Ch. 15  Kinematics of Rigid Bodies

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15.0 Introduction

Categories of rigid-body motion
1) Translation --- if any straight line inside the body keeps the same direction during motion
   - All the particles forming the body move along parallel paths.
     - straight lines → rectilinear translation (Fig. 15.1.)
     - curved lines → curvilinear translation (Fig. 15.2.)
15.0 Introduction

Categories of rigid-body motion

ii ) Rotation about a fixed axis --- particles move in parallel planes along circles centered on the same axis (Fig 15.3.)

→ axis of rotation, the particles on the axis have zero velocity / acceleration.

Fig. 15.4.(a) --- curvilinear translation, all particles moving along parallel circles
(b) ---rotation, all particles moving along concentric circles.

→ each particle moves in a given plane → plane motion
15.0 Introduction

Categories of rigid-body motion

iii ) General plane motion--- all the particles move in parallel planes.
   neither a rotation nor a translation
   two examples ---- Fig. 15.5.

iv ) Motion about a fixed point --- three-dimensional motion of a rigid body attached at a
   fixed point (Fig. 15.6.)
15.0 Introduction

Categories of rigid-body motion

v ) General motion ---- which does not fall in any of the categories above

Rotation about a fixed axis → angular velocity, angular acceleration velocity, acceleration of a given point ---- position vector + angular velocity

acceleration

General plane motion --- gears, connecting rods, pin-connected linkages.

velocity of a point B of the slab --- sum of

the velocity of the ref. point A.

the velocity of B relative to a frame of ref. translating with A

(Moving with A, but not rotating)

---- same approach used for acceleration

Alternative methods --- instantaneous center of rotation

use of parametric expressions

Motion of a particle relative to a rotating frame of ref. , Coriolis acceleration
15.1A Translation

Rigid body in translation. A,B ; two particles in it, (Fig. 15.7.(a))

\[ \overrightarrow{r}_A, \overrightarrow{r}_B \] - - - position vectors of A and B with respect to fixed frame of ref.

\[ \overrightarrow{r}_{B/A} \] - - - vector joining A and B

\[ \overrightarrow{r}_B = \overrightarrow{r}_A + \overrightarrow{r}_{B/A} \] \hspace{1cm} (15.1.)

Differentiate w.r.t. Translation ---- \[ \overrightarrow{r}_{B/A} \] must maintain a constant direction, also const. magnitude

\[ \Rightarrow \overrightarrow{r}_{B/A} = 0 \]

\[ \overrightarrow{v}_B = \overrightarrow{v}_A \] \hspace{1cm} (15.2.)

Differentiate once more

\[ \overrightarrow{a}_B = \overrightarrow{a}_A \] \hspace{1cm} (15.3.)

All particles have the same velocity / acceleration at any given instant.
15.1A Translation

Curvilinear translation - - - velocity / acceleration change direction / magnitude

Rectilinear translation - - - all particles move along a straight line, velocity / acceleration keep the same direction.
15.1B Rotation about a Fixed Axis

Rigid body rotating about a fixed axis AA’

P · · · · · point of the body, \( \mathbf{r} \) = position vector w.r.t. a fixed frame of ref.  
frame is centered at O on AA’, Z axis coincides with AA’ (Fig. 15.8.)

B : projection of P on AA’, P will describe a circle of center B, radius of \( r \sin \phi \), \( \phi \)  
: angle formed by \( \mathbf{r} \) and AA’.

Angular coordinate \( \theta \) · · · completely defines the position of P  
and the entire body positive when  
viewed as counterclockwise from A’

unit · · · radians(rad), degrees(°), revolutions(r)

\[ 1r = 2\pi \text{ rad} = 360° \]
15.1B Rotation about a Fixed Axis

Length of the arc \( \Delta s \)

\[
\Delta s = (BP) \Delta \theta = (r \sin \phi) \Delta \theta
\]

\[
v = \frac{ds}{dt} = r \dot{\theta} \sin \phi \tag{15.4.}
\]

Independent of \( P \)

Vector perpendicular to the plane containing \( AA' \) and \( \vec{r} \)

\[
\Rightarrow \quad \vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \tag{15.5.}
\]

Angular velocity \( \vec{\omega} = \omega \vec{k} = \dot{\theta} \vec{k} \tag{15.6.} \)

right-hand rule from the rotation of a body
obeys the parallelogram law of addition(vector quantities)
15.1B Rotation about a Fixed Axis

