Ch. 17
PLANE MOTION OF RIGID BODIES ENERGY AND MOMENTUM METHOD

Prof. SangJoon Shin

17.0 Introduction

Method of work and energy
Method of impulse and momentum → Plane motion of the rigid bodies

- Method of work and energy
  - Work of a force and a couple
  - Kinetic energy of a rigid body in plane motion
  - Problems involving displacement and velocities
  - Principle of conservation of energy

- Principle of impulse and momentum
  - Problem involving velocities and time
  - conservation of angular momentum

- Eccentric impact of rigid bodies
  - coefficient of restitution → Colliding bodies moving freely / Colliding bodies partially constrained

17.1A Principle of Work and Energy for a Rigid Body

- Main advantage → { Work of a force / Kinetic energy of a particle } → Scalar quantities

- Assumption------ rigid body is made of a large number n of particles of mass Δm_i

$$T_1 + U_{1\to2} = T_2 \qquad (17.1)$$

$$T = \frac{1}{2}\sum_i^n \Delta m_i v_i^2$$ → Positive scalar quantities

$U_{1\to2}$: work of all the forces acting on the various particles of the body (internal + external)

→ Total work of the internal forces =0

[Example] two particle A,B $\vec{F}$ and $-\vec{F}$ (Fig. 17.1)

displacement $d\vec{r}$, $d\vec{r}'$ different, but the component along AB must be equal.
→ total work of the internal forces =0

⇨ $U_{1\to2}$ reduces to the work of the external forces only

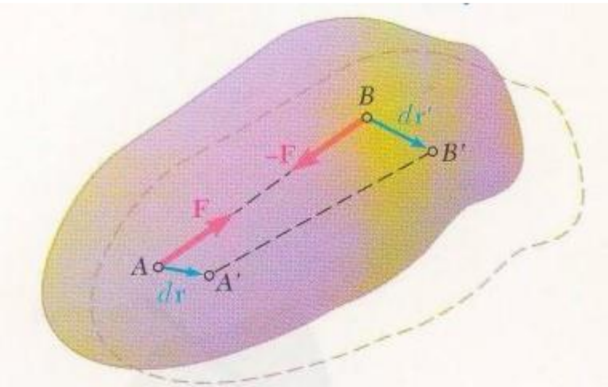

Fig. 17.1

17.1B Work of Forces Acting on a Rigid Body

$$U_{1\to 2} = \int_{A_1}^{A_2} F \circ d\vec{r} \qquad (17.3)$$

$$= \int_{s_1}^{s_2} (F \cos\alpha) ds \qquad (17.3')$$

α: the angle which the force forms with direction of motion

s : variable of integration, which measures the distance traveled by A

- Work of a couple

  …….. don't need to consider separately the work of each of the two forces forming the couple

Fig. 17.2 ….two forces $\vec{F}$ and $-\vec{F}$ forming a couple of moment $\vec{M}$ small displacement

$A \to A'$ $B \to B''$

→ Two part

1. A, B undergo equal displacement $d\vec{r}_1$

   ……..sum of the work of $\vec{F}$ and $-\vec{F}$ is zero

2. A′ remains fixed. $B' \to B''$, $d\vec{r}_2$, $ds_2 = rd\theta$

   ……only $\vec{F}$ works, $dU = Fds_2 = \underline{Frd\theta}$ → M

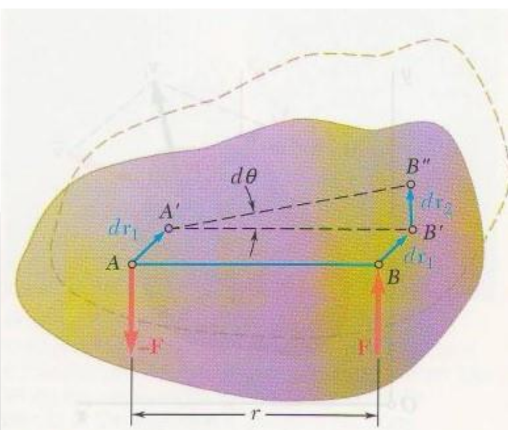

Fig. 17.2

$$dU = Md\theta \qquad (17.4)$$

$$U_{1\to 2} = \int_{\theta_1}^{\theta_2} Md\theta \qquad (17.5)$$

17.1B Work of Forces Acting on a Rigid Body

When $\vec{M}$ is constant, $U_{1\to2} = M(\theta_2 - \theta_1)$ (17.6)

