}\int_{t_1}^{t_2} F_y\,dt + \vec{k}\int_{t_1}^{t_2} F_z\,dt \qquad (13.29)$$

→ the areas under the F_x, F_y, F_z curves against t (Fig. 13.16)

case of constant magnitude and direction $\vec{F}$, impulse = $\vec{F}(t_2 - t_1)$

same direction as $\vec{F}$

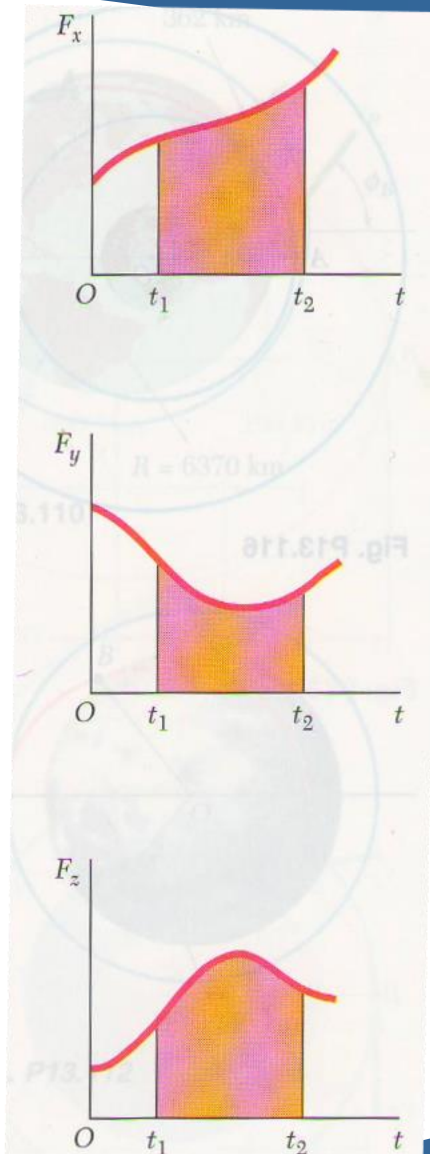

Fig. 13.16

13.3A Principle of Impulse and Momentum

- Unit: $N \cdot s = (kg \cdot m / s^2) \cdot s = kg \cdot m / s$

  ↑ Linear momentum

  Principle of Impulse and Momentum – final momentum $m\overrightarrow{v_2}$ can be obtained by adding vectorially its initial momentum $m\overrightarrow{v_1}$ and the impulse of $\vec{F}$ during its time

$$m\vec{v}_1 + \overrightarrow{Imp}_{1\to2} = m\vec{v}_2 \qquad (13.30)$$

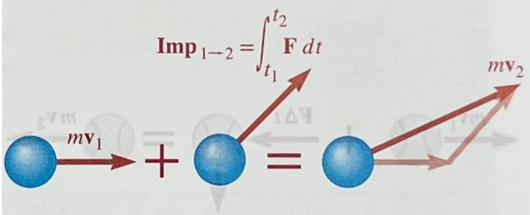

Fig. 13.17 Impulse–momentum diagram. Initial momentum plus impulse of a force **F** equals final momentum.

  Momentum, impulse – vector quantities

  → corresponding components

$$(mv_x)_1 + \int_{t_1}^{t_2} F_x\, dt = (mv_x)_2 \qquad (13.31)$$

- Several forces / Several particles

$$m\vec{v}_1 + \sum \overrightarrow{Imp}_{1\to2} = m\vec{v}_2 \qquad (13.32)$$

  → can apply Eq. (13.32) to each particle or, can add vectorially the momenta of all the particles and the impulses of all the forces involved

$$\sum m\vec{v}_1 + \sum \overrightarrow{Imp}_{1\to2} = \sum m\vec{v}_2 \qquad (13.33)$$

13.3A Principle of Impulse and Momentum

- Action and Reaction

→ impulse of the forces of action/reaction cancel out

only the external forces need be considered

→ impulse is always zero

In case of work, sum of action/reaction's work becomes zero

only when inextensible cords or links (constrained to move through equal distance)

- No external forces or sum of the external forces = 0

$$\sum m\vec{v}_1 = \sum m\vec{v}_2 \qquad (13.34)$$

- Boat example (Fig. 13. 18)