Acceleration \( \cdots \) differentiate Eq.(15.5.)

\[
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})
\]

\[
= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}
\]

\[
= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}
\]

\( \Rightarrow \alpha : \text{angular acceleration} \)

\[
\vec{a} = \alpha \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (15.8)
\]

\[
\begin{align*}
\text{Tangential component} \quad \text{Normal component}
\end{align*}
\]

Angular acceleration

\[
\vec{a} = \alpha \vec{k} = \dot{\omega} \vec{k} = \ddot{\theta} \vec{k} \quad (15.9)
\]

along the axis of rotation
15.1B Rotation about a Fixed Axis

Rotation of a Representative slab

- velocity of any given point ---
  \[ \vec{v} = \omega \hat{k} \times \vec{r} \quad (15.10.) \]
  magnitude - - - \( v = r \omega \)
  direction - - - by rotating \( \vec{r} \) through 90° in the sense of rotation

- acceleration - - -
  \[ \vec{a} = \alpha \hat{k} \times \vec{r} - \omega^2 \vec{r} \quad (15.11.) \]
  \[ \vec{a}_t = \alpha \hat{k} \times \vec{r}, \quad |\vec{a}_t| = r \alpha \]
  counterclockwise if \( \alpha(+) \), clockwise if \( \alpha(-) \)
  \[ \vec{a}_n = -\omega^2 \vec{r}, \quad |\vec{a}_n| = r \omega^2 \]
  always opposite to \( \vec{r} \), i.e., toward \( O \)
15.1C Equations defining the Rotation about a Fixed Axis

More often, motion is specified by angular acceleration $\alpha$ as a function of $t$, or a function of $\theta$ or $\omega$

\[ \omega = \frac{d\theta}{dt} \]  
\[ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \]

(15.12.) \rightarrow (15.13.)

\[ \alpha = \omega \frac{d\omega}{dt} \]  

(15.14.)

Integration $\rightarrow$

i) Uniform rotation --- angular acceleration $= 0$

\[ \theta = \theta_o + \omega t \]  

(15.15.)

ii) Uniformly accelerated rotation --- $\alpha = \text{constant}$

\[ \omega = \omega_o + \alpha t \]

\[ \theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2 \]  

(15.16.)

\[ \omega^2 = \omega_o^2 + 2\alpha (\theta - \theta_o) \]
15.2A General Plane Motion

General plane motion ---- always sum of a translation and a rotation.

[example]

i) Rolling wheel --- rolling motion = combination of simultaneous translation and rotation

ii) Sliding rod--- motion = translation(horizontal) + rotation about A(Fig.15.13(a)) or translation(vertical) + rotation about B(Fig. 15.13(b))
15.2A General Plane Motion

Displacement of two particles A and B (Fig. 15.14.)

two parts ---- \[
\begin{align*}
&\text{A1, B1 → A2, B1': translation} \\
&\text{A2, B1' → A2, B2: rotation about A}
\end{align*}
\]

Relative motion of B w.r.t. a frame attached at A, of fixed orientation

→ rotation, B will appear to describe an arc of circle centered at A.
15.2B Absolute velocity of a particle of the slab

\[ \vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \]  
\[ \vec{v}_{B/A} = \omega \vec{k} \times \vec{r}_{B/A}, \quad |\vec{v}_{B/A}| = r \omega \]  
\[ (15.17) \]

Translation of the slab

Rotation of the slab about A

Angular velocity of the slab w.r.t. axes of fixed orientation

Position vector of B relative to A

(15.15) \rightarrow (15.17) = \vec{v}_B = \vec{v}_A + \omega \vec{k} \times \vec{r}_{B/A}  

(15.17')
15.2B Absolute velocity of a particle of the slab

Sliding rod of Fig. 15.13

--- $\mathbf{v}_A$ known, fixed $\mathbf{v}_B, \mathbf{\omega}$ in terms of $\mathbf{v}_A, l, \theta$

motion = translation with $A$ + simultaneous rotation about $A$ (Fig.15.16.)

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$ (15.17)

$\mathbf{v}_{B/A}$ direction is known, but its magnitude $l\omega$ is unknown.

$\leftarrow \mathbf{v}_B$ direction is known

$$\mathbf{v}_B = v_A \tan \theta, \quad \omega = \frac{v_{B/A}}{l} = \frac{v_A}{l \cos \theta}$$ (15.19)
15.2B Absolute velocity of a particle of the slab

---- same result obtained by using B as a point of ref. (Fig.15.17.)

\[ \vec{v}_{A/B}, \vec{v}_{B/A} : \text{same magnitude, opposite sense} \]

Angular velocity \( \vec{\omega} \) ---- same no matter which the ref. is at a or b.

