Linear System and State Space Formulation

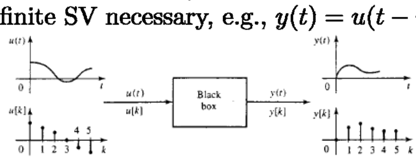
Chen Ch. 2-5

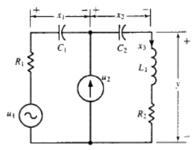
Dongjun Lee (이동준)

Department of Mechanical & Aerospace Engineering
Seoul National University

Basic Concepts

- **System:** a collection of elements, that interacts with its environment via a set of input $u \in \Re^m$ and output $y \in \Re^p$.
  - Causal system (current output depends only on past/current inputs, not future (e.g., $y = \frac{1}{s}u$) vs acausal system (e.g., $y = su$).
  - Static (memoryless) system $y(t) = h(u(t))$ vs dynamic (with memory) system $y(t) = h(t, x(t_o), u([t_o; t])$.
- **State:** $x(t_1) \in \Re^n$ of a causal system at t_1 is the informaiton needed, together with the input $u : [t_1, t_2]$, to uniquely define the output y at t_2.
  - State $x(t_1)$ at t_1 contains all the information related to the past input history of $u : [-\infty, t_1]$ to define $y(t)$, $\forall t \geq t_1$.
  - State is not unique (e.g., $x_1 + x_2, x_1 - x_2, x_3$).
  - Dimension of system = dimension of minimal state vector $x \in \Re^n$.
- Finite-dimensional system (only finite SV needed) vs infinite-dimensional system (infinite SV necessary, e.g., $y(t) = u(t - \tau)$).





Dynamical Systems

- State-space system representation:

  (Continuous-time): $\dot{x}(t) = f(x(t), u(t), t), \quad y(t) = h(x(t), u(t), t)$
  (Discrete-time): $x(k+1) = f(x(k), u(k), k), \quad y(k) = h(x(k), u(k), k)$

  where $t \in \Re$ is time, $k \in \mathcal{Z}$ is discrete-time index, $x \in \Re^n$ is state vector, $u \in \Re^p$ is input vector, and $y \in \Re^m$ output vector, with f, h being state dynamics map and output map.

- **State transition map:** given $x_o = x(t_o)$ and $u([t_o, t_1])$, $x(t_1)$ is uniquely determined, i.e., $\boxed{x(t_1) = s(t_1, t_o, x_o, u([t_o; t_1])}$

  - Semi-group property: $\forall t_2 \geq t_1 \geq t_o$, $x_o \in \Re^n$ and $u([t_o; t_2])$,

  $$\begin{aligned} s(t_2, t_o, x_o, u([t_o; t_2]) &= s(t_2, t_1, s(t_1, t_o, x_o, u(t_o, t_1)), u([t_1; t_2]) \\ &= s(t_2, t_1, x_1, u([t_1; t_2]) \end{aligned}$$

  - Ex) solution by integration: $x(t_1) = x(t_o) + \int_{t_o}^{t_1} f(x(\tau), u(\tau), \tau) d\tau$.
  - Output map: $y(t) = h(s(t, t_o, x_o, u([t_o; t])), u(t), t) = h'(t, t_o, x_o, u([t_o; t]))$.

Linear Dynamical System

- Linear system property: given $(x_i(t_o), u_i([t_o; t])) \to (x_i(t), y_i(t))$, $t \geq t_o$,
  - $\alpha(x_i(t_o), u_i([t_o; t])) \to \alpha(x_i(t), y_i(t))$ for constant α (homogeneity).
  - $(x_1(t_o) + x_2(t_o), u_1 + u_2) \to (x_1(t) + x_2(t), y_1(t) + y_2(t))$ (additivity).
  - **Superposition property:** for any real constants α, β,

    $$\boxed{(\alpha x_1(t_o) + \beta x_2(t_o), \alpha u_1 + \beta u_2) \to (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)}$$
  - Note the scaling applies both to IC and input.

- For **linear dynamics** $\dot{x} = f(x, u, t)$ (i.e., $f(\alpha x_1 + \beta x_2, \alpha u_1 + \beta u_2, t) = \alpha f(x_1, u_1, t) + \beta f(x_1, u_1, t)$), we have linear state-transition map:

  $$s(t, t_o, \alpha x_1^o + \beta x_2^o, \alpha u_1 + \beta u_2) = \alpha s(t, t_o, x_1^o, u_1) + \beta s(t, t_o, x_2^o, u_2)$$

  Further, with **linear output map** (i.e., $h(\alpha x_1 + \beta x_2, \alpha u_1 + \beta u_2, t) = \alpha h(x_1, u_1, t) + \beta h(x_2, u_2, t)$),

  $$\begin{aligned} h(s(t, t_o, \alpha x_1^o + \beta x_2^o, \alpha u_1 + \beta u_2), t) &= y(t, t_o, \alpha x_1^o + \beta x_2^o, \alpha u_1 + \beta u_2) \\ &= \alpha y(t, t_o, x_1^o, u_1) + \beta y(t, t_o, x_2^o, u_2) \end{aligned}$$

- LTV or LTI system: $\dot{x} = A(t)x + B(t)u$, $y = C(t)x + D(t)u$.

LTV and LTI Systems

- State-space representation of linear systems:

$$\boxed{\dot{x} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)}$$

where $A(t) \in \Re^{n\times n}$, $B(t) \in \Re^{n\times p}$, $C(t) \in \Re^{m\times n}$, $D(t) \in \Re^{m\times p}$ are state, input, output and direct feedthrough matrices. If these are constants $\Rightarrow$ linear time-invariant (LTI); else $\Rightarrow$ linear time-varying (LTV).