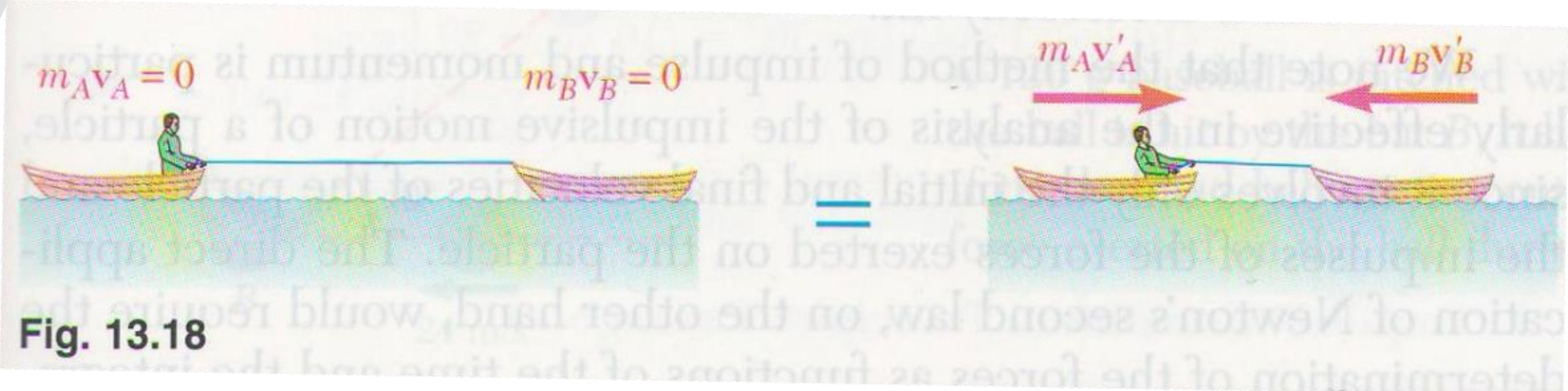

Fig. 13.18

→ initially at rest resistance of the water neglected

the only external forces – weights, buoyant forces (balanced with each other)

$$\sum m\vec{v}_1 = \sum m\vec{v}_2$$

$$0 = m_A \vec{v}'_A + m_B \vec{v}'_B$$

← moves toward each other with velocities inversely proportional to their masses

"Equipollent" – Blue equals signs in Fig. 13.18

the same resultant and moment resultant

"equivalent" (red equals signs) – the same effect

13.3B Impulsive motion

- Impulsive force → acting during a very short time interval, large enough to produce a definite change in momentum

  ➡ Impulsive motion

  (ex) baseball and bat

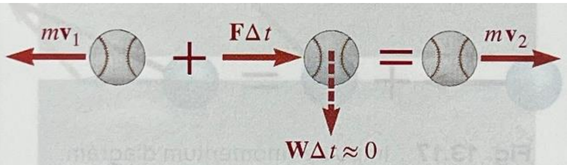

Fig. 13.19 When an impulsive force (i.e., a large force that acts over a short time) acts on a system, we can often neglect non-impulsive forces, such as weight.

Eq. (13.32) ⟶ $$m\vec{v}_1 + \vec{F}\Delta t = m\vec{v}_2 \qquad (13.35)$$

Non-impulsive force – may be neglected since $\vec{F}\Delta t$ is very small

(ex) weight of the body, the force exerted by a spring

Unknown reactions – may or may not be impulsive

→ should be included in Eq. (13.35)

- Baseball example → impulse of baseball weight – neglected

  impulse of bat weight – neglected

  impulse of reaction of player's hand – should be included

  ↳ not be negligible if the ball is incorrectly hit

13.3B Impulsive motion

- Impulsive force ➡ impulsive motion

  → require the determination of the forces as function of the time and integration of eqns. over the time interval Δt

- All the external forces acting on the various particles are non-impulsive

$$\sum m\vec{v}_1 = \sum m\vec{v}_2$$

  → total momentum of the particles is conserved

- Collision example→ total momentum is conserved, but their total energy is generally NOT conserved

13.4A Impact

- Impact → collision between two bodies occurs in a very small interval if time during which two bodies exert relatively large forces on each other
- Line of impact (LOI) → common normal to the surface in contact during impact
- Central impact → mass centers of the two bodies are isolated on LOI
  otherwise – eccentric (Sec. 17.12)
  └→ Only considered in the present scope
- direct impact → velocities of the two particles are directed along LOI (Fig. 13. 20a)
  oblique impact → either or both particles move along a line other than LOI (Fig. 13. 20b)

