Dynamics

CHAPER 16.
Plane Motion of Rigid Bodies
: Forces and Acceleration

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16.0 Introduction

Kinetics of rigid bodies

- relations between the forces acting on a rigid body
  shape and mass of the body
  motion produced

Kinetics of the particle

- Mass can be concentrated in one point, and all the forces acting at that point

→ Shape of the body, exact location of the points of application of the force will now be accounted.
Motion of a body as a whole, motion about its mass center

- Approach

Consider rigid bodies as made of large number of particles,
Use the relations obtained in Chap. 14

Eq. (14.16) $\sum \vec{F} = m\vec{\bar{a}}$ external force, acceleration of G

Eq. (14.23) $\sum \overrightarrow{M_G} = \dot{\overrightarrow{H_G}}$ moments of external forces, angular momentum about G

16.0 Introduction

- Limits of the results in this chapter (except Sec 16.1A)

i) Plane motion

ii) Rigid bodies: only plane slabs which are symmetrical with respect to the ref. plane
(→ principal centroidal axis of inertia perpendicular to the ref. plane)

→ Plane motion of nonsymmetrical three-dimensional bodies
motion in three-dimensional space } Chap. 18

- Angular momentum of a rigid body → $\dot{\vec{H}}_G = \bar{I}\vec{\alpha}$

- Sec 16.1B

External forces acting on a rigid body $\xrightarrow{equivalent} m\vec{\bar{a}} + \bar{I}\,\vec{\alpha}$

- Principle of transmissibility

- Free-body-diagram and kinetic diagram

→ solution of all problems involving plane motion of rigid bodies

- Connected rigid bodies, involving translation, centroidal rotation, unconstrained motion

- Noncentroidal rotation, rolling motion, other partially constrained motion

16.1A Equation of Motion for a Rigid Body

- Rigid body acted upon by several external forces (Fig. 16.1)

Assume that the body is made of a large number of *n* of particles of mass

$$\Delta m_i (i = 1, 2, \cdots, n)$$

Apply the results in Chap. 14

- Motion of mass center G with respect to the Newtonian frame *Oxyz*

$$\sum \vec{F} = m\vec{\bar{a}} \qquad (16.1)$$

Motion of the body relative to the centroidal frame *Gx'y'z'*

$$\sum \overrightarrow{M_G} = \dot{\overrightarrow{H_G}} \qquad (16.2)$$

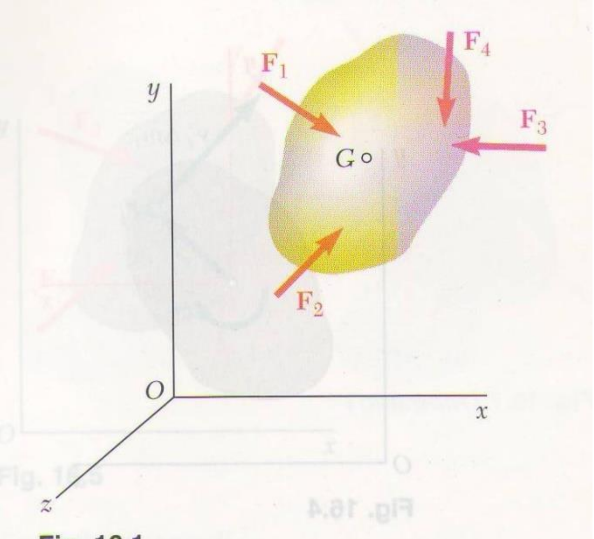


Fig. 16.1

$\dot{\overrightarrow{H_G}}$: angular momentum about G of the system of particles forming the body
→ angular momentum of the rigid body about G

16.1A Equation of Motion for a Rigid Body

Eqs. (16.1), (16.2)
: the system of the external forces $\xrightarrow{equipollent}$ system of $m\bar{\vec{a}}$ attached at G and $\dot{\vec{H}}_G$
the couple (Fig. 16. 3)

Eqs. (16.1), (16.2) → apply in the most general case of the motion of a rigid body

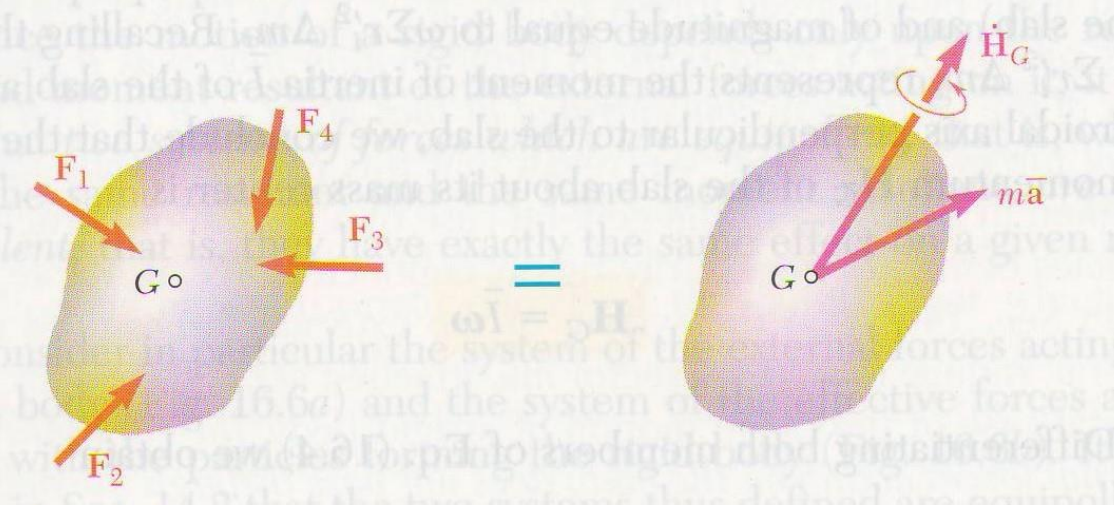


Fig. 16.3

- But, in this chapter, restricted to the plane motion
  Plane motion – each particle remains at a constant distance from a fixed ref. plane