- **Jacobian linearization** of $\dot{x} = f(x,u,t)$, $y = h(x,u,t)$ around nominal trajectory $\dot{\bar{x}} = f(\bar{x},\bar{u},t)$, $\bar{y} = h(\bar{x},\bar{u},t)$ or equilibrium (with $\dot{\bar{x}} = 0$).
  - Define deviations $\tilde{x} := x - \bar{x}$, $\tilde{u} := u - \bar{u}$, $\tilde{y} := y - \bar{y}$. Then,

$$\dot{\tilde{x}} = \dot{x} - \dot{\bar{x}} = f(\bar{x}+\tilde{x}, \bar{u}+\tilde{u}, t) - f(\bar{x},\bar{u},t) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}(t),\bar{u}(t)} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{\bar{x}(t),\bar{u}(t)} \tilde{u}$$

$$\tilde{y} = y - \bar{y} = h(\bar{x}+\tilde{x}, \bar{u}+\tilde{u}, t) - h(\bar{x},\bar{u},t) = \left.\frac{\partial h}{\partial x}\right|_{\bar{x}(t),\bar{u}(t)} \tilde{x} + \left.\frac{\partial h}{\partial u}\right|_{\bar{x}(t),\bar{u}(t)} \tilde{u}$$

  - Ex) $\dot{x}_1 = x_2, \dot{x}_2 = -\frac{g}{L}\sin x_2$; $\dot{x} = t\sin x + u^2 \sin t$, $y = \cos tx + e^{-t}u$.

- Realization of $Y(s) = H(s)U(s)$. Then, $H(s) = H_{sp}(s) + H(\infty)$ with $D = H(\infty)$. Further, $H_{sp}(s) = \frac{1}{D(s)}[N_1 s^{n-1} ... + N_n]$. Then, can obtain A, B, C, D via $Y(s) = [N_1 s^{n-1} + ... + N_n]X(s)$ and $X(s) = \frac{1}{D(s)}U(s)$.

Linear System Response

- Consider linear system $\dot{x} = A(t)x + B(t)u$ with superposition property:

$$s(t, t_o, \alpha x_1^o + \beta x_2^o, \alpha u_1 + \beta u_2) = \alpha s(t, t_o, x_1^o, u_1) + \beta s(t, t_o, x_2^o, u_2)$$

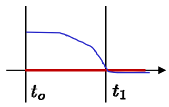
- Then, we can write the state response $x(t)$ as composed of:

$$\boxed{x(t) = s(t, t_o, x_o, u) = s(t, t_o, x_o + 0, 0 + u) = s(t, t_o, x_o, 0) + s(t, t_o, 0, u)}$$

  - **zero-input response** $x_{u=0,x_o}(t) = s(t, t_o, x_o, 0)$ (i.e., response $x(t)$ only due to IC x_o with no input) and
  - **zero-state response** $x_{u,x_o=0} = s(t, t_o, 0, u)$ (i.e., response only due to input u with zero IC).
  - Recall SISO TF case: $Y(s) = H(s)U(s) + H(s)[\text{IC}]$.

- **Fundamental matrix:** A matrix $X(t) \in \Re^{n\times n}$ is a fundamental matrix of $\dot{x} = A(t)x$, if

$$\boxed{\dot{X}(t) = A(t)X(t)}$$



with non-singular $X(t_o)$ for some t_o.

  - $X(t)$ is non-singular $\forall t$: if singular at $t_1 > t_o$ $\rightarrow \exists$ non-zero ν s.t., $x(t_1) := X(t_1)\nu = 0$. Then, $x(t) = X(t)\nu$ is a solution, thus, w/ $t \rightarrow t_o$, $x(t_o) = X(t_o)\nu = 0 \rightarrow$ contradiction (no information collapse).

Transition Matrix

- Consider the LTV system

$$\dot{x} = A(t)x + B(t)u$$

with zero-input response $s(t, t_o, x_o, 0)$. Then, since linear (e.g., using $(X(t), X(t_o))$), it can be represented by **transition matrix** $\Phi(t, t_o) \in \Re^{n\times n}$:

$$\boxed{s(t, t_o, x_o, 0) = \Phi(t, t_o)x_o}$$

  - $\Phi(t, t_o) = X(t)X^{-1}(t_o)$ with $\Phi(t_o, t_o) = I$.
  - $\frac{\partial}{\partial t}\Phi(t, t_o) = A(t)\Phi(t, t_o)$ (i.e., Φ is a fundamental matrix).
  - $\Phi(t, t_o) = \Phi(t, t_1)\Phi(t_1, t_o)$.
  - $\Phi^{-1}(t, t_o) = \Phi(t_o, t)$.
  - $\Phi(t, t_o)$ is non-singular $\forall t \geq t_o$ (i.e., no information collapse).
  - $\frac{\partial}{\partial t_o}\Phi(t, t_o) = -\Phi(t, t_o)A(t_o)$.
- LTI system: $\frac{d}{dt}\Phi(t, t_o) = A\Phi(t, t_o)$, $\Phi(t_o, t_o) = I \to \Phi(t, t_o) = e^{A(t-t_o)}$.
- LTV system: $\frac{d}{dt}\Phi(t, t_o) = A(t)\Phi(t, t_o) \to \Phi(t, t_o) \neq e^{\int_{t_o}^t A(\tau)d\tau}$.

with $e^{At} = I + At + \frac{A^2t^2}{2!} + ...$

Formulas for Transition Matrix

- Consider transition matrix $\Phi(t, t_o)$ of LTV system $\dot{x} = A(t)x + B(t)u$ with

$$\boxed{\frac{\partial}{\partial t}\Phi(t, t_o) = A(t)\Phi(t, t_o), \quad \Phi(t_o, t_o) = I}$$