No work……..
- Forces applied to a fixed point
- Acting in a direction perpendicular to the displacement

[Ex] - Reaction at a frictionless pin
- Reaction of a frictionless surface
- Weight of a body when its c.g. moves horizontally

+ a rigid body rolls without sliding a friction force does no work

(∵ velocity $\vec{v}_c$ of the point contract *C* is zero)

$$dU = Fds = F(v_c dt) = 0$$

17.1C Kinetic Energy of a Rigid Body in Plane Motion

$\vec{v}_i$ sum of
- $\vec{\bar{v}}$ of the mass center *G*
- $\vec{\bar{v}}_i'$ relative to $Gx'y'$ attached at *G*

- Kinetic energy of the rigid body

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n}\Delta m_i v_i'^2 \qquad (17.7)$$

$$v_i' = r_i' \, \omega$$

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\Big(\underbrace{\sum_{i=1}^{n} r_i'^2 \Delta m_i}_{\bar{I}}\Big)\omega^2 \qquad (17.8)$$

$$= \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\,\omega^2 \qquad (17.9)$$

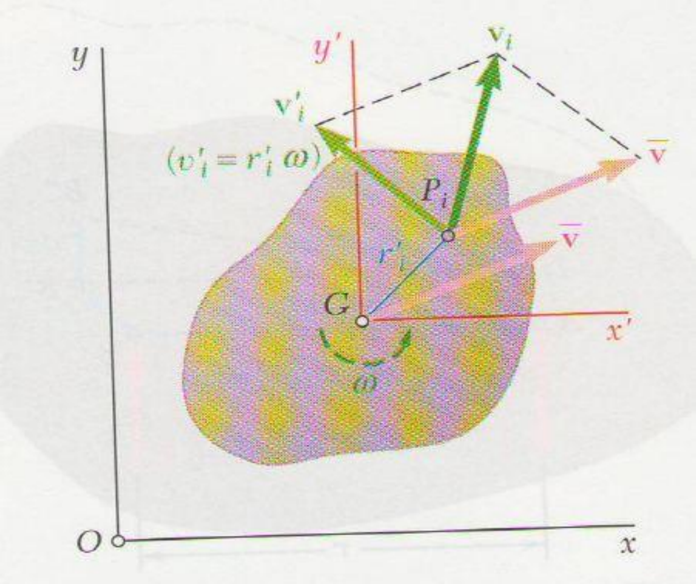

Fig. 17.3

- translation ($\omega = 0$) ⟶ $\frac{1}{2}m\bar{v}^2$

  centroidal rotation ($\bar{v} = 0$) ⟶ $\frac{1}{2}\bar{I}\,\omega^2$

- Rigid body in general plane motion
  1. $\frac{1}{2}m\bar{v}^2$ associated with the motion of *G*
  2. $\frac{1}{2}\bar{I}\,\omega^2$ associated with rotation about *G*

