Ch. 14 System of Particles

Prof. SangJoon Shin

14.0 Introduction

System of particles : motion of a large number of particles considered together.

- systems consisting of well-defined particles.
- variable systems – continually gaining or losing particles

Newton's 2nd Law → system of particles

- "Effective Forces" : external forces acting on various particles
  → equipollent to the system of effective forces.
  (both system have the same resultant and moment resultant about any given point)
- Resultant = rate of change of linear momentum
- Moment resultant = rate of change of angular momentum

Mass center : motion of that point

Conditions under which linear/angular momentum are conserved

Application of work-energy principle

- Impulse-momentum

Particles of a system are rigidly connected (→ rigid body)

- kinetics of rigid bodies (Ch. 16 ~ 18)

14.0 Introduction

Variable System of Particles

- Steady stream of particles
  - ex) a stream of water diverted by a vane,
    flow of air through a jet engine
- Systems which gains mass continually or loses
  - ➢ determine the thrust developed by a rocket

14.1A Application of Newton's Laws to the Motion of a System of Particles

System of n particles

Newton's 2nd Law -> each individual particle

- Particle $P_i,\ 1 \le i \le n,\ m_i$ mass, acceleration $\vec{a}_i$ with respect to Newtonian frame

Internal force $\vec{f}_{ij}$, exerted on P_i by another particle P_j (Fig. 14.1)

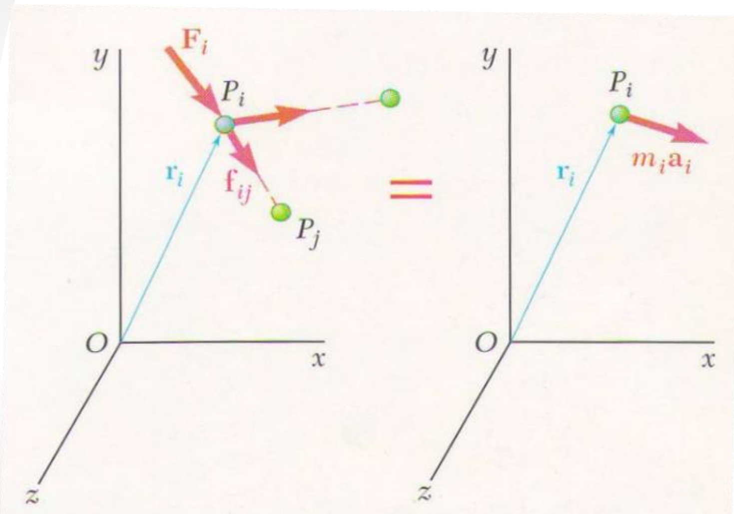

Fig. 14.1

Resultant $= \sum_{j=1}^{n} \vec{f}_{ij}$

(where $\vec{f}_{ii}$ has no meaning, and assumed zero)

Newton's 2nd Law for P_i

$$\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i \tag{14.1}$$

$m_i \vec{a}_i$: effective forces of the particle

Taking the moment about *O*

$$\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n} (\vec{r}_i \times \vec{f}_{ij}) = \vec{r}_i \times m_i \vec{a}_i \tag{14.2}$$

$\begin{cases} \text{n equations of the type (14.1)} \\ \text{n equations of the type (14.2)} \end{cases}$

14.1A Application of Newton's Laws to the Motion of a System of Particles

$\vec{F}_i$, $\vec{f}_{ij}$ form a system equivalent to that of the effective forces (Fig. 14.2)

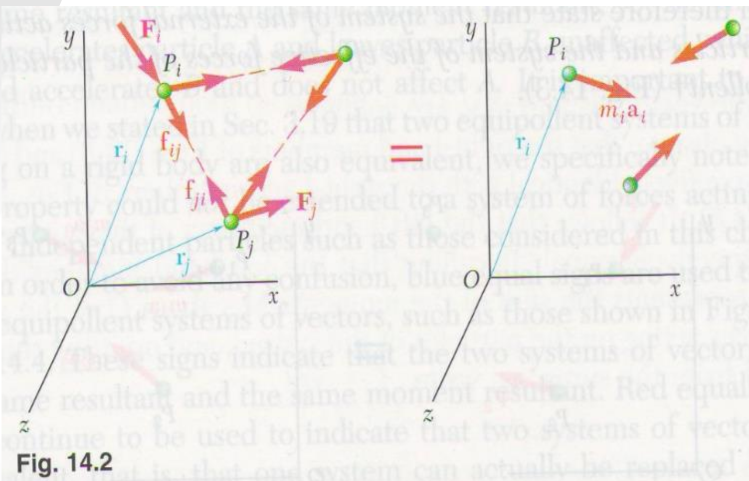

Fig. 14.2

Internal force $\vec{f}_{ij}$: according to Newton's 3rd law, $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are equal and opposite, and have the same line of action.