- **Picard iteration:**
  - Implicit integration equation: $\Phi(t, t_o) = I + \int_{t_o}^t A(\tau)\Phi(\tau, t_o)d\tau$.
  - k-th iteration: $\Phi_{k+1}(t, t_o) = I + \int_{t_o}^t A(\tau)\Phi_k(\tau, t_o)d\tau$ w/ $\Phi_1(t, t_o) = I$.
  - Peano-Baker formula:

$$\Phi(t, t_o) = I + \int_{t_o}^t A(\tau_1)d\tau_1 + \int_{t_o}^t A(\tau_1)\left[\int_{t_o}^{\tau_1} A(\tau_2)d\tau_2\right]d\tau_1 + ...$$

- For the LTI system with $\dot{x} = Ax + Bu$,
  - Peano-Baker formula:

$$\Phi(t, t_o) = I + \frac{A(t-t_o)}{1!} + \frac{A^2(t-t_o)^2}{2!} + ... = e^{A(t-t_o)}$$

  - Laplace transform: $s\Phi(s) - I = A\Phi(s) \to \Phi(s) = (sI - A)^{-1}$.
  - $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \to (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix} \to \Phi(t, 0)$

Jordan Form and Multiplicity

- Consider LTI system $\dot{x} = Ax + Bu$ with transition matrix

$$\boxed{\Phi(t,0) = e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots}$$

- Recall the eigen-problem of A, i.e., $A\nu_i = \lambda_i \nu_i$, where $\lambda_i \in \mathcal{C}$ and $\nu_i \in \mathcal{C}^n$ are eigenvalue and eigenvector. Then, for each λ_i,

algebraic multiplicity of λ_i $\geq$ geometric multiplicity of λ_i

i.e., order of $(s - \lambda_i)$ in CE $\geq$ number of independent eigenvectors.

- **Jordan form:** for $A \in \Re^{4\times4}$ with deficient λ_2,

$$A = T\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}T^{-1} \Rightarrow e^{At} = T\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix}T^{-1}$$

from $AT = \Lambda T$, where $T \in \Re^{4\times4}$ is collection of (generalized) eigenvectors.

  - Stable if all $\lambda_i \in$ LHP; marginally stable if all $\lambda_i \in$ LHP except some non-deficient λ_i on jw-axis; unstable if some $\lambda_i \in$ RHP or deficient λ_i on jw-axis (if deficiency $= 2 \rightarrow$ growth w/ t^2: $[0,0;0,0]$, $[0,1;0,0]$).

Similarity Transformation

- Consider the LTI system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

- Define coordinate transformation via $x = T\bar{x}$ with non-singular matrix T. We may then think of $\bar{x}_i$ as the component of x along T_i.

- **Similarity transformation:** with $x = T\bar{x}$, can transform the dynamics s.t.,

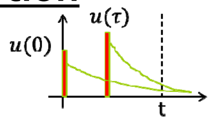
$$\dot{\bar{x}} = \bar{A}\bar{x}(t) + \bar{B}u(t), \quad y = \bar{C}\bar{x}(t) + Du$$

where $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{C} = CT$. This similarity transformation is equivalent with:

  - Same eigenvalues: $\text{CE}(\bar{A}) = \det[\lambda I - \bar{A}] = \det[T^{-1}]\det[\lambda I - A]\det[T] = \det[\lambda I - A] = \text{CE}(A)$
  - Same transfer function: $\bar{H}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + D = CT(sI - T^{-1}AT)^{-1}T^{-1}B + D = C(SI - A)^{-1}B + D = H(s)$.
  - Other properties (e.g., controllability, observability).
  - If all eigenvalues are distinct, we may then achieve **modal decomposition**: with $x = Tz$, T collection of eigenvectors,

$$\dot{z}_i = \lambda_i z_i + \bar{B}_i u, \quad \text{with mode } T_i \in \Re^n \text{ and modal freq. } \lambda_i$$

Zero-State Response and Convolution



- Consider LTV system:

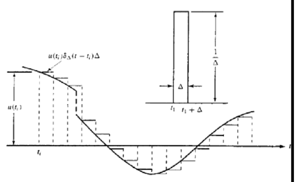
$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$

with state response given by zero-input and zero-state responses:

$$x(t) = \Phi(t, t_o)x(t_o) + s(t, t_o, 0, u([t_o; t]))$$

- To compute the zero-state response, using the unit impulse $\delta_\Delta(t - t_k)$, write

$$u(t) \approx \sum_k u(t_k)\delta_\Delta(t - t_k)\Delta$$



- Then, from the linearity of state-transition map s,

$$s(t, t_o, 0, u) \approx \sum_k s(t, t_o, 0, u(t_k)\delta_\Delta(t - t_k))\Delta$$

$$= \sum_k \Phi(t, t_k)B(t_k)u(t_k)\Delta \rightarrow \boxed{\int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \text{ (as } \Delta \rightarrow 0)}$$

since, with zero-state condition (i.e., $x(t_k) = 0$), $x(t_{k+1}) = \int_{t_k}^{t_{k+1}} [A(t)x(t) + B(t)u(t_k)\delta_\Delta(t - t_k)]dt = B(t_k)u(t_k)$, which will propagate to t via state transition matrix $\Phi(t, t_{k+1})$.