- Rigid bodies

Only plane slabs and bodies which are symmetrical with respect to the ref. plane
Further studies → Chap. 18

16.1B Angular Momentum of a Rigid Body in Plane Motion

- Eq. (14.24) of Sec 14.1D (pp. 928)

$\overrightarrow{H_G}$ can be computed by taking the moments about G of the momenta of the particles in their motion with respect to either of the frames *Oxy* or *Gx'y'* (Fig 16.4)

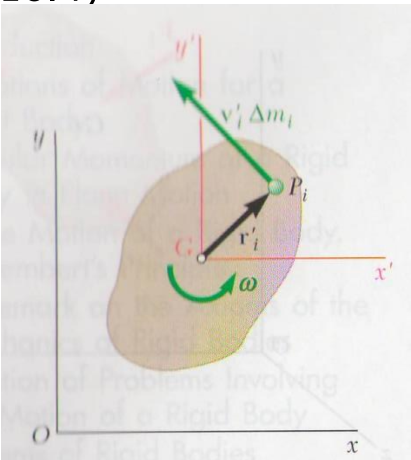


Fig. 16.4

$$\overrightarrow{H_G} = \sum (\vec{r}_i{}' \times \vec{v}_i{}' \Delta m_i) \qquad (16.3)$$

$\vec{r}_i{}'$: position vector

$\vec{v}_i{}' \Delta m_i$: linear momentum

} of the particle P_i relative to centroidal frame *Gx'y'*

- Since the particle belongs to the slab, $\vec{v}_i{}' = \vec{\omega} \times \vec{r}_i{}'$

$$\overrightarrow{H_G} = \sum \left[\vec{r}_i{}' \times \underbrace{\left(\vec{\omega} \times \vec{r}_i{}' \right) \Delta m_i}\right]$$

same direction as $\vec{\omega}$ (perpendicular to the slab)

$$\left|\vec{H}_G\right| = \omega \underbrace{\sum \vec{r}_i{}'^2 \Delta m_i}] \longrightarrow \overrightarrow{H_G} = \bar{I}\vec{\omega} \qquad (16.4) \xrightarrow{\text{Differentiate}} \dot{\overrightarrow{H_G}} = \bar{I}\dot{\vec{\omega}} = \bar{I}\vec{\alpha} \qquad (16.5)$$

Moment of inertia: $\bar{I}$

16.1B Angular Momentum of a Rigid Body in Plane Motion

rate of change of the angular momentum = a vector of the same direction of $\vec{\alpha}$, of magnitude $\bar{I}\vec{\alpha}$

→ valid results for the plane motion of rigid bodies which are symmetrical with respect to the ref. plane

However, do not apply to nonsymmetrical bodies or three-dim. motion.

16.1C Plane Motion of a Rigid Body

- rigid slab of mass m under several external forces $\vec{F}_1, \vec{F}_2, \cdots$ (Fig. 16.5)

Eq. (16.5) $\dot{\vec{H}}_G$ → (16.2), in scalar form

$$\sum \vec{F}_x = m\vec{\bar{a}}_x, \quad \sum \vec{F}_y = m\vec{\bar{a}}_y, \quad \sum \vec{M}_G = \bar{I}\alpha \quad (16.6)$$

- Eq. (16.6)

acceleration of G and its angular acceleration $\vec{\alpha}$ are easily obtained once { the resultant of external forces / their moment resultant about G }

with initial conditions, $\bar{x}, \bar{y}, \theta$ can be obtained by integration

→ Motion of the slab is completely defined by

$$\sum \vec{F} \text{ and } \sum \vec{M}_G$$

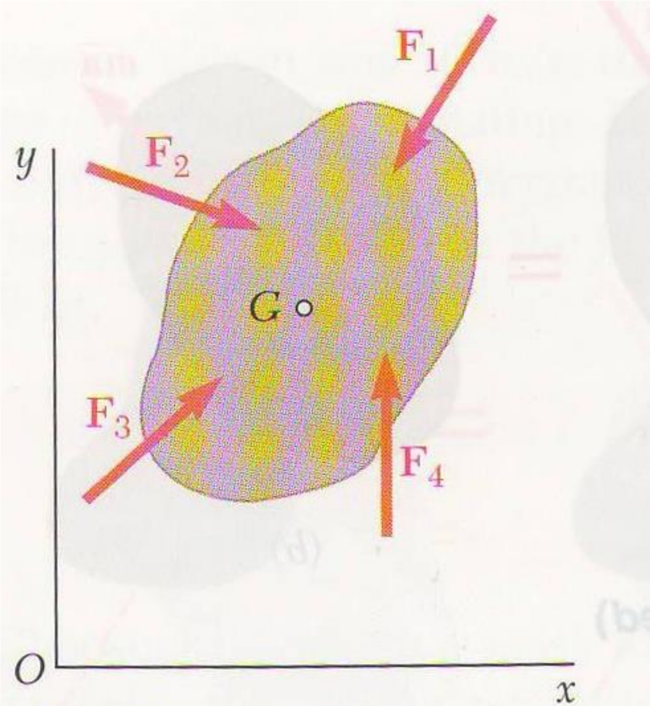


Fig. 16.5

16.1C Plane Motion of a Rigid Body

- System of particles which are not rigidly connected (Chap. 14)

specific external forces as well as internal forces

- Rigid body

only depends upon the resultant and moment resultant of external forces
→ two systems are equipollent, also equivalent