- $\vec{f}_{ij} + \vec{f}_{ji} = 0$
- sum of moments about *O*

$$\vec{r}_i \times \vec{f}_{ij} + \vec{r}_j \times \vec{f}_{ji} = \vec{r}_i \times \underbrace{\boxed{(\vec{f}_{ij} + \vec{f}_{ji})}}_{0} + (\vec{r}_j - \vec{r}_i) \times \vec{f}_{ji} = 0$$

collinear

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{f}_{ij} = 0, \quad \sum_{i=1}^{n}\sum_{j=1}^{n} (\vec{r}_i \times \vec{f}_{ij}) = 0 \qquad (14.3)$$

- Resultant and the moment resultant of the internal forces of the system are zero.

14.1A Application of Newton's Laws to the Motion of a System of Particles

Eq (14.1) : summing the left-hand and right-hand members, and considering the first of Eqs (14.3),

$$\sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} m_i \vec{a}_i \tag{14.4}$$

similarly,

$$\sum_{i=1}^{n} (\vec{r}_i \times \vec{F}_{ij}) = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{a}_i) \tag{14.5}$$

$$\underbrace{\text{The system of external forces } \vec{F}_i \qquad \text{The system of the effective forces } m_i \vec{a}_i}_{\text{The same resultant and moment resultant, “equipollent” (Fig. 14.3)}}$$

→ d'Alembert's Principle

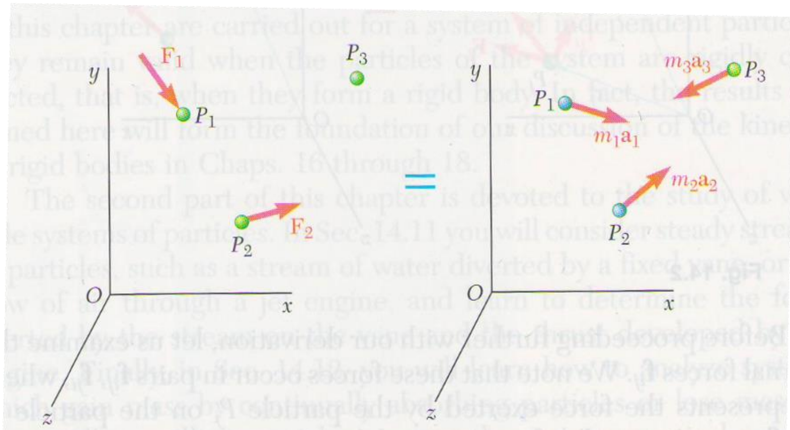

Fig. 14.3

14.1A Application of Newton's Laws to the Motion of a System of Particles

Eq (14.3) : The system of the internal forces $\vec{f}_{ij}$ is equipollent to zero.
It does not mean that the internal forces have no effect on the particles.

[Example] Sun and the planets : gravitational forces are internal, equipollent to zero.
However, these forces are still responsible for the motion of the planets about the sun.

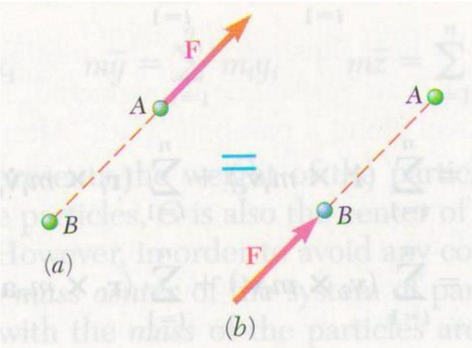

Fig. 14.4

Two systems of external forces (Fig. 14.4)

- Same resultant and moment resultant
- Not the same effect on a given system of particles

((a) accelerates A, leaves B unaffected;
(b) accelerates B, leaves A unaffected.)

Sec. 3.19 : Two equipollent system of forces acting on a rigid body
(1) equivalent
(2) not extended to a set of independent particles

Blue signs - equipollent (same resultant and moment resultant)

Red signs - equivalent (can actually be replaced by each other)

14.1B Linear and Angular Momentum of a System of Particles

Condensed form of Eq.(14.4) and (14.5)

Linear momentum $\vec{L}$: sum of the linear momenta of the various particles.

$$\overrightarrow{L} = \sum_{i=1}^{n} m_i \overrightarrow{v_i} \tag{14.6}$$

Angular momentum $\overrightarrow{H_o}$

$$\overrightarrow{H_o} = \sum_{i=1}^{n} (\vec{r_i} \times m_i \overrightarrow{v_i}) \tag{14.7}$$