\( \vec{\omega} \) of a rigid body in plane motion is independent of its ref. point.
15.2B Absolute velocity of a particle of the slab

---- same result obtained by using B as a point of ref. (Fig.15.17.)

Several moving parts --- pin-connected...

i) The points where two parts are connected must have the same absolute velocity

ii) Gears---- teeth in contact must have the same absolute velocity

iii) Parts which slide on each other --- relative velocity must be considered.
15.3 Instantaneous center of rotation in plane motion

At any given instant, the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis → “instantaneous axis of rotation”,

C : “instantaneous center of rotation”

---- $v_A, \omega$ can be obtained by rotating the slab with $\omega$ at a distance $r = v_A / \omega$ from A (Fig.15.18(b))

The velocities of all the other particles would be the same as originally defined.

→ As far as the velocities are concerned, the slab seems to rotate about the instant considered.
15.3 Instantaneous center of rotation in plane motion

Two ways to define the instantaneous center $C$

i) Directions of $\vec{V}_A$, $\vec{V}_B$ are known and different ---- drawing perpendicular lines to $\vec{V}_A$ and $\vec{V}_B$, the point where the two lines intersect (Fig.15.19(a))

ii) $\vec{V}_A$, $\vec{V}_B$ are perpendicular to $\vec{A}\vec{B}$ and their magnitudes are known ---- by intersecting $\vec{A}\vec{B}$ with the line joining the extremities of $\vec{V}_A$, $\vec{V}_B$ (Fig.15.19(b))

iii) IF $\vec{V}_A$ and $\vec{V}_B$ were parallel in Fig.15.19(a)) or if $\vec{V}_A$, $\vec{V}_B$ had the same magnitude in Fig.15.19(b)) ---- $C$ would be at an infinite distance, $\vec{\omega}$ would be zero.
15.3 Instantaneous center of rotation in plane motion

Sliding rod with the instantaneous center

C can be obtained by drawing perpendicular to $\vec{v}_A$ and $\vec{v}_B$

The velocities of all the particles are the same as if the rod rotated about C.

If $v_A$ is known, $\omega = \frac{v_A}{AC} = \frac{v_A}{l \cos \theta}$

$$v_B = (BC)\omega = l \sin \theta \frac{v_A}{l \cos \theta} = v_A \tan \theta$$

Only absolute velocities are involved.

C inside the rigid body ---- at that instant, its velocity is zero. But will probably different from zero at $t + \Delta t$

⇒ C doesn’t have zero acceleration.

⇒ Acceleration can not be determined as if the slab were rotating about C.
Any plane motion = a translation of an arbitrary ref. point + simultaneous rotation → determine the acceleration of the points of the slab

\[ \vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \quad (15.21.) \]

Relative acceleration

- Tangential component: \[ \overrightarrow{(a_{B/A})_t} = \alpha \vec{k} \times \vec{r}_{B/A} \], \[ |(a_{B/A})_t| = r\alpha \] (15.22.)
- Normal component: \[ \overrightarrow{(a_{B/A})_n} = -\omega^2 \vec{r}_{B/A} \], \[ |(a_{B/A})_n| = r\omega^2 \] (15.22.)

\[ (15.21.) \rightarrow \vec{a}_B = \vec{a}_A + \alpha \vec{k} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \quad (15.21') \]
15.4A Abs. and Rel. Acceleration in Plane Motion

Sliding AB (Fig. 15.23.)

\[ v_A , \, a_A \, \text{known} , \, \text{determine} \, \, a_B , \, \alpha \]

\[ a_B = a_A + a_{B/A} \]

\[ a_A + (a_{B/A})_n + (a_{B/A})_t \]  \hspace{1cm} (15.23.)

\[ l\alpha \] , but no way to tell it is directed to left or right

Toward A

Both possible sense of \( a_B \) in Fig. 15.23
Eq. (15.23) → Fig. 15.24

four possible vector polygons --- depending upon the sense of \( \overrightarrow{a_A} \) and relative magnitude of \( a_A \) and \( (a_{B/A})_n \)

\( \omega \) also has to be known ← either method from Sec. 15.2 or 15.3.

Then, \( a_B \) and \( \alpha \) can be obtained by x and y components.