Zero-State Response: Proof

- Consider LTV system:

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$

and denote by $x(t)$ as the zero-state response under $u(t)$ with $x(t_o) = 0$. We want to show this $x(t) = s(t, t_o, 0, u)$ is the same as

$$\boxed{z(t) := \int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau}$$

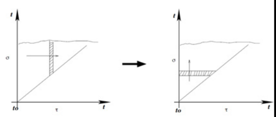
- Note first that $z(t_o) = 0$. Also, using $\Phi(t, \tau) = I + \int_\tau^t A(\sigma)\Phi(\sigma, \tau)d\sigma$,

$$z(t) = \int_{t_o}^{t} \left[I + \int_\tau^t A(\sigma)\Phi(\sigma, \tau)d\sigma \right] B(\tau)u(\tau)d\tau$$

$$= \int_{t_o}^{t} B(\tau)u(\tau)d\tau + \int_{t_o}^{t} \int_\tau^t A(\sigma)\Phi(\sigma, \tau)d\sigma B(\tau)u(\tau)d\tau$$

$$= \int_{t_o}^{t} B(\tau)u(\tau)d\tau + \int_{t_o}^{t} A(\sigma) \int_{t_o}^{\sigma} \Phi(\sigma, \tau)B(\tau)u(\tau)d\tau d\sigma$$

$$= \int_{t_o}^{t} B(\tau)u(\tau)d\tau + \int_{t_o}^{t} A(\sigma)z(\sigma)d\sigma$$



implying $\dot{z} = A(t)z(t) + B(t)u(t)$ with $z(t_o) = 0 \Rightarrow z(t) = x(t)$.

Output Response

- Consider LTV system:

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$

with total state-response given by:

$$x(t) = s(t, t_o, x_o, u) = \Phi(t, t_o)x_o + \int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

- Output response is then given by:

$$\boxed{y(t) = C(t)\Phi(t, t_o)x_o + C(t)\int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)}$$

which is linear (i.e., $\alpha(x_1 + u_1) + \beta(x_2 + u_2) \to \alpha y_1 + \beta y_2$), and consists of free-response (zero-input output) and force-response (zero-state output).

- LTI system: with $\Phi(t, t_o) = e^{A(t-t_o)}$, and $t_o = 0$ and $x(0) = x_o$,

$$y(t) = Ce^{At}x_o + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$
$$Y(s) = C(sI - A)^{-1}x_o + [C(sI - A)^{-1}B + D]U(s)$$

where $H(s) = [C(sI - A)^{-1}B + D]$ is the **transfer matrix**.

Impulse Response and Transfer Matrix

- Consider LTV system with its output response:

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$
$$y(t) = C(t)\Phi(t, t_o)x_o + C(t)\int_{t_o}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

- **Impulse response matrix**: with zero initial state and unit-impulse input through the j-th input channel $u_j(t) = [0; ...; \delta(t - t'); 0; ...0]$,

$$y_j(t) = [C(t)\Phi(t, t')B(t') + D(t)\delta(t - t')]_j \in \Re^m, \quad t \geq t'$$

i.e., $y_j(t)$ is a m-dimensional column vector. By collecting this, we can construct $m \times p$ impulse response matrix:

$$\boxed{h(t, t') = C(t)\Phi(t, t')B(t') + \delta(t - t') \cdot D(t)}$$

whose j-th column is output from j-th channel unit-impulse input.

- **LTI system**: with $t' = 0$, $h_{\text{impulse}}(t) = Ce^{At}B + \delta(t) \cdot D$. Taking Laplace TF,

$$H_{\text{impulse}}(s) = C(sI - A)^{-1}B + D$$

which is transfer matrix computed before (or directly from $\dot{x} = Ax + Bu, y = Cx + Du$) $\Rightarrow$ transfer matrix is Laplace TF of IR-matrix.

DT-LTV System and Exact Discretization

- Consider LTV discrete-time system:

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad y(k) = C(k)x(k) + D(k)u(k)$$

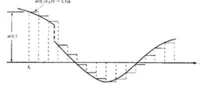
where $k \in \mathcal{Z}$ is the time-index, $x(k) \in \Re^n$, $u(k) \in \Re^p$, $y(k) \in \Re^m$, $A(k) \in \Re^{n\times n}$, $B(k) \in \Re^n \times p$, $C(k) \in \Re^{m\times n}$ and $D(k) \in \Re^{m\times p}$.

- Simple discretization: with $\dot{x}(t) \approx (x((k+1)T) - x(kT))/T$,

$$x(k+1) = (1 + TA)x(k)x(k) + TB(k)u(k), \quad y(k) = Cx(k) + Du(k)$$

- Exact discretization w/ piecewise-continuous input $u(k)$:
  - Response of CT-LTI system: $x(k) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau$.

$$x(k+1) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$



$$= e^{AT}x(k) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau = e^{AT}x(k) + \int_0^T e^{A\tau}d\tau Bu(k)$$

  - $x(k+1) = A_d x(k) + B_d u(k)$, $y(k) = Cx(k) + Du(k)$, with $A_d = e^{AT}$, $B_d = \int_0^T e^{A\tau}d\tau B$. Also, $B_d = A^{-1}(A_d - I)B$ if A is nonsingular.

DT-LTV System State Response

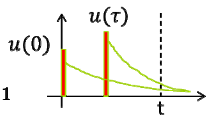
- LTV discrete-time system:

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad y(k) = C(k)x(k) + D(k)u(k)$$

- The system is still linear with

$$x(k_1) = s(k_1, k_o, x_o, u([k_o; k_1-1]) = s(k_1, k_o, x_o, 0) + s(k_1, k_o, 0, u([k_o : k_1-1]))$$

- State-propagation of LTV-DT system:



$$\begin{aligned} x_k &= A_{k-1}x_{k-1} + B_{k-1}u_{k-1} \\ &= A_{k-1}A_{k-2}x_{k-2} + A_{k-1}B_{k-2}u_{k-2} + B_{k-1}u_{k-1} \\ &\vdots \\ &= \left[\Pi_{i=k_o}^{k-1} A(i)\right] x(k_o) + \sum_{i=k_o}^{k-1} \left[\Pi_{j=i+1}^{k-1} A(j)\right] B(i)u(i) \\ &= \Phi(k, k_o)x(k_o) + \sum_{i=k_o}^{k-1} \Phi(k, i+1)B(i)u(i) \end{aligned}$$

consisting of zero-input response w/ state transition matrix $\Phi(k, k_o) = A(k-1)A(k-2)...A(k_o) \in \Re^{n\times n}$ and zero-state respons (i.e., convolution).