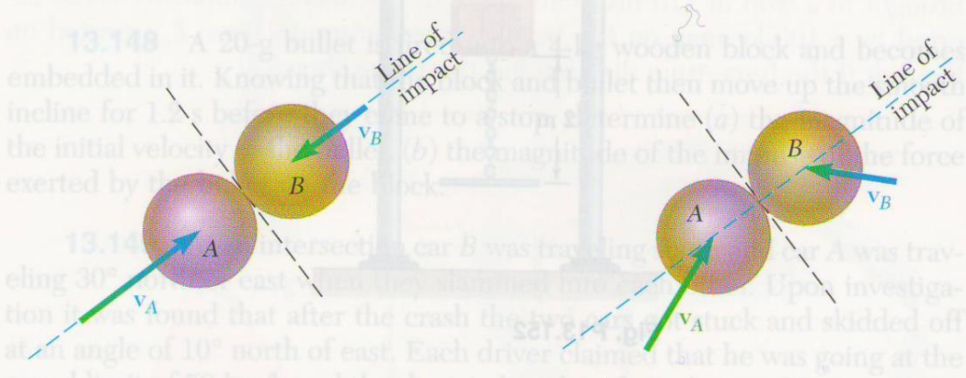

Fig. 13.20 (a) Direct central impact (b) Oblique central impact

13.4A Direct Central Impact

Fig. 13.21 a → $A(m_A)$, $B(m_B)$ moving in the same straight line

$\vec{v}_A$ and $\vec{v}_B$ to the right

b → they strike, same velocity $\vec{u}$

c → period of restitution $\vec{v}'_A$ and $\vec{v}'_B$

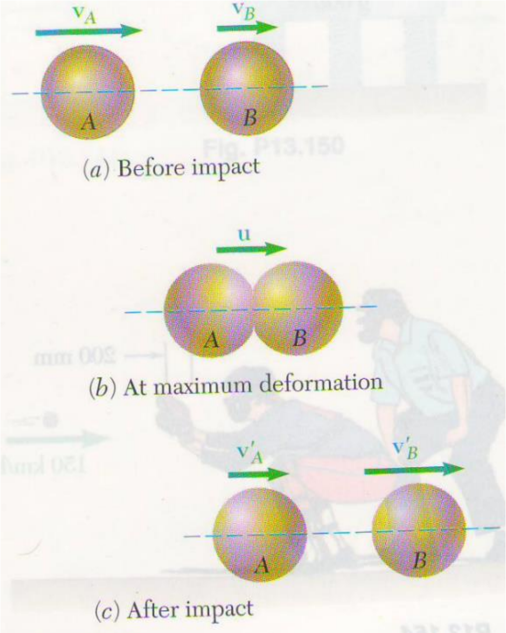

Fig. 13.21

- No impulsive, external force

  conservation of total momentum

$$m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v}'_A + m_B\vec{v}'_B$$

→ scalar components

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \qquad (13.37)$$

- 2nd relation between $\vec{v}'_A$ and $\vec{v}'_B$

  → particle A, principle of impulse and momentum period of deformation only impulsive force ➡ $\vec{P}$ exerted by B

$$m_A v_A - \int P dt = m_A u \qquad (13.38)$$

period of restitution, $\vec{R}$ exerted by B

$$m_A u - \int R dt = m_A v'_A \qquad (13.39)$$

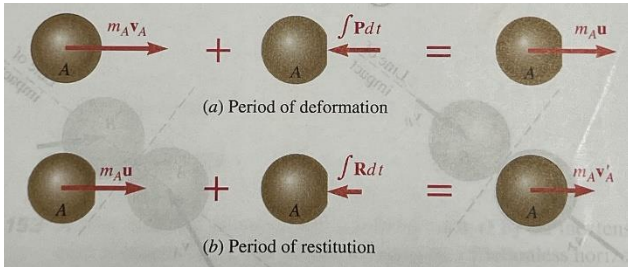

Fig. 13.22 Impulse–momentum diagram for particle A during (a) the period of deformation, and (b) during the period of restoration.

13.4A Direct Central Impact

- In general, $\vec{R} \neq \vec{P}$, $\int R dt < \int P dt$
- Coefficient of restitution $e = \dfrac{\int R dt}{\int P dt}$

$0 < e < 1$, ~ two materials involved

~ impact velocity, shape, size of the two colliding bodies

(13.38), (13.39) → (13.40) $e = \dfrac{u - v'_A}{v_A - u}$ (13.41)

for B, $e = \dfrac{v'_B - u}{u - v_B}$ (13.42)

(13.41), (13.42) ➡ quotients equal. also equal to the quotient obtained by adding, respectively, their numerators and denominators

$$e = \frac{(u - v'_A) + (v'_B - u)}{(v_A - u) + (u - v_B)} = \frac{v'_B - v'_A}{v_B - v_B}$$

$$v'_B - v'_A = e(v_B - v_B) \qquad (13.43)$$

→ relative velocity after impact = relative velocity before impact × e

13.4A Direct Central Impact

- Velocity after impact → can be obtained by Eqns. (13.37) and (13.43)

Two particular cases

i) $e = 0$, perfectly plastic impact

$$\rightarrow \quad v'_B = v'_A \qquad\qquad v'_A = v'_B = v'$$

$$m_A v_A + m_B v_B = (m_A + m_B) v' \qquad (13.44)$$

ii) $e = 1$, perfectly elastic impact

$$\rightarrow \quad v'_B - v'_A = v_A - v_B$$

Impulses by each particle during the period of deformation and restitution are equal.