From Fig.15.24(a),

\[ x \text{ component : } 0 = a_A + l\omega^2 \sin \theta - l\alpha \cos \theta \]
\[ y \text{ component : } -a_B = -l\omega^2 \cos \theta - l\alpha \sin \theta \]

Or, direct measurement on the vector polygon.

(careful on \( a_A \) and \( (a_{B/A})_n \))
If the extremities were moving along curved tracks, necessary to resolve

To normal, tangential components $\rightarrow$ six different vectors.

Several moving parts $\rightarrow$ the point where two parts are connected

$\rightarrow$ must have the same absolute acceleration

* Meshed gears $\rightarrow$ teeth in contact $\rightarrow$

the same tangential acceleration

normal component $\rightarrow$ different
Sec 11.4B --- rate of change of a vector is the same w.r.t. a fixed frame or a frame in translation.

how about w.r.t. a rotating frame of reference?

Fig. 15.26. ---- fixed frame $OXYZ$

rotating frame $Oxyz$, $\vec{\Omega}$: angular velocity

$\dot{\vec{Q}}(t)$ : vector function

$$
\begin{align*}
\dot{\vec{Q}}_{oxyz} \\
\dot{\vec{Q}}_{oxyz}
\end{align*}
$$

Fig. 15.26
15.5A Rate of change of a vector w.r.t. a Rotating Frame

\[ \vec{Q} = Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k} \quad (i, j, k: \text{unit vectors}) \quad (15.27) \]

Differentiate, rate of change of \( \vec{Q} \), w.r.t. rotating frame \( Oxyz \)

\[ (\dot{\vec{Q}})_{oxyz} = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k} \quad (15.28) \]

Differentiate \( \vec{Q} \), w.r.t. the fixed frame \( OXYZ \), \( \vec{i}, \vec{j}, \vec{k} \) are variable

\[ (\dot{\vec{Q}})_{oxyz} = \dot{Q}_x \vec{i} + \dot{Q}_y \vec{j} + \dot{Q}_z \vec{k} + Q_x \frac{d\vec{i}}{dt} + Q_y \frac{d\vec{j}}{dt} + Q_z \frac{d\vec{k}}{dt} \quad (15.29) \]

\[ = \vec{\Omega} \times \vec{Q} \quad (15.30) \]

Rate of change of \( \vec{Q} \) w.r.t. rotating frame \( Oxyz \)

\[ (\dot{\vec{Q}})_{OXYZ} = (\dot{\vec{Q}})_{oxyz} + \vec{\Omega} \times \vec{Q} \quad (15.31) \]

Induced by the rotation of \( Oxyz \)
(15.29)의 부연설명

\[(\dot{\mathbf{Q}})_{Oxyz}\]

는 만일, 벡터 \(Q\)가 frame Oxyz에 fixed 되어 있어 \((\dot{\mathbf{Q}})_{Oxyz} = 0\)

이 된다면 (15.29)식의 마지막 3개 항과 같이 됨.

그러한 경우 \((\dot{\mathbf{Q}})_{Oxyz}\) 는 \(\mathbf{Q}\) 이 tip에 위치하여 있는 particle의 velocity를 나타내게 되며,

이는 frame Oxyz에 rigidly attached 되어 있는 절점의 velocity를 의미함.

Frame Oxyz는 Frame OXYZ에 대하여 \(\mathbf{\Omega}\) 의 angular velocity로 회전하고 있으므로

마지막 3개 항 = \(\mathbf{\Omega} \times \mathbf{Q}\)
When $\vec{Q}$ is represented by the unit vectors in $Oxyz$, Eq.(15.31) simplifies the determination of $\dot{(\vec{Q})_{Oxyz}}$, since no need to separate computation of the derivatives of the unit vectors defining the rotating frame.
Fig. 15.27. ---- two frames of ref, in the plane of the figure

fixed frame $OXY$, rotating frame $Oxy$

$\dot{(\vec{r})}_{OXY}$ : rate of change of $\vec{r}$, w.r.t. a fixed frame
15.5B Plane Motion of a Particle Relative to a Rotating Frame

\( (\mathbf{r})_{Oxy} : \) w.r.t. the rotating frame \( Oxy \)

\( \vec{\Omega} : \) angular velocity of the frame \( Oxy \) w.r.t. \( OXY \)

\[
\mathbf{v}_P = (\mathbf{r})_{OXY} = \vec{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxy}
\]