17.1C Kinetic Energy of a Rigid Body in Plane Motion

- Noncentroidal Rotation

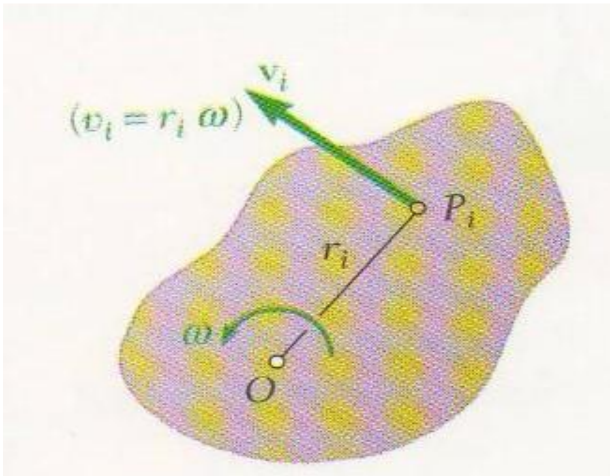

Fig. 17.4

$$T = \frac{1}{2}\sum_{i=1}^{n}\Delta m_i \left(v_i'\right)^2 = \frac{1}{2}\sum_{i=1}^{n}\Delta m_i \left(r_i\omega\right)^2 = \frac{1}{2}\underbrace{\left(\sum_{i=1}^{n} r_i^2 \Delta m_i\right)}_{I_0}\omega^2$$

$$T = \frac{1}{2} I_0 \omega^2 \qquad (17.10)$$

-→ applicable only in noncentroidal rotation, prefer to use Eq.(17.9)

17.1D Systems of Rigid Bodies

- Add all the kinetic energies of all the particles, and al the forces involved

$$T_1 + U_{1\to 2} = T_2 \qquad (17.11)$$

$U_{1\to 2}$: all the forces (internal + external)

- Problems involving
  - Pin-connected members
  - Blocks and pulleys connected by inextensible cords
  - Meshed gears

→ work of the internal forces is zero, $U_{1\to 2}$ reduces to work of the external force
($\because$ forces in each pair move through equal distance)

17.1E Conservation of Energy

- Work of a conservative forces ⟶ A change in potential energy

  [Example] – weigh of a body, force exerted by a spring

- Principle of work and energy ⟶ Modified form

$$T_1 + V_1 = T_2 + V_2 \qquad (17.12)$$

---sum of the kinetic energy and the potential energy of the system remains constant

(k.e = transition term $\frac{1}{2}m\bar{v}^2$, rotational term $\frac{1}{2}\bar{I}\omega^2$)

[Example] slender rod AB

length L, mass m, extremeties connected blocks of negligible mass sliding along horizontal and vertical tracks released with no initial velocity from a horizontal position (Fig. 17.5 (a))

⟶ Angular velocity after rotating θ (Fig 17.5 (b))?

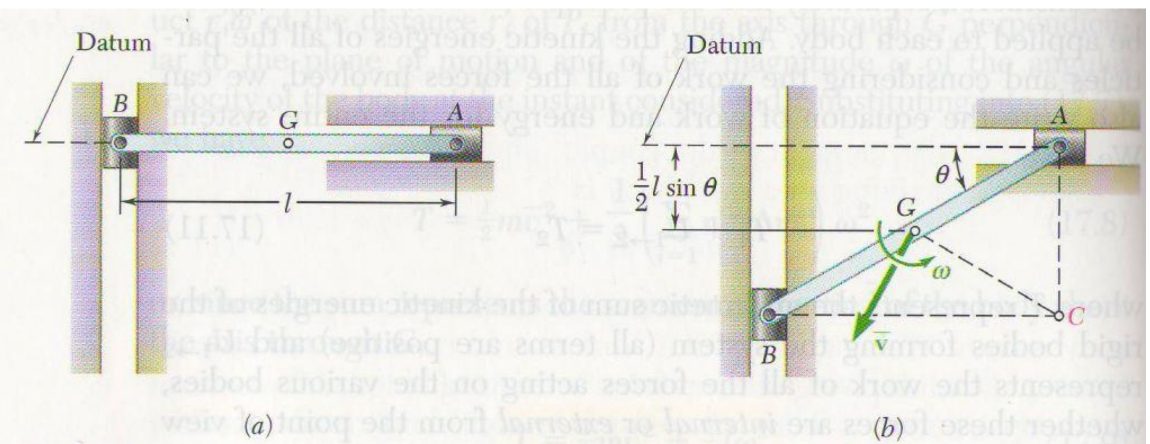

Fig. 17.5

17.1D Conservation of Energy

[sol]