- Sec. 14.1 A (Fig. 16. 6)

(a) system of external forces
(b) system of effective forces associated with the particles } equipollent

But, since the particles form a rigid body → equivalent (red equal sign in Fig. 16.6)

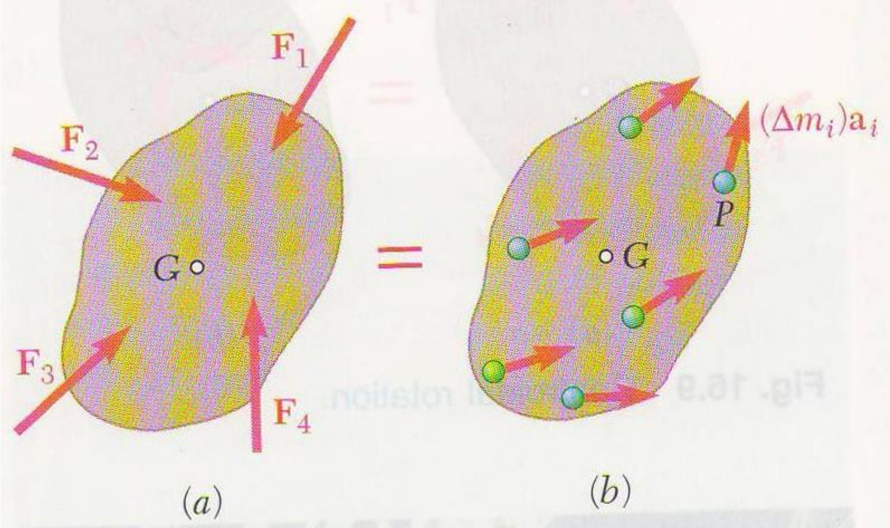


Fig. 16.6

16.1C Plane Motion of a Rigid Body

- Fig. 16. 7

effective forces → $m\vec{\bar{a}}$ attached at G and a couple of moment $\bar{I}\vec{\alpha}$

i) Translation

angular acceleration = 0,

effective forces → $m\vec{\bar{a}}$ attached of G (Fig. 16.8)