DT-LTV System Response

- LTV discrete-time system:

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad y(k) = C(k)x(k) + D(k)u(k)$$

- LTV-DT system state response:

$$x(k) = \Phi(k,k_o)x(k_o) + \sum_{i=k_o}^{k-1} \Phi(k,i+1)B(i)u(i), \quad \Phi(k,k_o) := \Pi_{i=k_o}^{k-1} A(i)$$

- LTV-DT system output response:

$$y(k) = C(k)\Phi(k,k_o)x_o + C(k)\sum_{i=k_o}^{k-1} \Phi(k,i+1)B(i)u(i) + D(k)u(k)$$

$$= C(k)\sum_{i=k_o}^{k-1} \Phi(k,i+1)B(i)u(i) + D(k)u(k) \quad (\text{if } x_o = 0)$$

$$\boxed{= \sum_{i=k_o}^{k} \left[C(k)\Phi(k,i+1)B(i) + \delta(k-i)D(i)\right]u(i) = \sum_{i=k_o}^{k} H(k,i)u(i)}$$

where $H(k,i) = 0\ \forall k < i$, and $H(k,i) := C(k)\Phi(k,i+1)B(i) + D(i)\delta(k-i)$ is impulse response matrix with unit-impulse $\delta(k-i)$ at $i = k$.

DT-LTV System Transition Matrix

- DT LTV system:

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad y(k) = C(k)x(k) + D(k)u(k)$$

- LTV-DT system state response:

$$x(k) = \Phi(k,k_o)x(k_o) + \sum_{i=k_o}^{k-1} \Phi(k,i+1)B(i)u(i), \quad \Phi(k,k_o) := \Pi_{i=k_o}^{k-1} A(i)$$

- DT-LTV state-transition matrix:

$$\boxed{\Phi(k,k_o) := \Pi_{i=k_o}^{k-1} A(i) = A(k-1)A(k-2)...A(k_o+1)A(k_o)}$$

  - $\Phi(k_1,k_o)$ exists and is unique $\forall k_1 \geq k_o$.
  - $\Phi(k+1,k_o) = A(k)\Phi(k,k_o)$, $\Phi(k_1,k-1) = \Phi(k_1,k)A(k-1)$.
  - If $k_1 < k_o$, $\Phi(k_1,k_o)$ exists & unique iff $A(k)$ invertible $\forall k_1 \leq k < k_o$.
  - $\Phi^{-1}(k_1,k_o)$ exsits and is given by

$$\Phi^{-1}(k_1,k_o) = A^{-1}(k_o)A^{-1}(k_o)..A^{-1}(k_1-1)$$

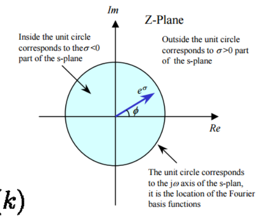
    iff $A(k)$ invertible $\forall k_o \leq k \geq k_1 - 1$.
  - $\Phi(k_2,k_o) = \Phi(k_2,k_1)\Phi(k_1,k_o)$ only for $k_o \leq k_1 \leq k_2$; $\forall k_o,k_1,k_2$ iff $A(k)$ invertible.

DT-LTI System Response

- DT-LTI system:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k)$$

- DT-LTI system response: with $\Phi(k,0) := \Pi_{i=0}^{k-1} A = A^k$,



$$x(k) = A^k x(0) + \sum_{i=k_o}^{k-1} A^{k-i-1} B(i) u(i)$$

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-i-1} Bu(i) + Du(k)$$

- Jordan form:

$$\boxed{A = T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} T^{-1} \Rightarrow A^k = T \begin{bmatrix} \lambda_1^k & 0 & 0 & 0 \\ 0 & \lambda_2^k & k\lambda_2^{k-1} & 0 \\ 0 & 0 & \lambda_2^k & 0 \\ 0 & 0 & 0 & \lambda_4^k \end{bmatrix} T^{-1}}$$

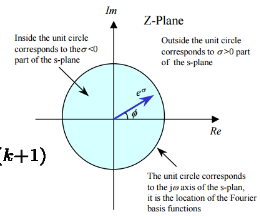
stable if all $|\lambda_i| < 1$; marginally stable if all $|\lambda_i| < 1$ with some $|\lambda_i| = 1$, yet, not deficient; unstable if some $|\lambda_i| > 1$ or some deficient $|\lambda_i| = 1$.

DT-LTI System Response and z-Transform

- DT-LTI system:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k)$$

- z-transform: $X(z) := \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} x(k)/z^k$.



- z-transfrom of $x(k+1)$:

$$\mathcal{Z}[x(k+1)] = \sum_{k=0}^{\infty} x(k+1) z^{-k} = z \sum_{k=0}^{\infty} x(k+1) z^{-(k+1)}$$

$$= z \left[\sum_{j=0}^{\infty} x(j) z^{-j} + x(0) - x(0) \right] = zX(z) - zx(0)$$

- z-transform of DT-LTI system (w/ $(zI-A)^{-1}z = \mathcal{Z}[A^k]$, $\frac{z}{z-a} = \mathcal{Z}[a^k]$),

$$\boxed{\begin{aligned} X(z) &= (zI-A)^{-1} zx(0) + (zI-A)^{-1} BU(z) \\ Y(z) &= C(zI-A)^{-1} zx(0) + C(zI-A)^{-1} BU(z) + DU(z) \end{aligned}}$$

- Transfer matrix $H(z) := C(zI-A)^{-1}B + D$ (i.e., z-transform of impulse response): stable if all poles are within unit circle, unstable if some are outside unit circle, marginally stable if on the circle.