$$T_1 = 0 \quad V_1 = 0$$

After rotating θ, G is at $\frac{1}{2} l \sin\theta$ below the reference level,

$$V_2 = -\frac{1}{2} W l \sin\theta = -\frac{1}{2} mgl \sin\theta$$

At this instant, instantaneous center of rotation at C,

$$\overline{C}\overline{G} = \frac{1}{2} l, \quad \overline{v}_2 = \frac{1}{2} l\omega$$

$$T_2 = \frac{1}{2} m\overline{v}_2 + \frac{1}{2} \overline{I}\omega_2^2 = \frac{1}{2} m \left(\frac{1}{2} l\omega \right)^2 + \frac{1}{2} \left(\frac{1}{12} ml^2 \right) \omega_2^2 \quad = \frac{1}{2} \frac{ml^2}{3} \omega^2$$

$$T_1 + V_1 = T_2 + V_2 \qquad 0 = \frac{1}{2} \frac{ml^2}{3} \omega^2 - \frac{1}{2} mgl \sin\theta$$

$$\omega = \left(\frac{3g}{l} \sin\theta \right)^{1/2}$$

17.1D Conservation of Energy

- To determine reactions at fixed axles, rollers, or sliding blocks, supplemented by Newton's 2nd law

Coupled analysis ………combined use of
- the method of work and energy
- The principle of equivalence of the external forces/moments and inertial terms

17.1E Power

$$Power = \frac{dV}{dt} = \vec{F} \cdot \vec{v} \tag{13.13}$$

- Rigid body rotating at $\vec{\omega}$ and acted upon by a couple of moment $\vec{M}$

$$Power = \frac{dV}{dt} = \frac{Md\theta}{dt} = M\omega \tag{17.13}$$

- The various units used to measure power, such as the watt and the horsepower, were defined in sec.13.1D

17.2A Principle of Impulse and Momentum for the Plane Motion of a Rigid Body

- Principle of impulse and momentum………well adapted to the problems involving time and velocity

  - only practicable method for impulsive motion or impact

- Section 14.1C …… the system of the momenta of the particle, at t_1

+ the system of the impulses of the external forces at $t_1 \sim t_2$

$\xleftrightarrow{\text{equipollent}}$ system of the momenta at t_2

For rigid body, $\xleftrightarrow{\text{equivalent}}$

System Momenta$_1$ + System External Impulse$_{1\text{->}2}$ = System Momenta$_2$ (17.14)

where,

$$\vec{L} = \sum_{i=1}^{n} \vec{v}_i \Delta m_i = m\vec{\bar{v}}; \vec{H}_G = \sum_{i=1}^{n} \vec{r}_i' \times \vec{v}_i \Delta m_i = \bar{I}\,\vec{\omega}$$

……. system of the momenta $\vec{v}_i \Delta m_i$

$\xleftrightarrow{\text{equivalent}}$ (Fig. 17.7)
- linear momentum $m\vec{\bar{v}}$ vectors attached at G
- Angular momentum couple $\bar{I}\,\vec{\omega}$