\( F : \) rotating frame

\( \mathbf{v}'_P : P' \) velocity in the slab which coincides with \( P \) at the instant

\[
\mathbf{v}_P = \mathbf{v}'_P + \mathbf{v}_{P/F}
\]

abs. velocity of \( P \) \quad vel. of \( P' \) of moving frame \( F \) coinciding with \( P \) \quad vel. of \( P \) relative to \( F \)

Fig. 15.28
15.5B Plane Motion of a Particle Relative to a Rotating Frame

Absolute acceleration --- rate of change of $\vec{v}_P$, w.r.t. $OXY$

$$\vec{a}_P = \ddot{\vec{v}}_P = \vec{\Omega} \times \vec{r} + \vec{\Omega} \times \dot{\vec{r}} + \frac{d}{dt}[(r)_{Oxy}]$$

(15.34)

$\vec{v}_P$

$$\begin{align*}
\vec{a}_P &= \vec{\Omega} \times \vec{r} + (r)_{Oxy} \quad \text{by Eq.(15.32)} \\
&= \vec{\Omega} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\vec{\Omega} \times (r)_{Oxy} + (r)_{Oxy}
\end{align*}$$

$\vec{a}_P$, $\vec{a}_C$: complementary acceleration

Coriolis acceleration $\quad \vec{a}_{P/F}$

(15.36)
15.5B Plane Motion of a Particle Relative to a Rotating Frame

\[ \vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/F} + \vec{a}_C \]

- \( \vec{a}_P \): abs. accel. of \( P \)
- \( \vec{a}_{P'} \): accel. of \( P' \) of moving frame \( F \) coinciding with \( P \)
- \( \vec{a}_{P/F} \): accel. of \( P' \) of relative to \( F \)
- \( \vec{a}_C \): accel. of \( P \) w.r.t. a frame in translation

Compared with Eq. (15.21) \( \vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \)

\[ \left| \vec{a}_C \right| = 2\Omega v_{P/F}, \text{ rotating } v_{P/F} \text{ through } 90^\circ \]

in the sense of rotation of the moving frame (Fig. 15.29)
15.5B Plane Motion of a Particle Relative to a Rotating Frame

Significance of \( \mathbf{a}_C \)

Abs. velocity of \( P \) at time \( t \) and \( t + \Delta t \)  (Fig. 15.30(b))

At \( t \), velocity components \( \mathbf{u} \), \( \mathbf{v}_A \), at \( t + \Delta t \) \( \mathbf{u}' \), \( \mathbf{v}'_A \)

- Fig 15.30(c), change in velocity during \( \Delta t \rightarrow \mathbf{R}'R', \mathbf{T}''T', \mathbf{T}'T' \)

- \( \mathbf{T}'' \) ---- change in the direction of \( \mathbf{v}_A \), \( \mathbf{T}''/\Delta t \) represents \( \mathbf{a}_A \) as \( \Delta t \rightarrow 0 \)

\[
\lim_{t \to 0} \frac{TT''}{\Delta t} = \lim_{t \to 0} v_A \frac{\Delta \theta}{\Delta t} = r\omega \omega = r\omega^2 = a_A
\]

\( \mathbf{R}'R' \) --- change in direction of \( \mathbf{u} \) due to the rotation

\( \mathbf{T}'T' \) --- change in magnitude of \( \mathbf{v}_A \) due to the motion of \( P \) along the rod

"combined effect of the relative motion of \( P \) and of the rotation of the rod"
15.5B Plane Motion of a Particle Relative to a Rotating Frame

- sum of these two $\rightarrow \overrightarrow{a_c}$

$$\overrightarrow{RR'} = u \Delta \theta, \ T''T' = v'_{A'} - v_A = (r + \Delta r) \omega - r \omega = \omega \Delta r$$

$$\lim_{t \to 0} \left( \frac{RR'}{\Delta t} + \frac{T''T'}{\Delta t} \right) = \lim_{t \to 0} \left( u \frac{\Delta \theta}{\Delta t} + \omega \frac{\Delta r}{\Delta t} \right) = u \omega + \omega u = 2 \omega u$$

Eqs. (15.33.), (15.36) $\rightarrow$ mechanism which contain parts sliding on each other

abs. and relative motions of sliding pins and collars.

$\overrightarrow{a_c}$ ---- useful in long-range projectiles, appreciably affected by the earth rotation.

* system of axes attached to the earth $\rightarrow$ not truly a Newtonian frame.

$\rightarrow$ rotating frame of ref., formulas derived in this section facilitate the study of the motion w.r.t. axes attached to the earth.