Stability and Norms

- **IO-stability (or BIBO-stability):** any bounded input produces bounded output $\Rightarrow$ related to zero-state response and transfer function.
- **Internal stability:** state evolution is bounded for any initial conditions with zero-input $\Rightarrow$ related to zero-input response.
- For LTI systems, internal stability implies IO-stability, but not vice versa (e.g., pz-cancelation: $\dot{x}_1 = x_2, \dot{x}_2 = 4x_1$, $y = x_2 - 2x_1 \Rightarrow \lambda(A) = \{2, -2\}$, $H(s) = 1/(s+2)$).
- For LTV systems, internal stability not necessarily implies IO-stability.
- Signal ∞-norm of $u(t) \in \Re^n$: $||u(\cdot)||_\infty := \sup_t ||u(t)||_\infty = \sup_t (\max_i |u_i(t)|)$, where $||x(t)||_\infty := \max_i |x_i|$ is vector ∞-norm.
- Matrix induced ∞-norm of $A \in \Re^{n\times n}$:

$$\boxed{||A||_{i,\infty} := \sup_x \frac{||Ax||_\infty}{||x||_\infty} = \max_j \sum_i |A_{ji}|}$$

i.e., $||Ax||_\infty \le \max_j \sum_i (|A_{ji}| \max_k |x_k|) =: ||A||_{i,\infty} ||x||_\infty$.

IO-Stability

$$u(t) \longrightarrow \boxed{h(t,\tau)} \longrightarrow y(t)$$

- **Def:** We say a system $H : u \mapsto y$ is IO-stable (or BIBO-stable) if $\exists\, M \ge 0$ s.t., for **any bounded** $u(t)$,

$$\boxed{||y(\cdot)||_\infty \le M \cdot ||u(\cdot)||_\infty}$$

- For LTV system: $\dot{x} = A(t)x + B(t)u, y = C(t)x + D(t)u$, zero-state response is given by:

$$y(t) = C(t)\int_{t_o}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \boxed{= \int_{t_o}^{t} h(t,\tau)u(\tau)d\tau}$$

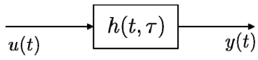
where $h(t,\tau) := C(t)\Phi(t,\tau)B(\tau) + \delta(t-\tau)\cdot D(t)$ is impulse response matrix.

- **Th. 5:** CT-LTV system is IO-stable iff

$$\boxed{\sup_t \int_{t_o}^{t} ||h(t,\tau)||_{i,\infty} d\tau \le M, \quad \forall t \ge t_o}$$

- For SISO LTI system: $||y(\cdot)||_\infty = \sup_t |\int_0^t h(t-\tau)u(\tau)d\tau| = \sup_t |\int_0^t h(\alpha)u(t-\alpha)d\alpha| \le u_{\max} \int_0^\infty |h(\alpha)|d\alpha \Rightarrow$ i.e., BIBO stable if $\int_0^\infty |h(\alpha)|d\alpha \le M < \infty$ (i.e., impulse response $h(t)$ is absolutely integrable).

IO-Stability Proof



Th. 5: CT-LTV system is IO-stable iff

$$\boxed{\sup_t \int_{t_o}^{t} ||h(t,\tau)||_{i,\infty} d\tau \le M, \quad \forall t \ge t_o}$$

- Sufficiency ($\Leftarrow$): from $y(t) = \int_{t_o}^{t} h(t,\tau)u(\tau)d\tau$, for all $t \ge t_o$,

$$||y(\cdot)||_\infty = \sup_t || \int_{t_o}^{t} h(t-\tau)u(\tau)d\tau||_\infty \le \sup_t \int_{t_o}^{t} ||h(t-\tau)u(\tau)||_\infty d\tau$$

$$\le \sup_t \int_{t_o}^{t} ||h(t,\tau)||_{i,\infty}||u(\tau)||_\infty d\tau \le ||u(\cdot)||_\infty \sup_t \int_{t_o}^{t} ||h(t,\tau)||_{i,\infty} d\tau$$

definition of induced norm   definition of signal norm

- Necessity ($\Rightarrow$): Suppose not. Consider SISO LTI for simplicity. Then,

$$|y(t)| = |\int_0^t h(\alpha)u(t-\alpha)d\alpha| \to \infty$$

if we choose bounded input $u(t)$ s.t., $u(t-\alpha) = \begin{cases} 1 & \text{if } h(\alpha) \ge 0 \\ -1 & \text{if } h(\alpha) < 0 \end{cases}$.

IO-Stability of CT-LTI System

- **Th. 5:** CT-LTV system is IO-stable iff

$$\sup_t \int_{t_o}^{t} ||h(t,\tau)||_{i,\infty} d\tau \le M, \ \forall t \ge t_o$$

- **Th. 5-M1:** CT-LTI system is IO-stable (or BIBO-stable) iff every $h_{ij}(t)$ is absolutely integrable.
  - The above condition can be rewritten by

$$\int_0^\infty ||h(t)||_{i,\infty} dt = \int_0^\infty \max_j \sum_i |h_{ji}(t)| dt \le M$$

with the remaining arguments going similarly as before.

- **Th. 5-M2:** CT-LTI system with a proper rational transfer matrix $H(s)$ is BIBO-stable iff every pole of every component $H_{ij}(s)$ has a negative real part.
  - For BIBO-stability, $H(s)$ should be strictly stable (i.e., poles strictly within LHS).
  - Poles of $H(s)$ is a subset of eigenvalues of A.