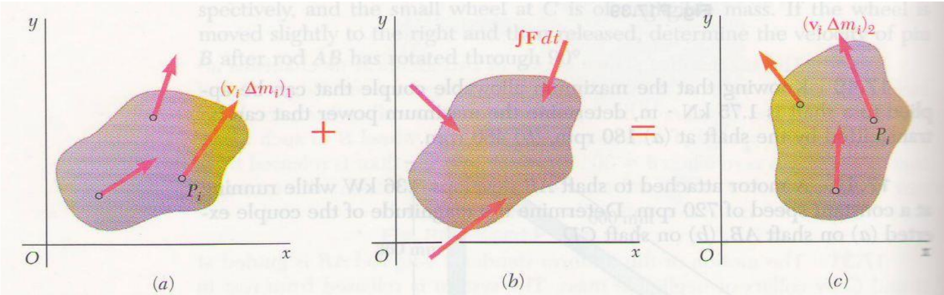

Fig. 17.6

17.2A Principle of Impulse and Momentum for the Plane Motion of a Rigid Body

Fig. 17.6 (a),(c) ⟶ Fig. 17.8

……. Impulse-momentum diagram: visual representation of Eq. (17.14)

- Three eqns of motion from Fig. 17.8

two…..summing and equating the *x* and *y* component

third…..summing and equating the moments about any given point

(coordinate is either fixed or translated with *G*)

Keep the same position relative to the coordinate axes during the interval of time

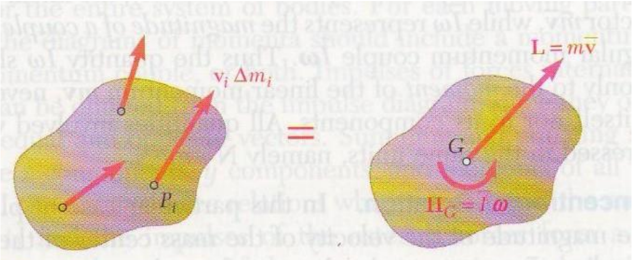

Fig. 17.7

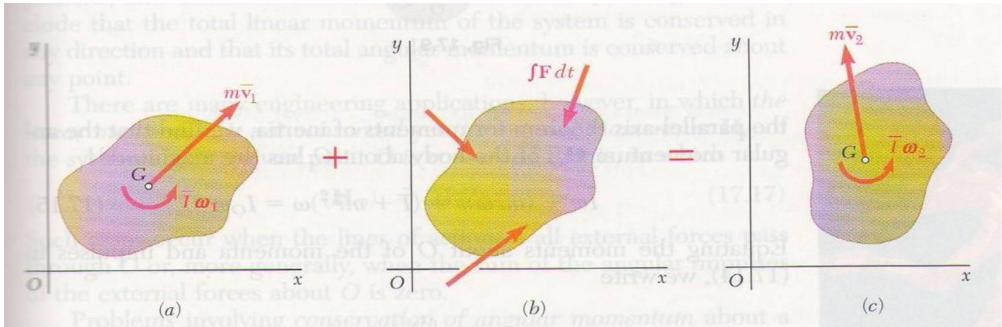

Fig. 17.8

17.2A Principle of Impulse and Momentum for the Plane Motion of a Rigid Body

- Sum of the moments about an arbitrary point *P*

$$\bar{I}\omega_1 + m\bar{v}_1 d_{\pm 1} + \sum \int_{t_1}^{t_2} M_P dt = \bar{I}\omega_2 + m\bar{v}_2 d_{\pm 2} \qquad (17.14')$$

$d_{\pm}$: perpendicular distance from *P* to the line of action of linear velocity of *G*

- Sum of the moments about C.G. of the body

$$\bar{I}\omega_1 + \sum \int_{t_1}^{t_2} M_G dt = \bar{I}\omega_2 \qquad (17.14'')$$

- Careful about avoid adding linear and angular momenta indiscriminately
- $\bar{I}\vec{\omega}$ should be added only to the moment of $m\vec{\bar{v}}$

17.2A Principle of Impulse and Momentum for the Plane Motion of a Rigid Body

- Noncentroidal Rotation

$$\bar{v} = \bar{r}\omega, \quad m\bar{v} = m\bar{r}\omega,$$

- Moments about *O* (Fig 17.9)

$$\left|\vec{H}_0\right| = \bar{I}\omega + (m\bar{r}\omega)\bar{r} = (\bar{I} + m\bar{r}^2)\omega = I_o\omega \qquad (17.15)$$