Differentiate (14.6)

$$\dot{\vec{L}} = \sum_{i=1}^{n} m_i \dot{\vec{v}}_i = \sum_{i=1}^{n} m_i \vec{a}_i \tag{14.8}$$

Differentiate (14.7)

$$\begin{aligned}\dot{\vec{H}}_o &= \sum_{i=1}^{n} (\dot{\vec{r}}_i \times m_i \overrightarrow{v_i}) + \sum_{i=1}^{n} (\vec{r}_i \times m_i \dot{\vec{v}}_i) \\ &= \sum_{i=1}^{n} (\overrightarrow{v_i} \times m_i \overrightarrow{v_i}) + \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{a}_i)\end{aligned}$$

$$\dot{\vec{H}}_o = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{a}_i) \tag{14.9}$$

14.1B Linear and Angular Momentum of a System of Particles

Combining with left-hand members of Eqs.(14.4) and (14.5)

$$\sum \vec{F} = \dot{\vec{L}} \tag{14.10}$$

$$\sum \overrightarrow{M_0} = \dot{\vec{H}}_o \tag{14.11}$$

Resultant = rate of change of linear momentum

Moment Resultant = rate of change of angular momentum about *O*

14.1C Motion of the Mass Center of a System of Particles

Mass center G, position vector $\vec{\bar{r}}$

$$m\vec{\bar{r}} = \sum_{i=1}^{n} m_i \vec{r}_i \qquad (14.12)$$

total mass $\sum_{i=1}^{n} m_i$

rectangular components

$$m\bar{x} = \sum_{i=1}^{n} m_i x_i, \quad m\bar{y} = \sum_{i=1}^{n} m_i y_i, \quad m\bar{z} = \sum_{i=1}^{n} m_i z_i \qquad (14.12')$$

- G is also center of gravity of the system of particles.

Differentiate Eq.(14.12)

$$m\dot{\vec{\bar{r}}} = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i$$

$$m\vec{\bar{v}} = \sum_{i=1}^{n} m_i \vec{v}_i \qquad (14.13)$$

velocity of the mass center G

$$\vec{L} = m\vec{\bar{v}} \qquad (14.14)$$

14.1C Motion of the Mass Center of a System of Particles

differentiate again

Eq.(14.10)

$$\vec{L} = m\vec{\bar{a}} \quad (14.15)$$

$$\sum \vec{F} = m\vec{\bar{a}} \quad (14.16)$$

: the motion of the mass center G of the system

→ the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point.

[Example] Exploding shell : Neglecting air resistance, a shell will travel along a parabolic path. After explosion, its fragments' mass center G will travel the same path. *G* moves as if the shell had not exploded.

External forces $\xleftrightarrow{\text{equipollent}}$ $m\vec{\bar{a}}$ attached at G ... *Wrong!!*

→ sum of the moments about G of the external forces is not zero, in general.

14.1D Angular Momentum of a System of Particles about its Mass Center

Centroidal frame of reference $Gx'y'z'$: translates with respect to Newtonian frame
Fig (14.5)

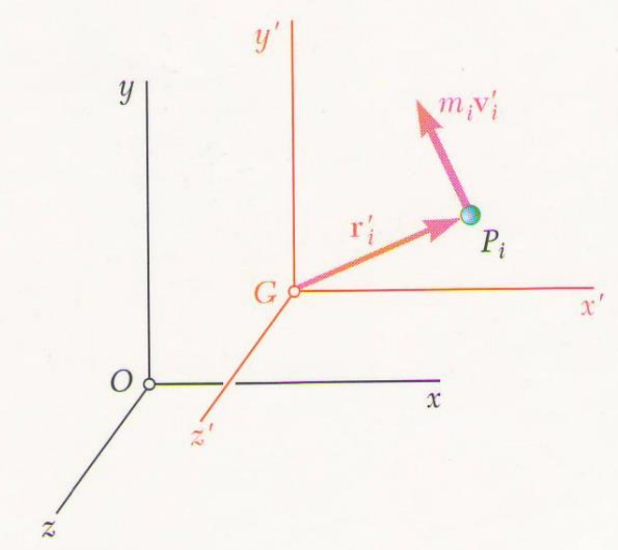

Fig. 14.5

- Although a centroidal frame is not a Newtonian one, fundamental relation still holds.
- $\overrightarrow{r_i'}, \overrightarrow{v_i'}$: position vector, velocity vector with respect to $Gx'y'z'$
- Angular momentum $\overrightarrow{H_G}'$ about mass center G