IO-Stability and CT-LTI Response

- **Th. 5.2:** Suppose a SISO CT-LTI system with impulse response $h(t)$ is BIBO-stable. Then,

$$y(t) \to aH(0) \;\text{ if } u(t) = a; \text{ and}$$
$$y(t) \to |H(jw_o)| \sin(w_o t + \angle H(jw_o)) \;\text{ if } u(t) = \sin w_o t$$

  – If $u(t) = a$, with $H(s) = \int_0^\infty h(\tau)e^{-s\tau} d\tau$,

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau = a \int_0^t h(\tau)d\tau \;\Rightarrow\; y(t) \to a \int_0^\infty h(\tau)d\tau = aH(0)$$

  – From $H(jw_o) = \int_0^\infty h(\tau)e^{-jw_o\tau}d\tau = \int_0^\infty h(\tau)[\cos w_o\tau - j\sin w_o\tau]d\tau$, $\mathrm{Re}[H(jw_o)] = \int_0^\infty h(\tau)\cos w_o\tau d\tau$ and $\mathrm{Im}[H(jw_o)] = -\int_0^\infty h(\tau)\sin w_o\tau d\tau$. Also, if $u(t) = \sin w_o t$, output is given by $y(t) = \int_0^t h(\tau)\sin w_o(t-\tau)d\tau = \sin w_o t \int_0^t h(\tau)\cos w_o\tau d\tau - \cos w_o t \int_0^t h(\tau)\sin w_o\tau d\tau$. Thus, from absolute integrability of $h(t)$, the integrands exist and

$$y(t) \to \sin w_o t \int_0^\infty h(\tau)\cos w_o\tau d\tau - \cos w_o t \int_0^\infty h(\tau)\sin w_o\tau d\tau$$
$$= \sin w_o t \mathrm{Re}[H(jw_o)] + \cos w_o t \mathrm{Im}[H(jw_o)] = |H(jw_o)|\sin(w_o t + \angle H(jw_o))$$

IO-Stability of CT-LTV and DT-LTI Systems

- **Th. 5-M3:** CT-LTV system with $h(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + \delta(t-\tau)D(t)$ is IO-stable (or BIBO-stable) iff $||D(\cdot)||_\infty \le M_1$ and

$$\int_{t_o}^t ||C(t)\Phi(t,\tau)B(\tau)||_{i,\infty} d\tau \le M_2, \quad \forall t \ge t_o$$

- Consider DT-LTV system, with its zero-state output response given by

$$y(k) = C(k)\sum_{i=k_o}^{k-1} \Phi(k, i+1)B(i)u(i) + D(k)u(k) = \sum_{i=k_o}^{k} h(k,i)u(i)$$

  with impulse response $h(k,i) = C(k)\Phi(k,i+1)B(i) + \delta(k-i)D(i)$. BIBO-stability would then have something with absolute summability of $h(k,i)$, i.e., $\sum_{i=0}^k ||h(k,i)||_{i,\infty} \le M$, $\forall k \ge i$.

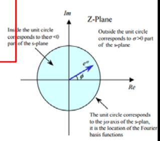
- **Th. 5-MD1:** DT-LTI system with impulse response matrix $h(k) \in \Re^{m\times p}$ is BIBO-stable iff $h_{ij}(k)$ is absolutely summable.

- **Th. 5-MD2:** DT-LTI system with proper rational transfer matrix $H(z)$ is BIBO-stable iff every pole of every $H_{ij}(z)$ has a magnitude less than 1.

IO-Stability and DT-LTI Response

- **Th. 5-D2:** Suppose a SISO DT-LTI system with impulse response sequence $h(k)$ is BIBO-stable. Then,

$$\boxed{\begin{aligned} &y(k) \to aH(1) \ \text{ if } u(k) = a; \text{ and} \\ &y(k) \to |H(e^{-jw_o})| \sin(w_o t + \angle H(e^{-jw_o})) \ \text{ if } u(k) = \sin w_o k \end{aligned}}$$



– If $u(k) = a$, with $H(z) = \sum_{i=1}^{\infty} h(k) z^{-k}$,

$$y(k) = \sum_{m=0}^{k} h_{k-m} u_m = \sum_{m=0}^{k} h_m u_{k-m} = a \sum_{m=0}^{k} h_m \ \Rightarrow \ y(k) \to a \sum_{m=0}^{\infty} h_m = aH(1)$$

– From $H(e^{-jw_o}) = \sum_{m=0}^{\infty} h_m e^{-jw_o m} = \sum_{m=0}^{\infty} h_m [\cos w_o m - j \sin w_o m]$, $\mathrm{Re}[H(e^{-jw_o})] = \sum_{m=0}^{\infty} h_m \cos w_o m$ and $\mathrm{Im}[H(e^{-jw_o})] = -\sum_{m=0}^{\infty} h_m \sin w_o m$.
Also, if $u(k) = \sin w_o k$, output is given by $y(k) = \sum_{m=0}^{k} h_m \sin w_o (k - m) = \sin w_o k \sum_{m=0}^{k} h_m \cos w_o m - \cos w_o k \sum_{m=0}^{k} h_m \sin w_o m$.
Thus, from absolute summability of $h(k)$, summations exist (w/ $k \to \infty$) and

$$\begin{aligned} y(k) &\to \sin w_o k \mathrm{Re}[H(e^{-jw_o})] + \cos w_o k \mathrm{Im}[H(e^{-jw_o})] \\ &= |H(e^{-jw_o})| \sin(w_o k + \angle H(e^{-jw_o})) \end{aligned}$$

Internal Stability

- Recall the zero-input responses of FD linear systems:

$$x(t) = \Phi(t, t_o) x(t_o), \quad x(k) = \Phi(k, k_o) x(k_o)$$

where $\Phi(t, t_o) = e^{A(t - t_o)}$ and $\Phi(k, k_o) = \Pi_{i=k_o}^{k-1} A$ for CT/DT LTI systems.