$$external\ forces \xleftrightarrow{equivalent} m\vec{\bar{a}}$$

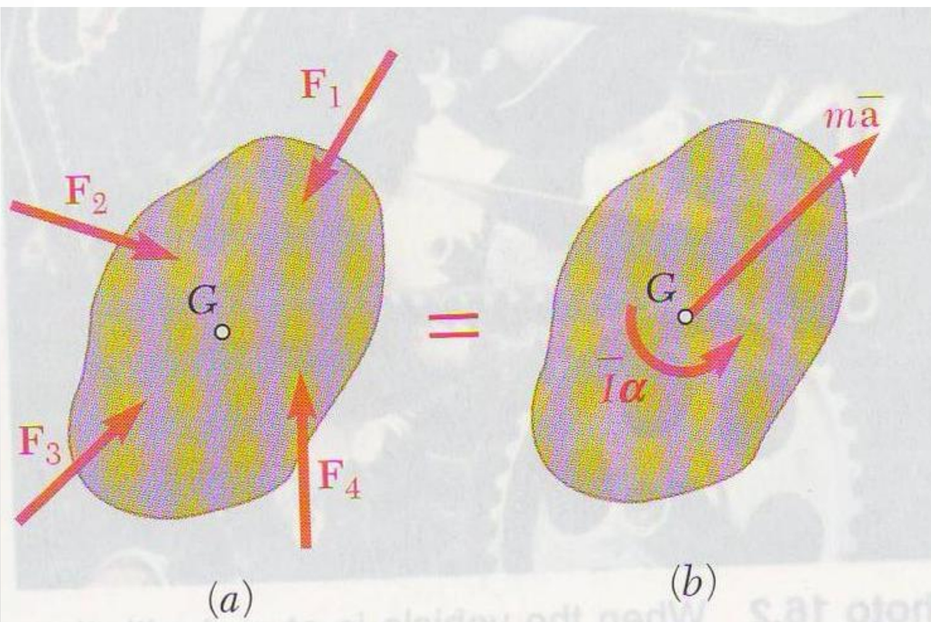


Fig. 16.7

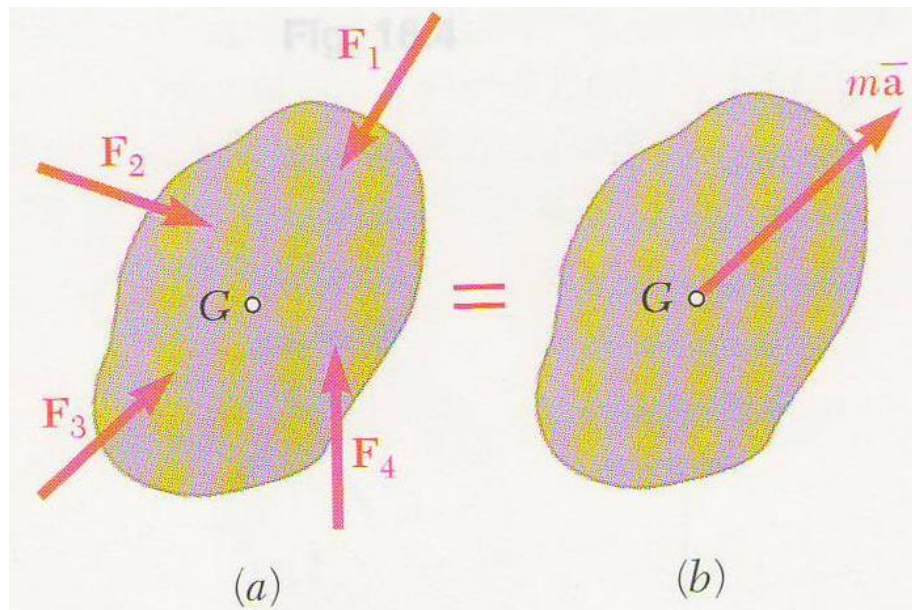


Fig. 16.8 Translation.

16.4 Plane Motion of a Rigid Body

ii) Centroidal Rotation
Rotating about a fixed axis perpendicular to the ref. plane and passing through G

→ centroidal rotation $\vec{\bar{a}} = 0$

effective forces → $\bar{I}\vec{\alpha}$ (Fig. 16.9) $external\ forces \xleftrightarrow{equivalent} couple\ of\ moment\ \bar{I}\vec{\alpha}$

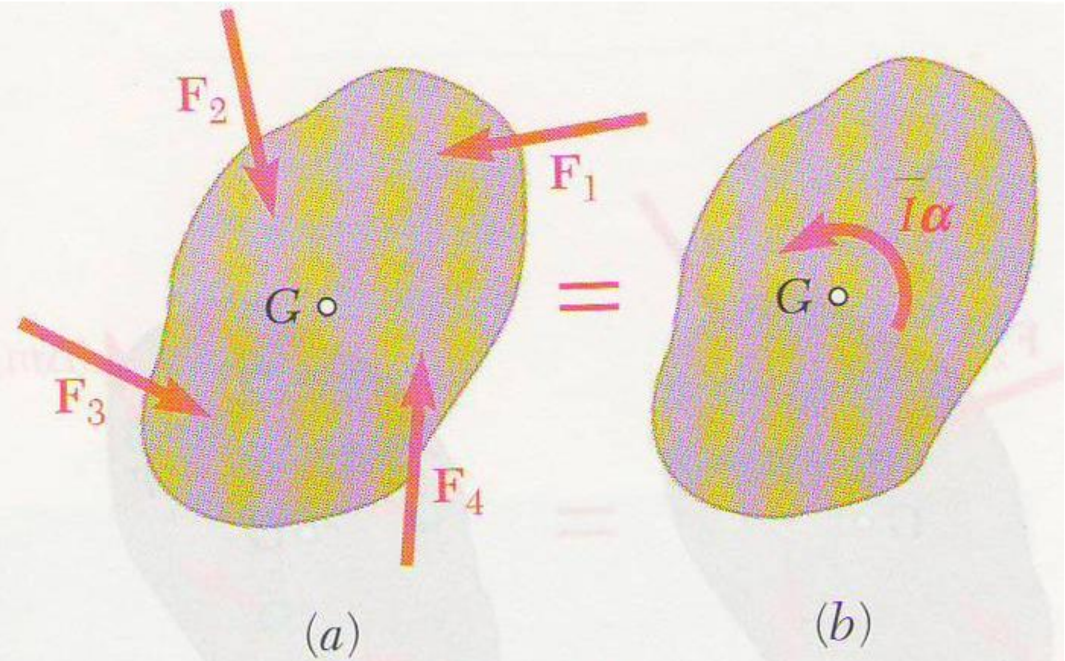


Fig. 16.9 Centroidal rotation.

16.1C Plane Motion of a Rigid Body

iii) General plane motion

general plane motion $\xleftrightarrow{\text{replaced}}$ sum of { translation, centroidal rotation }

mass center G is the ref. point ··· more restrictive than that of kinematics (Sec. 15.2A)

- First two eqns of Eq. (16.6)

already obtained in the general case of system of particles (not necessarily rigidly connected)

However, in the general case of the plane motion of a rigid body, the resultant of the external forces does NOT pass through G.

- Last eqn of Eq. (16.6)

still valid if the body were constrained through G

→ a rigid body in plane motion rotates about G as if G were fixed.