14.1D Angular Momentum of a System of Particles about its Mass Center

$$\overrightarrow{H_G}' = \sum_{i=1}^{n}(\overrightarrow{r_i}' \times m_i \overrightarrow{v_i}') \qquad (14.17)$$

differentiate

$$\dot{\vec{H}}_G' = \sum_{i=1}^{n}(\overrightarrow{r_i}' \times m_i \overrightarrow{a_i}') \qquad (14.18)$$

acceleration relative to moving frame

From Sec. 11.4D,

$$\overrightarrow{a_i} = \vec{\bar{a}} + \overrightarrow{a_i}'$$

$\boxed{\overrightarrow{a_i} = \vec{\bar{a}} + \overrightarrow{a_i}'}$

Absolute acceleration of P_i

Absolute acceleration of G

Relative acceleration of P_i with respect to $Gx'y'z'$

$$\dot{\vec{H}}_G' = \sum_{i=1}^{n}(\overrightarrow{r_i}' \times m_i \overrightarrow{a_i}) - (\sum_{i=1}^{n} m_i \vec{\bar{r}}_i') \times \vec{\bar{a}} \qquad (14.19)$$

$m_i \vec{\bar{r}}_i' = 0, \ \vec{\bar{r}}_i' = 0$ with respect to $Gx'y'z'$

replace $\overrightarrow{F_i} + \sum_{i=1}^{n}\overrightarrow{f_{ij}}$ from (14.1)

Again, moment resultant of $\overrightarrow{f_{ij}}$ about $G = 0$

14.1D Angular Momentum of a System of Particles about its Mass Center

$$\sum \overrightarrow{M_G} = \dot{\vec{H}}_G{}' \qquad (14.20)$$

: moment resultant about G of the external forces =
rate of change of the angular momentum about G

Eq.(14.17) : $\dot{\vec{H}}_G{}'$ is the sum of the moments about G of $m_i \overrightarrow{v_i}{}'$ (relative motion)
how about it with $m_i \overrightarrow{v_i}$ (absolute motion)

$$\overrightarrow{H_G} = \sum_{i=1}^{n} (\overrightarrow{r_i}{}' \times m_i \overrightarrow{v_i}) \qquad (14.21)$$

Remarkably, $\overrightarrow{H_G} = \overrightarrow{H_G}{}'$

From Sec. 11.4D,

$$\overrightarrow{v_i} = \vec{\bar{v}} + \overrightarrow{v_i}{}' \qquad (14.22)$$

$$\overrightarrow{H_G} = \underbrace{\left(\sum_{i=1}^{n} m_i \overrightarrow{r_i}{}' \right)}_{0} \times \vec{\bar{v}} + \sum_{i=1}^{n} (\overrightarrow{r_i}{}' \times m_i \overrightarrow{v_i}{}')$$

14.1D Angular Momentum of a System of Particles about its Mass Center

From Eq.(14.20),

$$\sum \overrightarrow{M_G} = \dot{\vec{H}}_G \qquad (14.23)$$

$\overrightarrow{H_G}$: moments about G of the momenta of the particles in their motion with respect to either the Newtonian frame $Oxyz$ or centroidal frame $Gx'y'z'$.

$$\overrightarrow{H_G} = \sum_{i=1}^{n} (\overrightarrow{r_i}' \times m_i \overrightarrow{v_i}) = \sum_{i=1}^{n} (\overrightarrow{r_i}' \times m_i \overrightarrow{v_i}') \qquad (14.24)$$

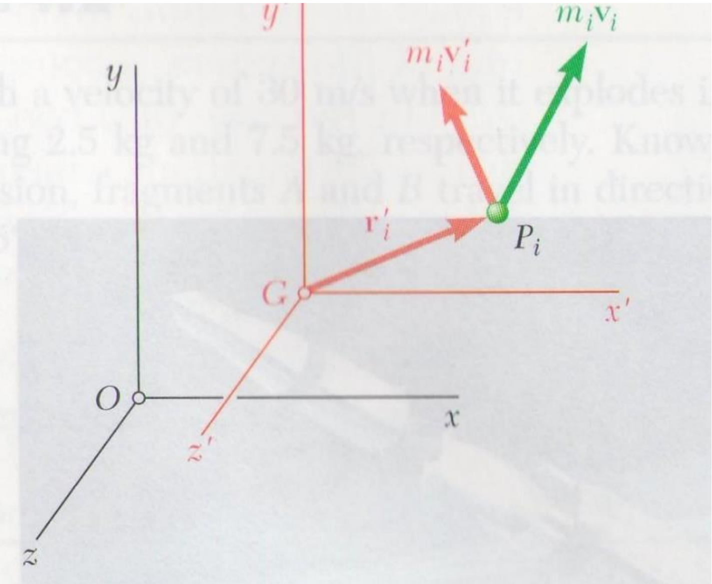

Fig. 14.6

14.1E Conservation of Momentum for a System of Particles

No external forces

$$\vec{L} = 0,\ \dot{\vec{H}}_o = 0$$

$$\rightarrow \vec{L} = \text{constant},\ \vec{H}_o = \text{constant} \qquad (14.25)$$

Central forces : moment about O of each external force can be zero, but any of the forces are non-zero. Second of Eq.(14.25) still holds.