- Moments about *O* of the momenta and impulses in Eq.(17.14)

$$I_o\omega_1 + \sum\int_{t_1}^{t_2} M_o dt = I_o\omega_2 \qquad (17.16)$$

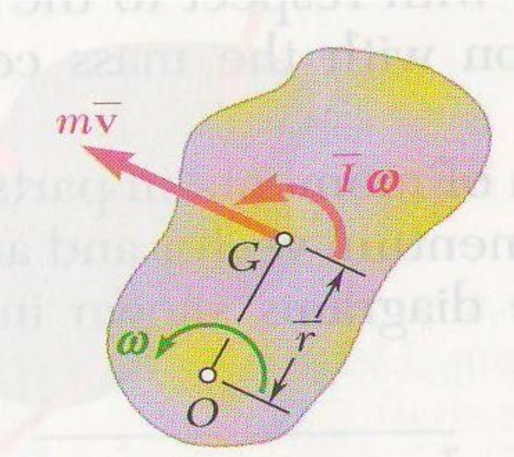

Fig. 17.9

⟶ can be used w.r.t the instantaneous axis of rotation under certain conditions

All problems of plane motion should be solved by the general method described earlier

17.2B Systems of Rigid Bodies

- Apply the principle of impulse and moment to each body separately (Sample Prob. 17.7)

- No more than three unknowns ⟶ Apply the principle to the system as a whole

  - Impulses of internal forces can be omitted

  - each equation should be checked to make sure that consistent units have been used.(Sampled Prob. 17.9 - 17.13)

17.2C Conservation of Angular Momentum

- No external force → System of momenta at t_1

  ←equipollent→ System of momenta at t_2

  ⇨ total linear momentum is conserved at any direction; total angular momentum is conserved

- Many engineering application: linear momentum is NOT conserved; Angular momentum conserved

$$\left(\vec{H}_P\right)_1 = \left(\vec{H}_P\right)_2 \qquad (17.17)$$

lines of action of all external forces pass….through *P*

sum of angular impulses of the external forces about *P* is zero

17.2C Conservation of Angular Momentum

- Problems involving the conservation of angular momentum about a point *P*
  - drawing impulse-momentum diagrams as described earlier
  - obtain Eq. (17.17) by summing and equating moments about *P* (Sample Prob. 17.9)

- (Sample Prob. 17.11) obtain two additional eqns. by summing and equating *x* and *y* components of the linear momentum. Then use those to determine two unknown linear impulses.

17.3 Eccentric Impact

- Central impact---- mass centers of the two colliding bodies are on the line of impact
- Eccentric Impact of two rigid bodies

---$\vec{v}_A$ and $\vec{v}_B$ before impact of the two points of contact *A*, *B* (Fig. 17.10 (a))

- Period of deformation ------ at its end, $\vec{v}_A$ and $\vec{v}_B$ will have equal components long the line of impact (Fig. 17.10 (b))

- Period of restitution -----at its end, $\vec{v}_A'$ and $\vec{v}_B'$ (Fig. 17.10 (c))