16.6 Solution of Problems Involving the Motion of a Rigid body

- Fundamental relation between the forces $\vec{F}_1, \vec{F}_2, \cdots$ and $\vec{\bar{a}}, \vec{\alpha}$

→ free-body-diagram eqn. (Fig. 16.7)

→ can be used to determine $\{$ $\vec{\bar{a}}, \vec{\alpha}$ from a given system of forces; forces which produce a given motion

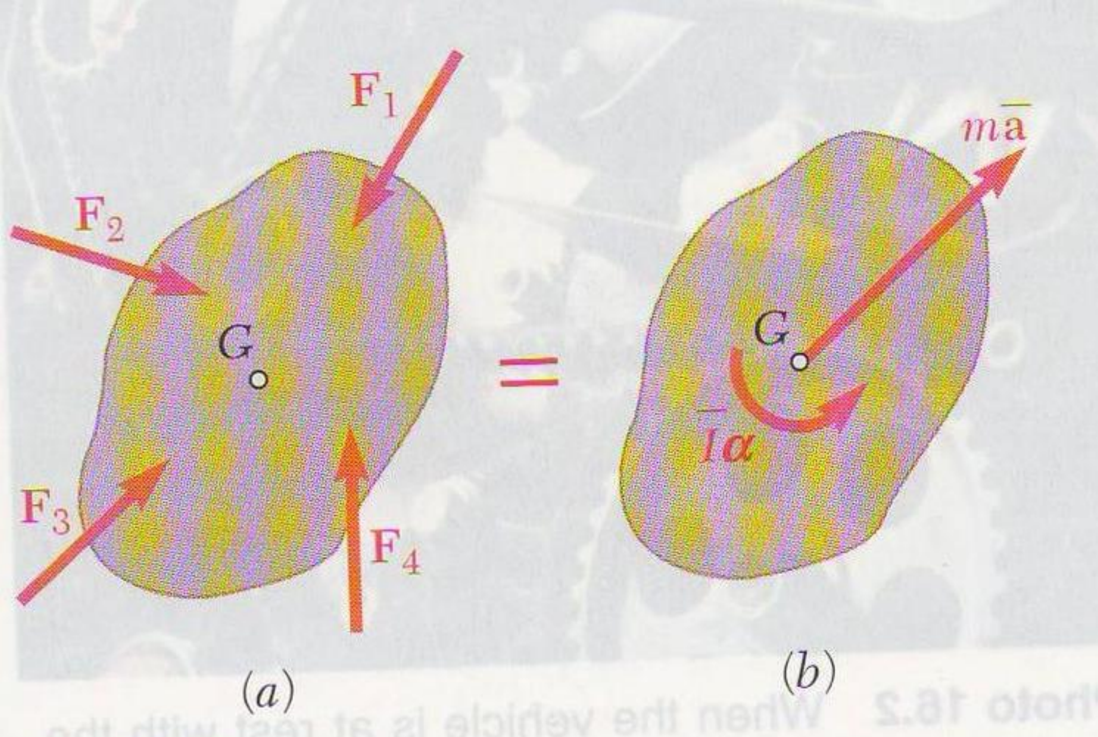


Fig. 16.7

16.1E Solution of Problems Involving the Motion of a Rigid body

- Sec. 16.1C
  fundamental relationship between the forces and $\vec{\bar{a}}$ of the mass center, and $\vec{\alpha}$ of the body
  ⟶ free-body diagram, kinetic diagram (Fig. 16.7)

- Statics
  Solution can be simplified by an appropriate choice of the point about which the moments of the forces are computed
  → derive the component or moment equations which fit best the solution from the fundamental relations

- FBD for rigid bodies: same steps as in Chap. 12, but draw forces at the location of action, label different dimensions when summing their moments

- KD for rigid bodies: $m\vec{\bar{a}}$ always on the mass center, and include $\bar{I}\vec{\alpha}$

- Steps for a pendulum in Fig. 16.10
  - isolate the body
  - define the axes
  - replace the constraints with support forces
  - applied forces/moments, body forces
  - label FBD with the dimensions

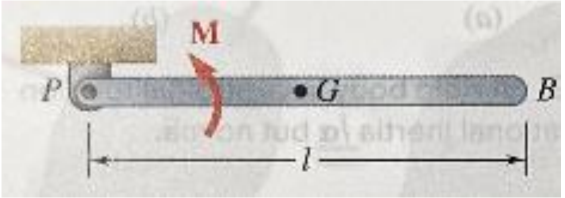
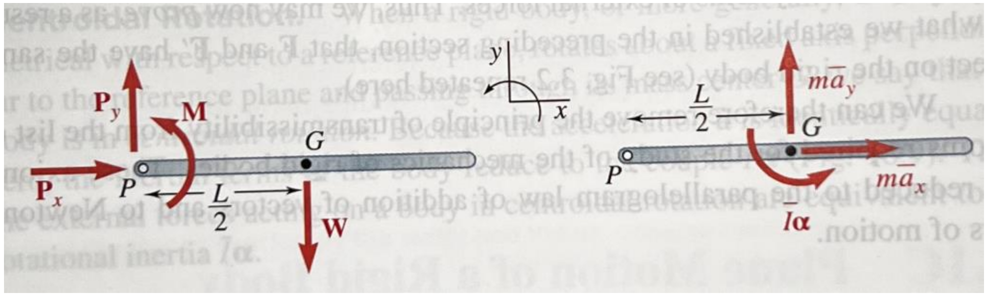


Fig. 16.10

16.1E Solution of Problems Involving the Motion of a Rigid body

Fig. 16.11



- Sum of moments about the mass center

$$+\odot\sum M_G = \bar{I}\alpha : M - P_y\left(\frac{L}{2}\right) = \bar{I}\alpha$$

- Alternatively, sum of moments about an arbitrary point P

$$+\odot\sum M_P = \bar{I}\alpha + m\bar{a}d_{\pm} : M - W\left(\frac{L}{2}\right) = \bar{I}\alpha + m\bar{a}_y\left(\frac{L}{2}\right) + m\bar{a}_x(0)$$