Particles move away from each other after impact with the same velocity with they approached each other before impact.

13.4A Direct Central Impact

total energy of the two particles is conserved

$$m_A(v_A - v'_A) = m_B(v'_B - v_B) \qquad (13.37')$$

$$v_A + v'_A = v_B + v'_B \qquad (13.45')$$

Multiplying (13.37′) and (13.45′) member by member

$$m_A(v_A - v'_A)(v_A + v'_A) = m_B(v'_B - v_B)(v'_B + v_B)$$

$$m_A {v_A}^2 - m_A(v'_A)^2 = m_B(v'_B)^2 - m_B {v_B}^2$$

$$\frac{1}{2}m_A {v_A}^2 + \frac{1}{2}m_B {v_B}^2 = \frac{1}{2}m_A(v'_A)^2 + \frac{1}{2}m_B(v'_B)^2 \qquad (13.46)$$

- In general $(e < 1)$, the total energy of the particles is not conserved

lost kinetic energy → { transformed in to heat

spent in generating elastic waves within the

two colliding bodies

13.4B Oblique Central Impact

- Oblique → velocities of the two bodies are not directed along the line of impact

  $\vec{v}'_A, \vec{v}'_B$ - unknown direction as well as magnitude

  → require four independent equation

- coordinate → n axis along the line of impact

  t axis along their common tangent

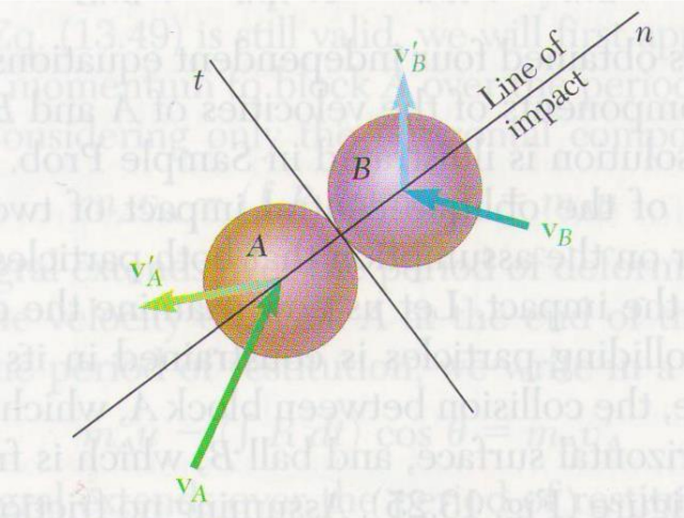

Fig. 13.23

13.4B Oblique Central Impact

- Assuming that the particles are perfectly smooth and frictionless,
  only impulses are due to internal forces directed along the line of impact (n axis)

i) Component of the momentum along t axis is conserved

→ t component of velocity of each particle remains unchanged

$$(v_A)_t = (v'_A)_t \ , \ (v_B)_t = (v'_B)_t \qquad (13.47)$$

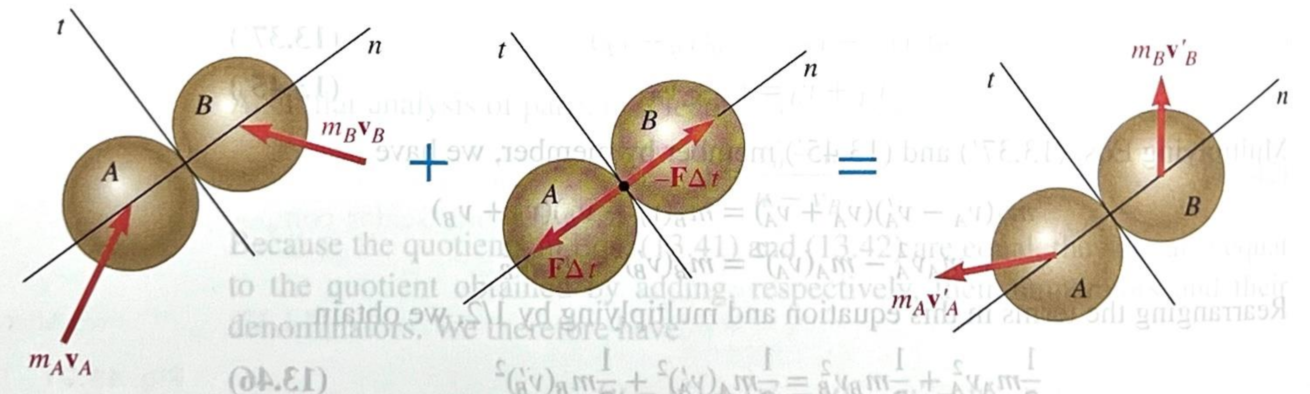

Fig. 13.24 Impulse–momentum diagram for an oblique impact. By including the internal impulses as equal and opposite, you also have the impulse–momentum diagram for each individual object (just ignore the other object).