Sum of external forces = 0, from Eq.(14.14)

$$\vec{\bar{v}} = \text{constant} \qquad (14.26)$$

- mass center G moves in a straight line and of a constant speed.

Sum of the moments about G = 0 from Eq.(14.23)

$$\overrightarrow{H_G} = \text{constant} \qquad (14.27)$$

14.2A Kinetic Energy of a System of Particles

Kinetic energy T : sum of the kinetic energy of the various particles.

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \qquad (14.28)$$

Centroidal frame of Reference

- convenient to consider separately $\begin{cases} \text{the motion of the mass center } G \\ \text{the motion of the system relative to } G \end{cases}$

$$\boxed{v_i^2 = \vec{v}_i \cdot \vec{v}_i}$$

$$\vec{v}_i = \vec{\bar{v}} + \vec{v}_i{}' \qquad (14.22)$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n}(m_i \vec{v}_i \cdot \vec{v}_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}[m_i (\vec{\bar{v}} + \vec{v}_i{}') \cdot (\vec{\bar{v}} + \vec{v}_i{}')]$$

$$= \frac{1}{2}\underbrace{\left(\sum_{i=1}^{n} m_i\right)}_{m}\bar{v}^2 + \vec{\bar{v}} \cdot \underbrace{\sum_{i=1}^{n} m_i \vec{v}_i{}'}_{=0\ (\because\ m\vec{\bar{v}}' = 0)} + \frac{1}{2}\sum_{i=1}^{n} m_i v_i{}'^2$$

14.2A Kinetic Energy of a System of Particles

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2 \tag{14.29}$$

- Kinetic energy T = kinetic energy of the mass center G (assuming the entire mass concentrated at G) + kinetic energy of the system in its motion relative to the frame $Gx'y'z'$

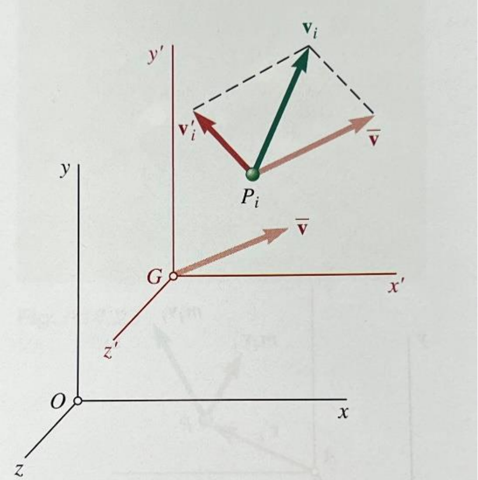

Fig. 14.7 A centroidal frame of reference $Gx'y'z'$ moving in translation with velocity $\bar{\mathbf{v}}$ with respect to a newtonian reference frame $Oxyz$.

14.2B Work Energy Principle

Can be applied to each particle P_i

$$T_1 + U_{1\to 2} = T_2 \qquad (14.30)$$

$U_{1\to 2}$: the work done by the internal force $\vec{f}_{ij}$ and the resultant external force $\vec{F}_i$ (must consider the work of the internal forces $\vec{f}_{ij}$ since the particle P_i and P_j in general undergo different displacements.)

All the forces are conservative,

$$T_1 + V_1 = T_2 + V_2 \qquad (14.31)$$

- Principle of conservation of energy

- When non-conservative force is involved

$$T_1 + V_{g1} + V_{e1} + U_{1\to 2}^{NC} = T_2 + V_{g2} + V_{e2} \qquad (14.30')$$

14.2C Principle of Impulse and Momentum

Integrating Eqs.(14.10) and (14.11) in t

$$\sum \int_{t_1}^{t_2} \overrightarrow{F}\, dt = \overrightarrow{L_2} - \overrightarrow{L_1} \tag{14.32}$$

$$\sum \int_{t_1}^{t_2} \overrightarrow{M_o}\, dt = \left(\overrightarrow{H_o}\right)_2 - \left(\overrightarrow{H_o}\right)_1 \tag{14.33}$$

- sum of the linear impulses of the external forces
  = change in linear momentum of the system
- sum of the angular impulses about O of the external forces
  = change in angular momentum about O

$\rightarrow$

$$\overrightarrow{L_1} - \sum \int_{t_1}^{t_2} \overrightarrow{F}\, dt = \overrightarrow{L_2} \tag{14.34}$$

$$\left(\overrightarrow{H_o}\right)_1 + \sum \int_{t_1}^{t_2} \overrightarrow{M_o}\, dt = \left(\overrightarrow{H_o}\right)_2 \tag{14.35}$$

14.2C Principle of Impulse and Momentum

Fig. 14.8 (a), (c) : momenta of particles at t_1 and t_2.