$\overline{d_{\pm}}$: Perpendicular distance from point P to the line of action of the resultant acceleration vector

- In statics: moment about a point P will be determined by a vector product

$$m\bar{a}\overline{d_{\pm}} = \vec{r}_{G/P} \times m\vec{a}$$

16.1E Solution of Problems Involving the Motion of a Rigid body

- Eq. (16.6) can be re-written as

$$\sum \vec{F}_x = m\vec{a}_x, \quad \sum \vec{F}_y = m\vec{a}_y,$$

and

$$\sum \vec{M}_G = \bar{I}\alpha \quad \text{or} \quad \sum M_P = \bar{I}\alpha + m\bar{a}d_{\pm} \quad \text{or} \quad \sum M_P = \bar{I}\alpha + \vec{r}_{G/P} \times m\vec{a}$$

16.1E Solution of Problems Involving the Motion of a Rigid body

- Advantage of free-body and kinetic diagram

  vectorial relationship between the forces applied and resulting linear and angular accelerations

i) pictorial representation → much clearer understanding of the effect of the forces
ii) two solution procedures
① analysis of kinematic and kinetic characteristics → free-body diagrams (Fig. 16.7)
② diagram → analyze various forces and vectors involved.

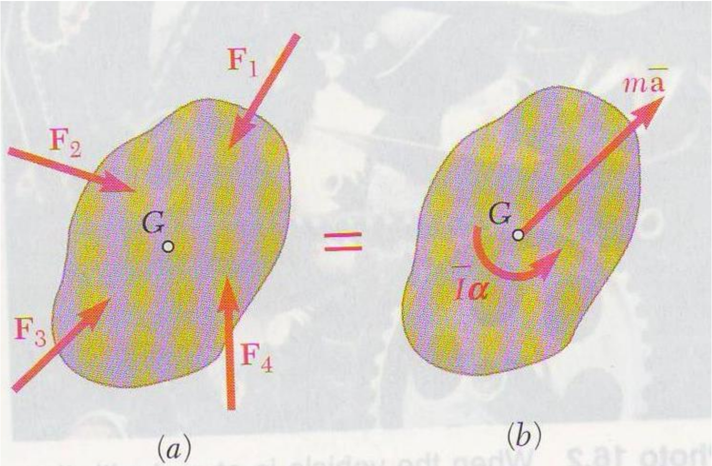


Fig. 16.7

16.1E Solution of Problems Involving the Motion of a Rigid body

iii) unified approach for the analysis of the plane motion of a rigid body

regardless of the particular type of motion involved

kinematics: may vary from one case to another

kinetics : consistently the same approach → diagram containing
- external $\vec{F}$
- $m\vec{\bar{a}}$ at G
- $\bar{I}\vec{\alpha}$

iv) resolution { translation, centroidal rotation } → basic concept for the study of mechanics

→ used again in Chap. 17 (method of work and energy, impulse and momentum)

v) extended to general three-dim. motion (Chap. 18)

16.1F Systems of Rigid Bodies

- The previous method → plane motion of several connected rigid bodies

for each point, a diagram similar to Fig. 16.7, eqns of motion obtained from these diagrams are solved simultaneously.

- Single diagram for the entire system (Sample Prob. 16.4)

internal forces can be omitted since they are equal and opposite forces → equipollent to zero

Eqns obtained by expressing that the system of external forces is equipollent to the system of internal terms → can be solved for the remaining unknowns (now NOT dealing with a single rigid body)

- Multiple rigid bodies:

$$\sum \vec{F} = \sum m_i \vec{\bar{a}}_i \quad \text{and} \quad \sum \overrightarrow{M_P} = \dot{\overrightarrow{H_P}}$$

$$\dot{\overrightarrow{H_P}} = \sum \bar{I}_i \vec{\alpha}_i + \sum m_i \vec{\bar{a}}_i \left(d_{\pm}\right)_i = \sum \bar{I}_i \vec{\alpha}_i + \sum \left[\left(\vec{r}_{G/P}\right)_i \times m_i \vec{\bar{a}}_i \right]$$

Sometimes, can be re-written as

$$\sum \vec{F} = \sum \vec{F}_{eff} \qquad \sum \overrightarrow{M_P} = \sum \left(\overrightarrow{M_P}\right)_{eff}$$

However, not possible to solve the problems involving more than 3 unknowns.

16.2 Constrained Plane Motion

- constrained motion

cranks (must rotate about a fixed axis), wheels (must roll without sliding)
connecting rods (must describe certain prescribed motion)

→ definite relation exists between $\begin{cases} \vec{\bar{a}} & \text{of mass center G} \\ \vec{\alpha} & \text{, angular acceleration} \end{cases}$

- Solution procedure

i) Kinematic analysis
Plane motion of a slender rod (Fig. 16.12)
length l, mass m, extremities connected to blocks of negligible mass
horizontal and vertical frictionless tracks, force $\vec{P}$ applied at A

from kinematics, $\vec{\bar{a}}$ can be determined from $\vec{P}$ given
→ wish to determine θ, ω, α required for this motion,
as well as N_A, N_B