13.4B Oblique Central Impact

- Assuming that the particles are perfectly smooth and frictionless,
  only impulses are due to internal forces directed along the line of impact (n axis)

  ii) Component of the total momentum along n axis is conserved

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n \qquad (13.48)$$

  iii) Component of the relative velocity along n axis after impact =
  Component of the relative velocity along n axis before impact $\times\ e$

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \qquad (13.49)$$

  ➡ 4 independent equations

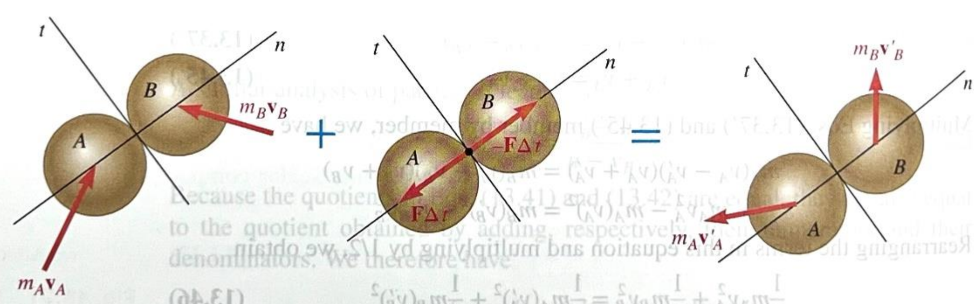

Fig. 13.24 Impulse–momentum diagram for an oblique impact. By including the internal impulses as equal and opposite, you also have the impulse–momentum diagram for each individual object (just ignore the other object).

13.4B Oblique Central Impact

- Case of constraints on one or both particles → example of block A and ball B

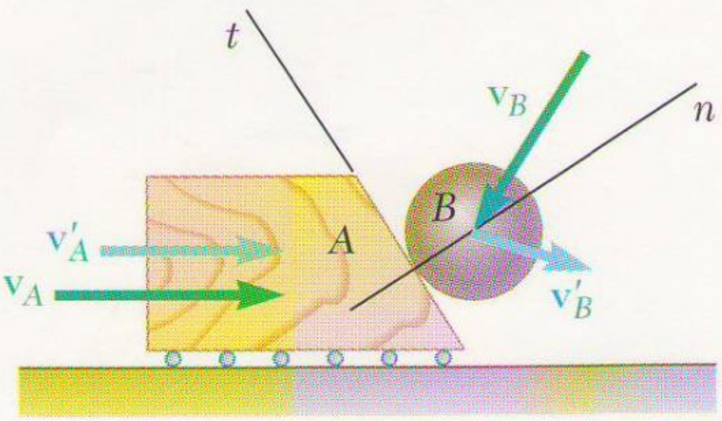

Fig. 13.25

Assuming no friction between block and B, or between block and the horizontal surface

impulses ➡ { Internal force $\vec{F}$ and $-\vec{F}$ along n axis
External force $\vec{F}_{ext}$ exerted by the surface

13.4B Oblique Central Impact

- 3 unknowns → magnitude of $\vec{v}_A'$, magnitude and direction of $\vec{v}_B'$

  i) Component of the momentum of ball B along t axis is conserved

$$(v_B)_t = (v_B')_t \qquad (13.50)$$

  ii) Component of the total momentum of A and B along the horizontal axis is conserved