(b) : sum of linear impulse, angular impulses about O.

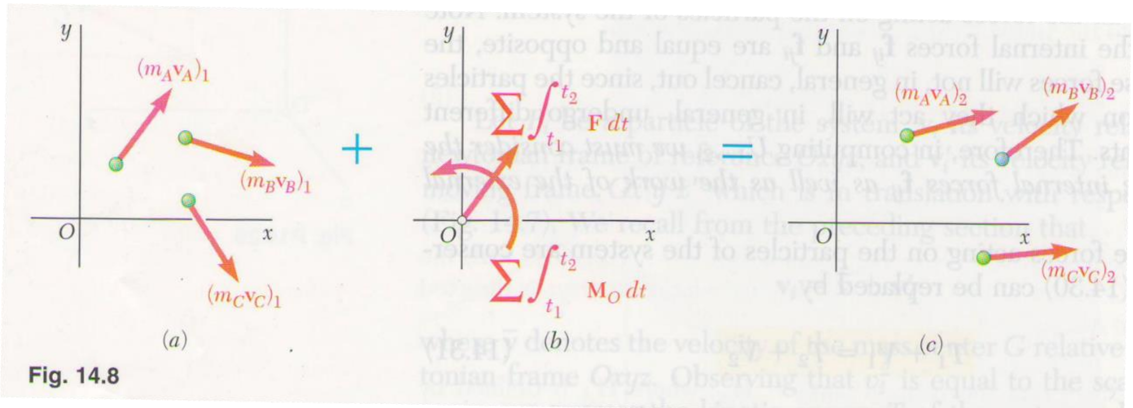

Fig. 14.8

→ remain valid in case of particles moving in space.

Eqs. (14.34), (14.35) : momenta at t_1 + impulse of external forces $\xleftrightarrow{\text{equipollent}}$ momenta at t_2

No external forces

$$\overrightarrow{L_1} = \overrightarrow{L_2} \tag{14.36}$$

$$(\overrightarrow{H_o})_1 = (\overrightarrow{H_o})_2 \tag{14.37}$$

- linear, angular momentum are conserved.

14.3 Variable Systems of Particles

Well-defined system : considered so far (does not gain or lose any particles during their motions)

Variable systems of particles : continually gaining or losing particles, or doing both.

ex) hydraulic turbine : to determine the forces exerted by a stream of water or rotating blades, particles of water in contact with the blades form on ever-changing system.

rocket : continual ejection of fuel particles.

Must find a way to reduce the analysis of a variable system of particles to that of an auxiliary constant system.

- steady stream of particles
- system gaining or losing mass

14.3A Steady Stream of Particles

Stream of water diverted by a fixed vane
A flow of air through a duck or a blower

- isolate these particles and denote by S (Fig. 14.9)

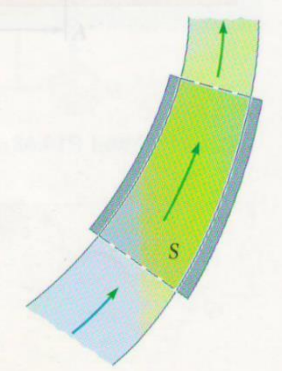

Fig. 14.9

S : a variable system of particles, continually gaining particles flowing in, and loses an equal number of particles flowing out.

→ The kinetics principles established so far cannot be directly applied

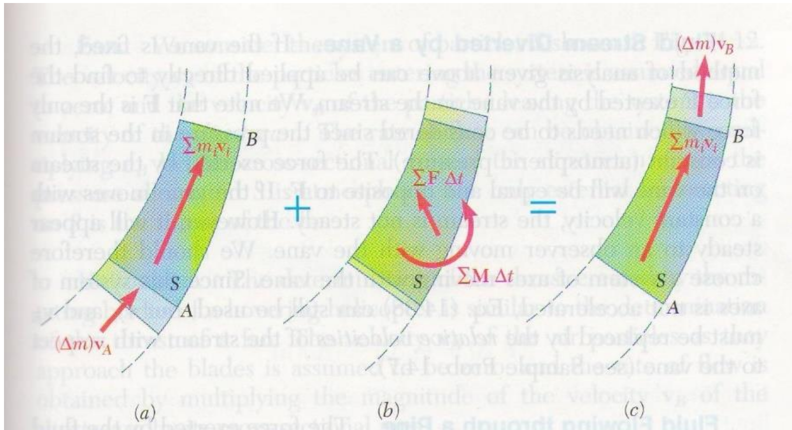

Fig. 14.10

Auxiliary system of particles remaining constant for Δt

i) At time t, S + the particles which will enter S during Δt (Fig. 14.10(a))

ii) At $t+\Delta t$, S + particles which have left S during Δt (Fig. 14.10(c))