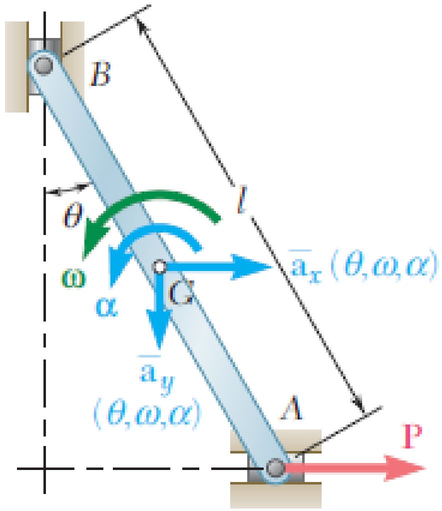


Fig. 16.12

16.2 Constrained Plane Motion

i) determine $\overline{a}_x, \overline{a}_y$ from kinematics

ii) apply FBD and KD (Fig. 16.13)

→ $\vec{P}, \overrightarrow{N_A}, \overrightarrow{N_B}$ can be determined.

[problem] Given $\vec{P}, \theta, \omega$, find $\overline{a}, \alpha, N_A, N_B$

(Sol) From kinematics express $\overline{\alpha_x}, \overline{\alpha_y},$ in terms of α

(First, express $\overrightarrow{\alpha_A}$ in terms of α . Then, express $\overline{\alpha_x}, \overline{\alpha_y}$ in terms of α .
Put the expressions into Fig. 16.13)
3 equations in terms of α, N_A, N_B and can be solved.

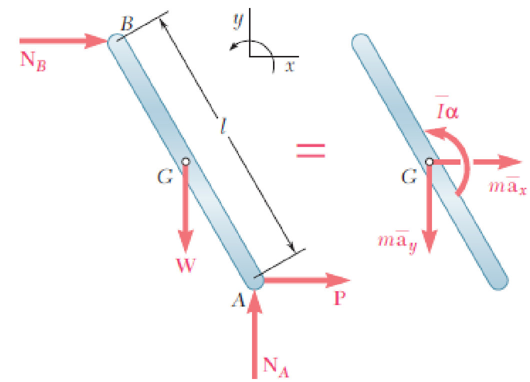


Fig. 16.13

- Several moving parts

the above approach can be used with each part of the mechanism

- two particular cases { translation: $\alpha = 0$
  centroidal rotation: $\overline{a} = 0$

- two other cases { noncentroidal rotation
  rolling motion of a disk/wheel } ← special comments

16.2 Constrained Plane Motion

I. Noncentroidal Rotation

rotation about a fixed axis which does not pass through its mass center

$$O \longleftarrow \bar{r} \longrightarrow G \qquad \text{(Fig. 16.14)}$$

ω, α : angular velocity and acceleration of line *OG*

$$\bar{a}_t = \bar{r}\alpha, \quad \bar{a}_n = \bar{r}\omega^2 \qquad (16.7)$$

→ ω, α of line *OG*: also represents the angular vel. and accel. of the body

→ Eq. (16.7): kinematic relation between
{ motion of the mass center *G*
{ motion of the body about *G*

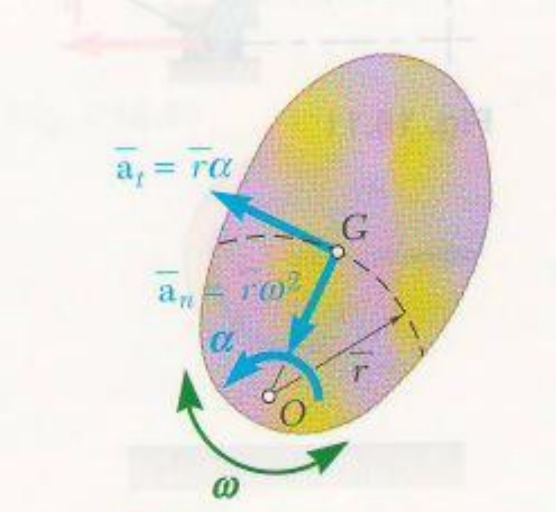


Fig. 16.14

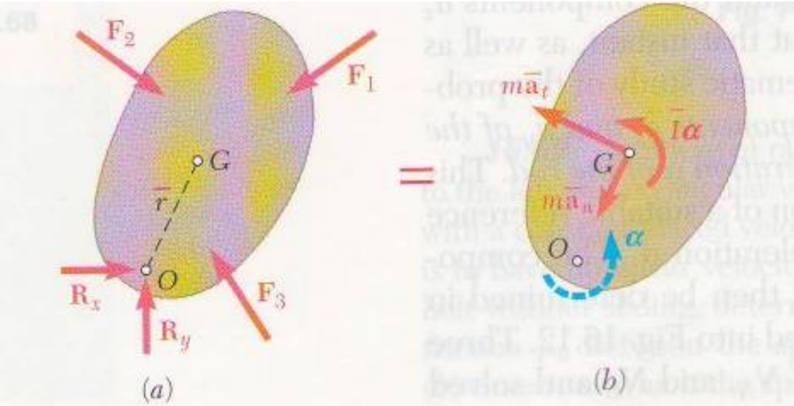


Fig. 16.15

16.2 Constrained Plane Motion

- Interesting relation

  moments about *O* from Figure 16.15

$$+\odot \sum M_0 = \bar{I}\alpha + (m\bar{r}\alpha)r = \underbrace{(\bar{I} + m\bar{r}^2)}_{I_0}\alpha$$

(parallel-axis theorem) (16.8)

$$\rightarrow \quad \sum M_0 = I_0 \alpha$$

Although (16.8) expresses an important relation between the moment of the external forces about the fixed point *O* and product $I_0\alpha$, we still need Eq. (16.1) to find the forces at *O*.