$$m_A v_A + m_B (v_B)_x = m_A v_A' + m_B (v_B')_x \qquad (13.51)$$

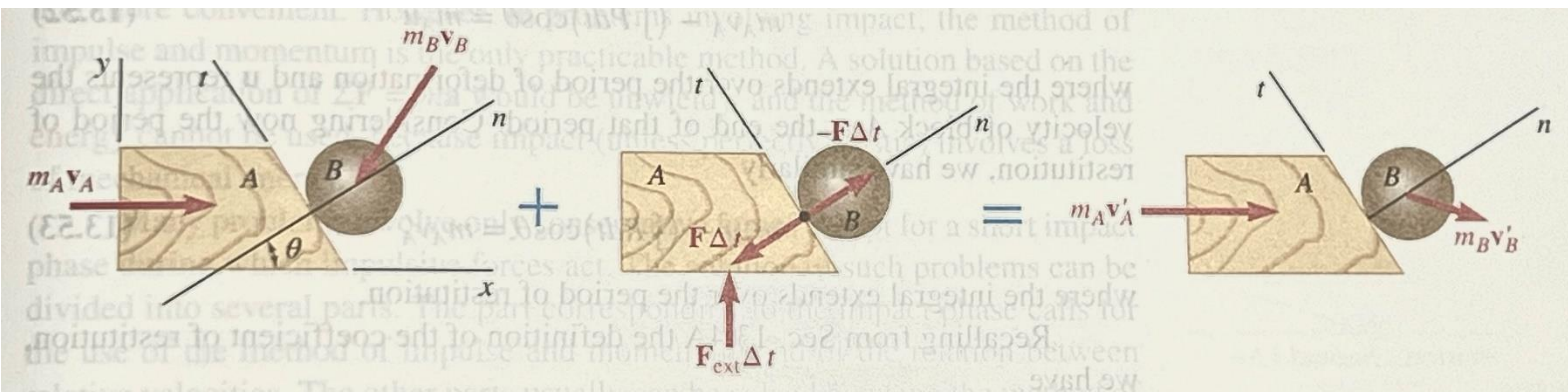

Fig. 13.26 Impulse–momentum diagram for a constrained impact between block *A* and ball *B*.

13.4B Oblique Central Impact

- 3 unknowns → magnitude of $\vec{v}'_A$, magnitude and direction of $\vec{v}'_B$

  iii) Component of the relative velocity of A and B after impact along n axis =
  Component of the relative velocity of A and B before impact along n axis × e

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \quad (13.49)$$

- Validity of Eq. (13.49) → cannot be established through a mere extension of the derivation given in Sec. 13.13
- Difference → block A in the present case is subject to the impulse exerted by the horizontal surface

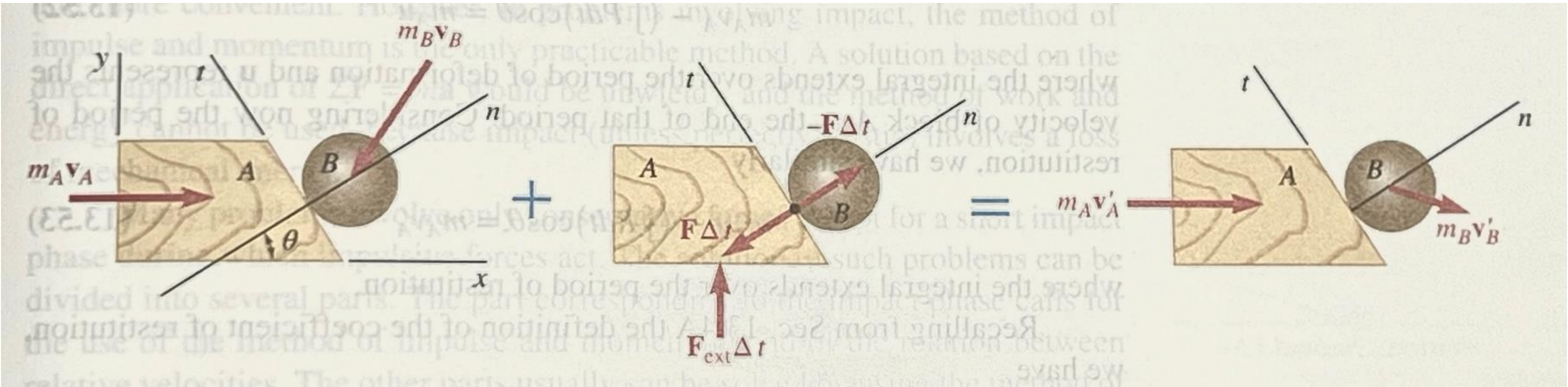

Fig. 13.26 Impulse–momentum diagram for a constrained impact between block *A* and ball *B*.

13.4B Oblique Central Impact

- Proof of validity of Eq. (13.49)