14.3A Steady Stream of Particles

The same particles are involved in both cases, can apply the principle of impulse and momentum

- total mass m of S remains constant Δt
- particles entering and leaving S during Δt must have the same mass: Δm
- $\vec{v}_A$: velocity of the particles entering S at $\mathbf{A}$
- $\vec{v}_B$: velocity of the particles leaving S at $\mathbf{B}$
- $(\Delta m)\vec{v}_A$: momentum of particles entering S
- $(\Delta m)\vec{v}_B$: momentum of particles leaving S
- $m_i\vec{v}_i$: momenta of the particles forming S

Fig. 14.10 : momenta + impulses $\overset{\text{equipollent}}{=}$ momenta

- $\sum m_i\vec{v}_i$ is formed on both sides → can be omitted

$$\rightarrow \begin{bmatrix} \text{the system formed by} \\ (\Delta m)\vec{v}_A \text{ entering } S \end{bmatrix} + \begin{bmatrix} \text{Impulse of the forces} \\ \text{entered on } S \text{ during } \Delta t \end{bmatrix} = \begin{bmatrix} (\Delta m)\vec{v}_B \text{ leaving } S \\ \text{during } \Delta t \end{bmatrix}$$

14.3A Steady Stream of Particles

$$(\Delta m)\overrightarrow{v_A} + \sum \vec{F}\Delta t = (\Delta m)\overrightarrow{v_B} \qquad (14.38)$$

$$\sum \vec{F} = \frac{dm}{dt}(\overrightarrow{v_B} - \overrightarrow{v_A}) \qquad (14.39)$$

vector difference

$$kg / s \cdot m / s \;\rightarrow\; kg \cdot m / s^2 ,\; N$$

i) Fluid stream diverted by a vane

- if the vane is fixed : analysis above can be applied directly to find the force $\vec{F}$ exerted by the vane on the stream
  - (1) $\vec{F}$ is the only force since the pressure in the stream is constant.
  - (2) force exerted by the stream on the vane will be equal and opposite to $\vec{F}$

- if the vane moves with a constant velocity : stream is not steady, but steady to an observer moving with vane.
  - (1) choose an axes system moving with vane. Eq.(14.38) still can be used since the axes system is not accelerated. But $\vec{v}_A$ and $\vec{v}_B$ must be replaced by the relative velocity with respect to the vane.

14.3A Steady Stream of Particles

ii) Flow flowing through a pipe

force exerted by the fluid on a pipe transition(bend, contraction) can be determined by considering S in transition. In general, the pressure will vary, forces exerted on S by the adjoining portions of the fluid should also be considered.

iii) Jet engine

- Air enters with no velocity and leaves with a high velocity.
- energy required to accelerate the air particles : obtained by burning fuel
- mass of the burned fuel : small enough compared with the air → neglected

  ⟹ analysis of a jet engine ≈ an air stream

  - can be considered as a steady stream if all velocities are measured with respect to the airplane.
- airstream enters with $\vec{v}$ (speed of the airplane), and leaves with $\vec{u}$ (relative velocity of the exhaust gas) (Fig. 14.11)
- intake and exhaust pressures are nearly atmospheric → only external force is the force exerted by the engine on the airstream. (equal and opposite to the thrust)

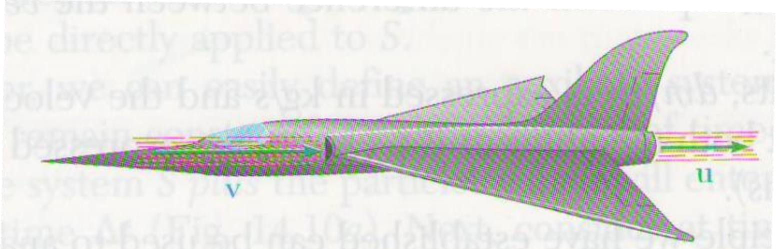

Fig. 14.11

14.3A Steady Stream of Particles

iv) Fan (Fig. 14.12)

- $\vec{v}_A$ entering the system is assumed zero.
- $\vec{v}_B$ leaving the system is that of the slipstream.
- rate of flow = v_B × cross-sectioned area of the slipstream.
- pressures all around S is atmospheric, the only external force on S is the thrust of the fan.