- Uniform rotation

α = 0, $-\bar{I}\alpha$ vanishes, $-m\overline{\overline{a}}_t$ vanishes

$-m\overline{\overline{a}}_n$ centrifugal force, represents the tendency to break away from axis of rotation

16.2 Constrained Plane Motion

II. Rolling Motion

- rolls without sliding

$\overline{\vec{a}}$ and $\overline{\vec{\alpha}}$ not independent. assuming balanced disk, $\bar{x}$ traveled by G during a rotation θ

$$\bar{x} = r\theta, \xrightarrow{diff.} \bar{a} = r\alpha \quad (16.9)$$

- system of the inertial terms $\xrightarrow{equivalent}$ $m\overline{\vec{a}}_t$ and $\bar{I}\vec{\alpha}$ (Fig. 16.17)

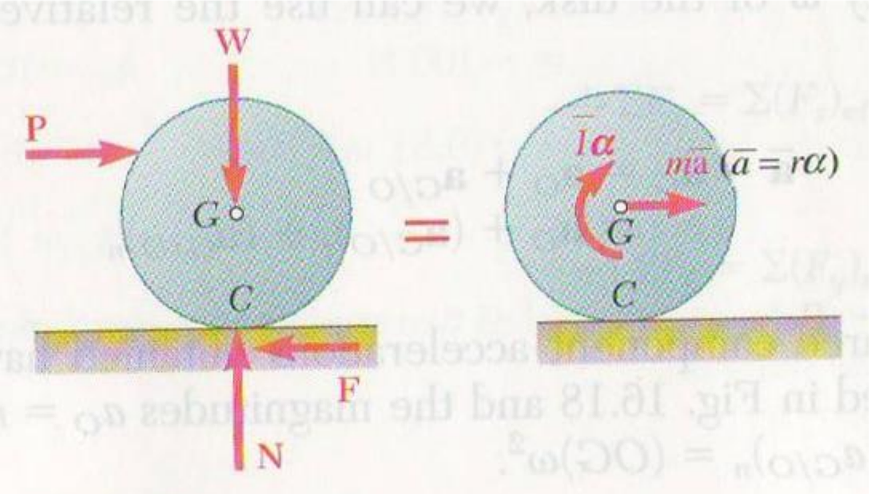


Fig. 16.17

16.2 Constrained Plane Motion

- rolls without sliding

no relative motion between the disk point and the ground.

regarding the friction force, a block resting on a surface

$$\left| friction \quad force \right| \leq \ \text{max}.\, F_m = \mu_s N$$

rolling disk, $\left| \vec{F} \right|$ can be independently of *N* by solving the eqns from Fig. 16.17

If sliding is impending, F reaches $\text{max}.\, F_m = \mu_s N$

If sliding and rolling, $F_k = \mu_k N, \ \bar{a}$ and α independent

- Three different cases

i) Rolling, no sliding : $F \leq \mu_s N, \quad \bar{a} = r\alpha$

ii) Rolling, sliding impending : $F = \mu_s N, \quad \bar{a} = r\alpha$

iii) Rotating and sliding : $F = \mu_k N,$ $\bar{a}$ and α independent

16.2 Constrained Plane Motion

→ First assume that rolling without sliding, find *F*

if $F \le \mu_s N$, assumption correct

$F > \mu_s N$, assumption incorrect, should be started again, assuming rotating and sliding

- Unbalanced disk: *G* does not coincide with *O* (geometric center)
  → (16.9) does not hold. However, a similar relation will hold

$$a_0 = r\alpha \qquad (16.10)$$

when it rolls without sliding,

For $\bar{a}$ in terms of ω, α , use the relative-acceleration formula

$$\vec{\bar{a}} = \vec{a}_G = \overrightarrow{a_0} + \overrightarrow{a_{G/O}} \qquad (16.11)$$

$$= \overrightarrow{a_0} + (\overrightarrow{a_{G/O}})_t + (\overrightarrow{a_{G/O}})_n$$

$$a_o = r\alpha, \quad \left|(\overrightarrow{a_{G/O}})_t\right| = (OG)\alpha, \quad \left|(\overrightarrow{a_{t/O}})_n\right| = (OG)\omega^2,$$

(Fig. 16.17)

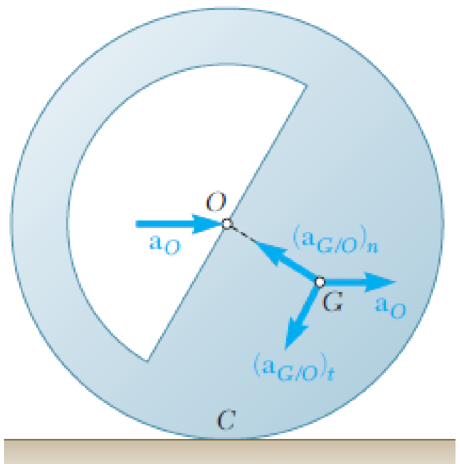


Fig. 16.17

Or by the relationship between two points on a rigid body

$$\vec{\bar{a}} = \overrightarrow{a_0} + \vec{\alpha} \times \vec{r}_{G/O} - \omega^2 \overrightarrow{r_{G/O}}$$

Q & A