- Coefficient of restitution

$$e = \frac{\int R dt}{\int P dt} \quad (17.18)$$

Sec 13.4……..relative velocities along the line of impact

$$(\vec{v}_B')_n - (\vec{v}_A')_n = e[(v_A)_n - (v_B)_n] \quad (17.19)$$

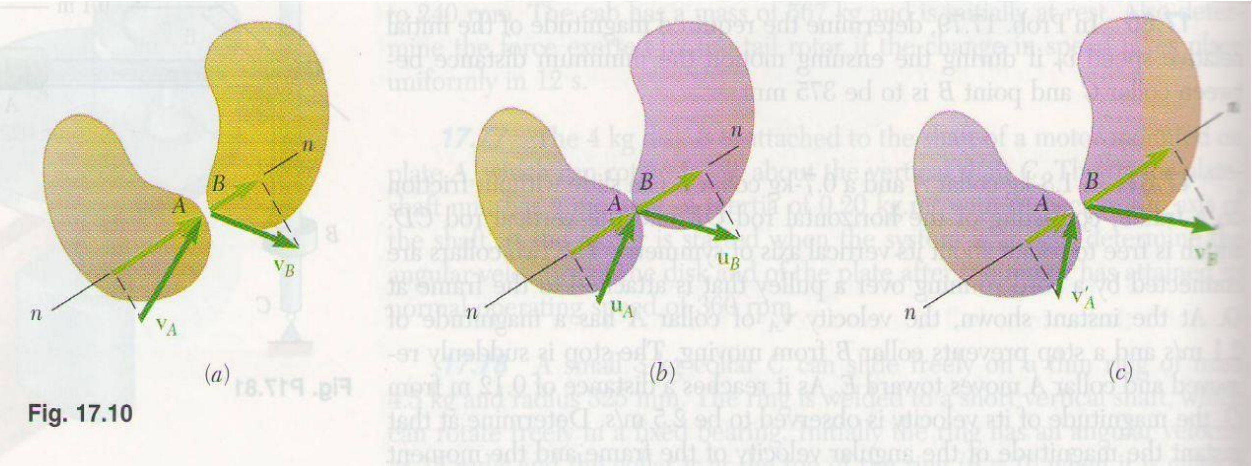

Fig. 17.10

17.3 Eccentric Impact

- For rigid body only impulsive force exerted during the impact are applied at *A*, *B*

Fig. 17.11 ….. Momentum and impulse diagram for the body *A* (period of deformation)

$\vec{\bar{v}}, \vec{\bar{u}}$ ………. Velocity of the mas center at the beginning and end of period of deformation

$\vec{\omega}, \vec{\omega}^*$ ………. Angular velocities

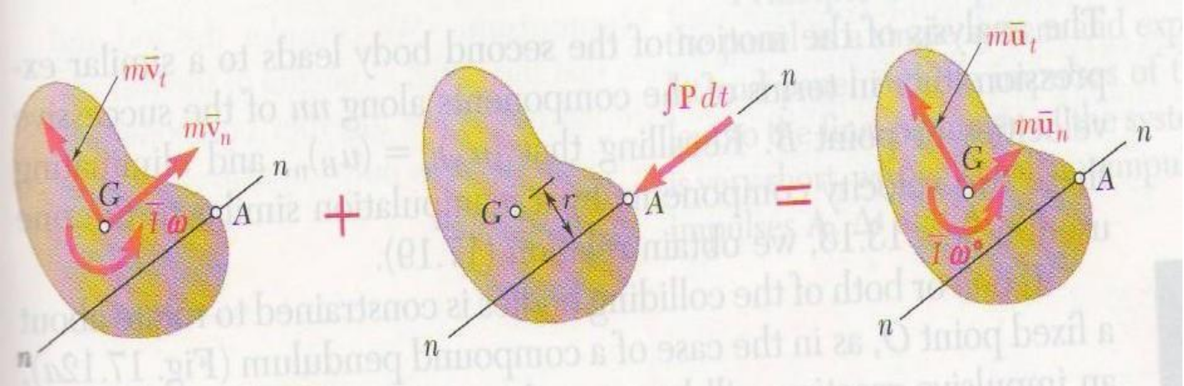

Fig. 17.11

- Components of the momenta along the line of impact *nn*

$$m\bar{v}_n - \int P dt = m\bar{u}_n \quad (17.20)$$

-moments about *G*

$$\bar{I}\omega - r\int P dt = \bar{I}\omega^* \quad (17.21)$$

r :perpendicular distance from *G* to the line of impact

- Period of restitution

$$m\bar{v}_n - \int R dt = m\bar{v}_n{}' \quad (17.22)$$

$$\bar{I}\omega^* - r\int R dt = \bar{I}\omega' \quad (17.23)$$

$\vec{\bar{v}}', \vec{\omega}'$ velocity of *G*, angular velocity after impact

17.3 Eccentric Impact

(17.20),(17.22) ⟶ (17.18)
(17.21),(17.23) ⟶ (17.18)

$$e = \frac{\bar{u}_n - \bar{v}_n'}{\bar{v}_n - \bar{u}_n} \qquad e = \frac{\omega^* - \omega'}{\omega - \omega^*} \times r \tag{17.24}$$