  Period of deformation (Fig. 13.27), only horizontal components

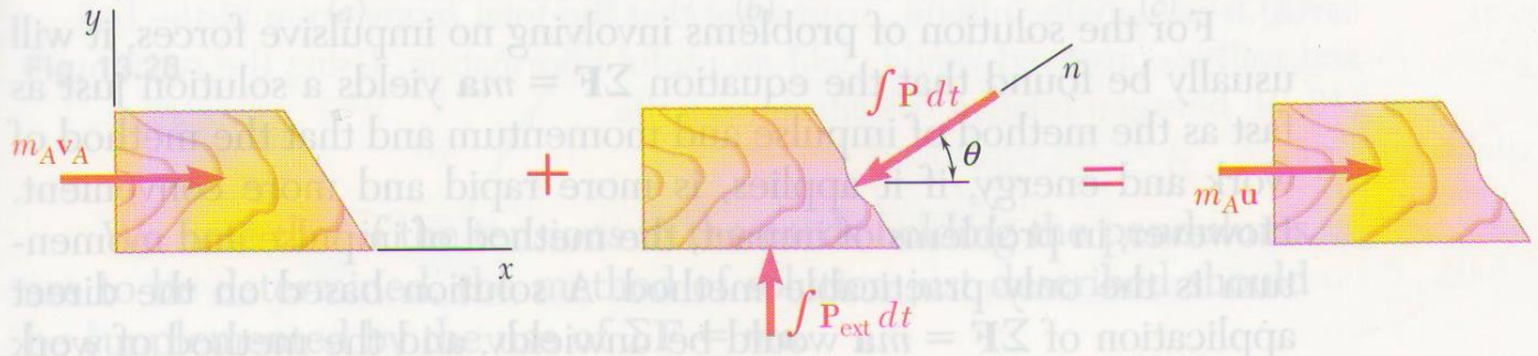

Fig. 13.27

$$m_A v_A - \left(\int P dt\right)\cos\theta = m_A u \qquad (13.52)$$

Period of restitution,

$$m_A u - \left(\int R dt\right)\cos\theta = m_A v'_A \qquad (13.53)$$

$$e = \frac{\int R dt}{\int P dt} = \frac{u - v'_A}{v_A - u}$$

13.4B Oblique Central Impact

multiplying all velocities by $\cos\theta$

$$e = \frac{u_n - (v'_A)_n}{(v_A)_n - u_n} \qquad (13.54)$$

→ Identical to Eq. (13.41)

except for n used to indicate components along the line of impact

B is unconstrained, Eq. (13.49) can be proved as in Eq. (13.43)

Eq. (13.49) remains valid when one of the particles is constrained

Or both particles are constrained

13.4C Problems Involving Energy and Momentum

- 3 different methods
  - Newton's 2nd law $\sum \vec{F} = m\vec{a}$
  - Method of work and energy
  - Method of impulse and momentum

  → choose the method best suited for given problem
  use different methods for various parts of a problem

- In general, method of work and energy → more expeditious than Newton's 2nd law
  However, to determine acceleration or normal force supplemented by Newton's 2nd law

- No impulsive forces, $\sum \vec{F} = m\vec{a}$ as fast as method of impulse and momentum
  Still method of work and energy → more rapid convenient

- Impact problem → method of impulse and momentum ➡ only practicable
  $\sum \vec{F} = m\vec{a}$ → unwieldy
  Principle of work and energy → not applicable since a loss of mechanical energy

13.4C Problems Involving Energy and Momentum

- Conservative forces, with a short impact phase (impulsive force)

  - Impact phase – method of impulse and momentum
  - Others – method of work and energy + $\sum \vec{F} = m\vec{a}$

- Pendulum example (Fig. 13.28)

  i) $A_1 \to A_2$ → principle of conservation of energy, determine $(v_A)_2$

  ii) $A \to B$ → principle of conservation of total momentum, relation between velocities, determine $(v_A)_3$, $(v_B)_3$

  ii) $B_3 \to B_4$ → principle of conservation of energy, determine max. elevation y_4 if tension in the cord, $\sum \vec{F} = m\vec{a}$