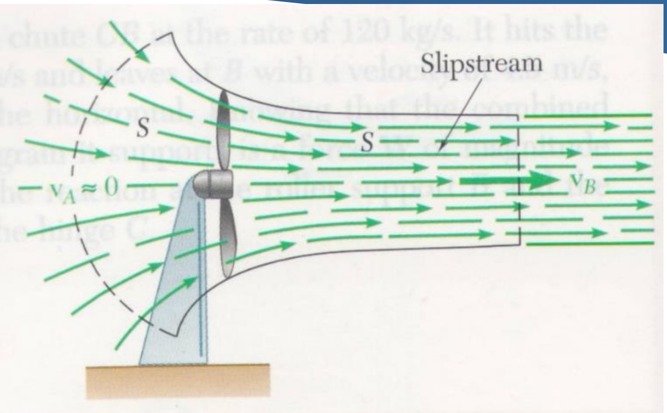

Fig. 14.12

v) Helicopter

- determination of the thrust created by the rotating blades of a hovering helicopter
  - similar to that of fan.
- $\vec{v}_A$ as approaching the blades is assumed zero.
- the rate of flow = $\left|\vec{v}_B\right|$ × cross-sectioned area of the slipstream.

14.3B Systems Gaining or Losing Mass

Fig. 14.13 : at Δt, mass m increase by Δm during Δt.

- Principle of impulse and momentum
  - at t, $S + \Delta m$ (where, $\vec{v}_a$: absolute velocity of the particles absorbed.)
  - at $t+\Delta t$, $m + \Delta m$, $\vec{v}+\Delta\vec{v}$

$$m\vec{v}+(\Delta m)\vec{v_a}+\sum\vec{F}\Delta t = (m+\Delta m)(\vec{v}+\Delta\vec{v}) \qquad (14.40)$$

excluding the forces exerted by the particles being absorbed

$$\sum\vec{F}\Delta t = m\Delta\vec{v}+\Delta m(\vec{v}-\vec{v_a})+\underbrace{(\Delta m)(\Delta\vec{v})}_{\text{second order, neglected}} \qquad (14.41)$$

relative velocity $\vec{u} = \vec{v}_a - \vec{v}$: with respect to S of the particles absorbed.

$\vec{v}_a < v$, $\vec{u}$ is directed left in Fig. 14.13

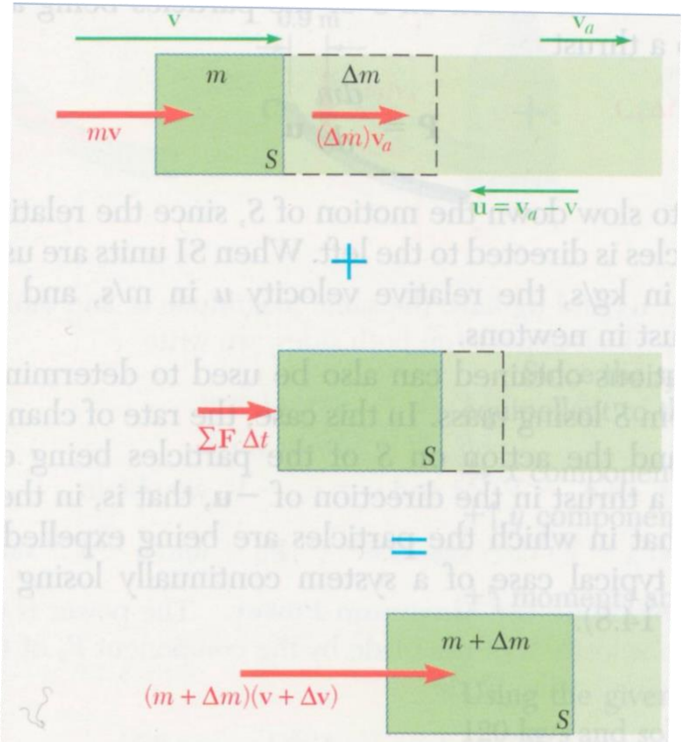

Fig. 14.13

14.3B Systems Gaining or Losing Mass

$$\sum \vec{F}\,\Delta t = m\Delta\vec{v} - (\Delta m)\vec{u}$$

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\vec{u} \qquad (14.42)$$

$$\sum \vec{F} + \underbrace{\frac{dm}{dt}\vec{u}} = m\vec{a} \qquad (14.43)$$

$\rightarrow \vec{P} = \frac{dm}{dt}\vec{u}$: the action on S of the particles being absorbed

$$[N] \leftarrow \frac{kg}{m}\cdot m/s$$

- tends to slow down the motion of S since $\vec{u}$ is directed to the left.

S losing mass

- $\frac{dm}{dt}$ is negative, $\vec{P}$ is in direction of $-\vec{u}$, thrust is in the same direction $\rightarrow$ rocket