(+)

$$e = \frac{\bar{u}_n + r\omega^* - (\bar{v}_n' + r\omega')}{\bar{u}_n + r\omega - (\bar{u}_n + r\omega^*)} \tag{17.25}$$

$$\bar{v}_n + r\omega = (v_A)_n \quad \bar{u}_n + r\omega^* = (u_A)_n \quad \bar{v}_n' + r\omega' = (v'_A)_n$$

$$e = \frac{(v_A)_n - (v_A')_n}{(v_A)_n - (u_A)_n} \tag{17.26}$$

- Second body ⟶ Similar expression for *e* in terms of the components along *nn* Of the successive velocities of *B*.

and eliminating these two velocities ⟶ (17.19)

17.3 Eccentric Impact

- Constrained to rotate about *O* ……. Compound pendulum (Fig. 17.12(a))

⟶ impulsive reaction will be exerted at *O* (Fig. 17.12. (b))

Apply Eq. (17.16) to the { period of deformation / period of restitution

$$\bar{I}\omega - r\int P dt = \bar{I}\omega^* \qquad (17.27)$$

$$\bar{I}\omega^* - r\int P dt = \bar{I}\omega' \qquad (17.28)$$

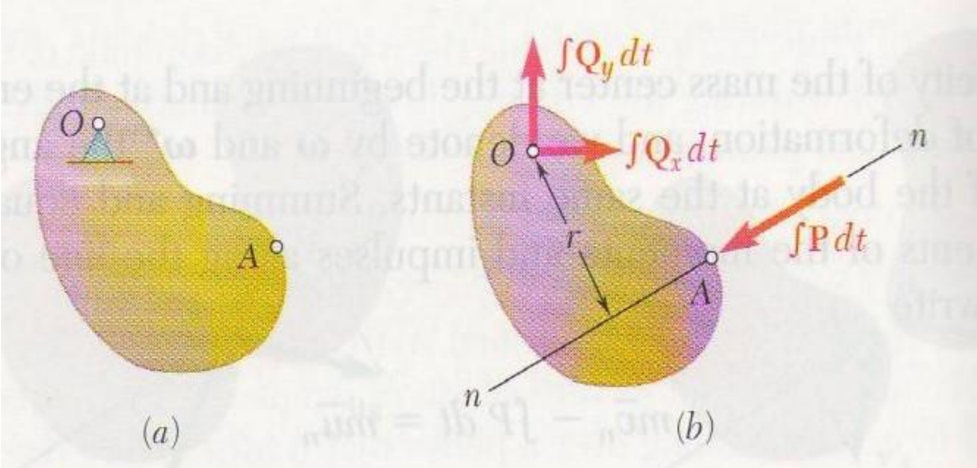

Fig. 17.12

:perpendicular distance from the fixed point *O* to the line of impact

(17.27),(17.28) ⟶ (17.18)

$$e = \frac{\omega^* - \omega'}{\omega - \omega^*} = \frac{r\omega^* - r\omega'}{r\omega - r\omega^*} = \frac{(v_A)_n - (v_A')_n}{(v_A)_n - (u_A)_n}$$

Components along *nn* of the successive velocities at *A*

17.3 Eccentric Impact

- Eq. (17.26) still holds

- Eq. (17.19) remains valid when constrained to rotate about *O*

  (17.19) in conjunction with one or several other equations obtained by the principle of impulse and momentum (Sample Prob. 17.11, 17. 13)