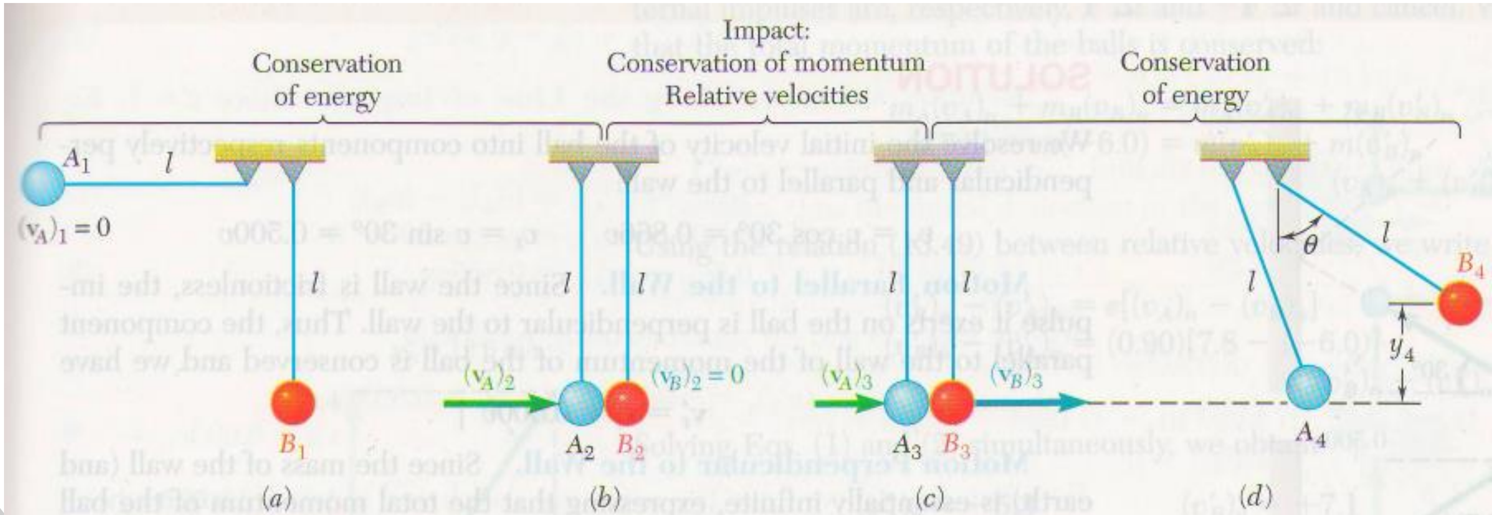

Fig. 13.28

13.4C Problems Involving Energy and Momentum

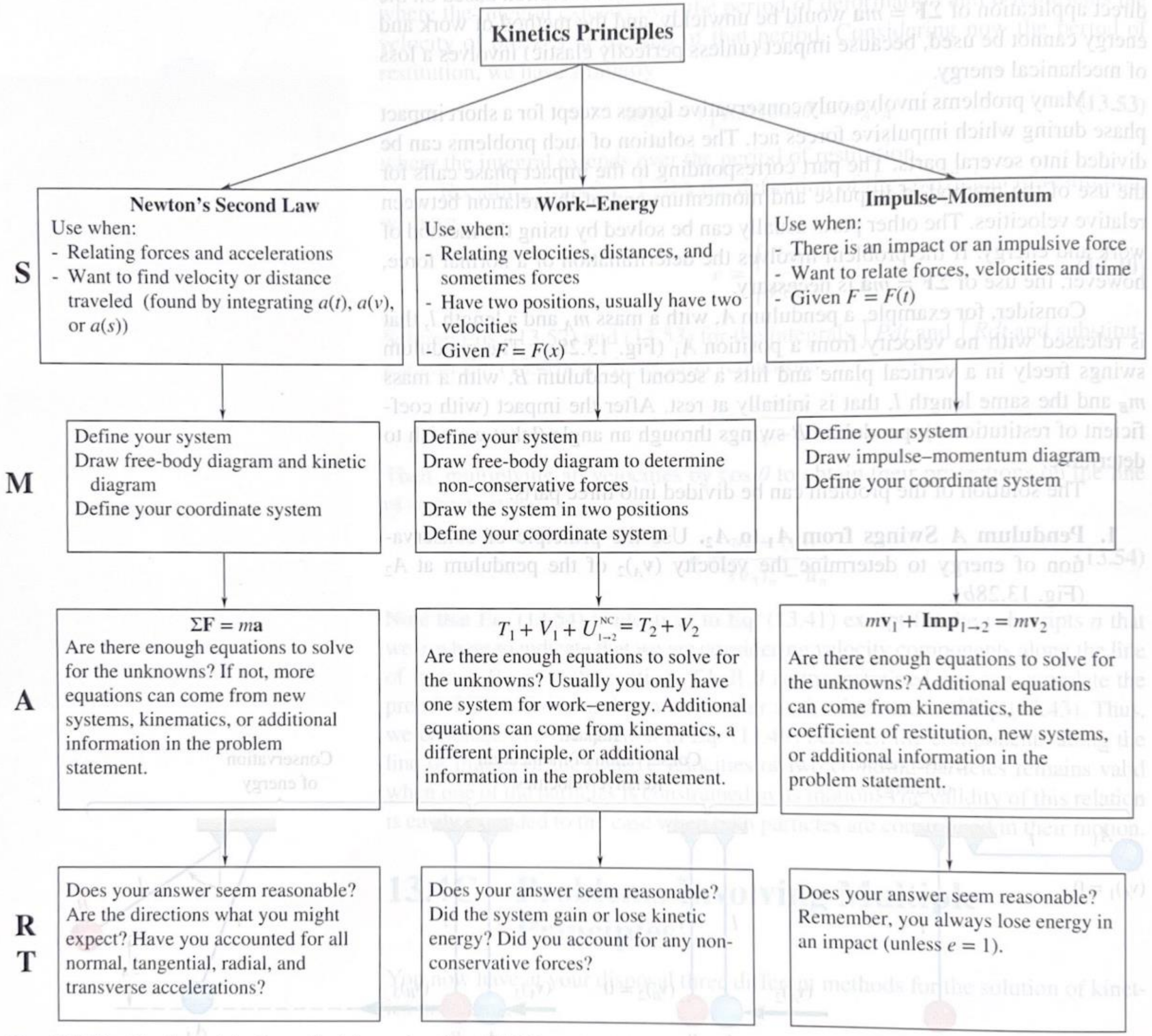

Fig. 13.29 The three kinetics principles using the SMART methodology.