- **Def. 5.1:** Linear systems is **Lyapnunov stable (or marginally stable)** if every bounded IC x_o produces bounded zero-input response x; **asymptotically stable** if, for all bounded ICs x_o, MS and $x \to 0$.

- For MS $\Rightarrow$ Φ should be bounded; For AS $\Rightarrow$ Φ should converge to zero.

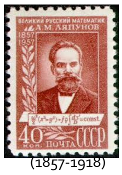
- **Th. 5-4:** CT-LTI system $\dot{x} = Ax$ is MS iff all $\lambda_i(A)$ are in LHP with some jw-axis being non-deficient; AS iff all $\lambda_i(A)$ are strictly within LHP.

- **Th. 5-D4:** DT-LTI system $x_{k+1} = Ax_k$ is MS iff all $\lambda_i(A)$ are in UC with some on UC being non-deficient; AS iff all $\lambda_i(A)$ strictly within UC.

$$e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & te^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} T^{-1}, \quad A^k = T \begin{bmatrix} \lambda_1^k & 0 & 0 & 0 \\ 0 & \lambda_2^k & k\lambda_2^{k-1} & 0 \\ 0 & 0 & \lambda_2^k & 0 \\ 0 & 0 & 0 & \lambda_4^k \end{bmatrix} T^{-1}$$

- Not applicable to LTV systems...

Lyapunov Stability - Definition

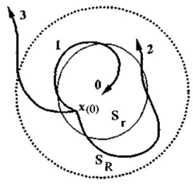


(1857-1918)

Def. 2.1: Consider an **autonomous** system

$$\dot{x} = f(x), \quad f(0) = 0$$

$f : \mathcal{D} \to \Re^n$ locally Lipschitz on $\mathcal{D}$ and $0 \in \mathcal{D}$. Then, equilibrium $x = 0$ is



- Lyapunov stable, if, $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ s.t.,

$$||x(0)|| < \delta \Longrightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0$$

- unstable, if it is not stable
- asymptotically stable, if it is stable and we can find $\delta' > 0$ s.t.

$$||x(0)|| < \delta' \Longrightarrow ||x(t)|| \to 0$$

- exponentially stable if $\exists \alpha, \gamma, \delta' > 0$ s.t.,

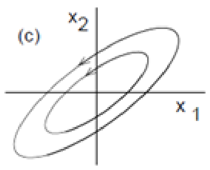
$$||x(0)|| < \delta' \Longrightarrow ||x(t)|| \leq \alpha ||x(0)|| e^{-\gamma t}$$

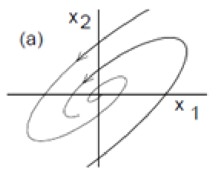
- globally asymptotically stable, if asymptotically stable for any $\forall x(0) \in \Re^n$.

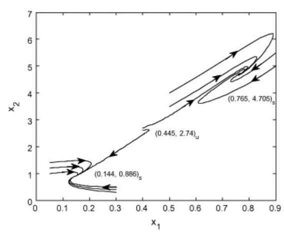
Lyapunov Stability - Examples

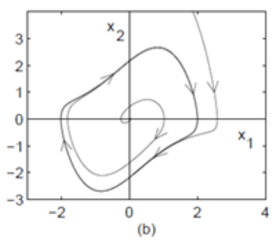
Lyapunov stable, if, for <u>any</u> $\epsilon > 0$, there exits $\delta(\epsilon) > 0$ s.t.,

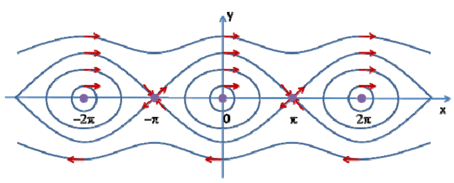
$$||x(0)|| < \delta \Longrightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0$$











satisfy defintion: 1) for some ϵ or 2) $\forall \delta$, $\exists \epsilon$

CT-LTI Lyapunov Theorem

Th. 5.5: $A \in \Re^{n\times n}$ is Hurwitz (AS) iff, for **any** $Q \succ 0$, $\exists$ a **unique** $P \succ 0$ s.t.

$$PA + A^T P = -Q \quad \text{(Lyapunov equation)}$$

- Lyapunov analysis: for $\dot{x} = Ax$, define $V = \frac{1}{2}x^T P x \Rightarrow \dot{V} = -x^T Q x < 0$ unless $V = 0 \Rightarrow V(t) \to 0$ (AS, in fact, ES).
- Can be used to find Lyapunov function: given $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, choose any $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succ 0$. Then, solve for $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix} \succ 0$.
- (Sufficiency $\Rightarrow$): Given $Q \succ 0$, define $P := \int_0^\infty e^{A^T t} Q e^{At} dt \succ 0$, which exists (from AS) and PD. Then,

$$\begin{aligned} PA + A^T P &= \int_0^\infty e^{A^T t} Q e^{At} A dt + \int_0^\infty A^T e^{A^T t} Q e^{At} dt \\ &= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) dt = e^{A^T t} Q e^{At}\Big|_0^\infty = -Q \end{aligned}$$

- (Necessity $\Leftarrow$): Define λ, v s.t., $Av = \lambda v \Rightarrow v^* A^* = v^* \bar{A}^T = v^* A^T = \bar{\lambda} v^*$. Further, $-v^* Q v = v^*(PA + A^T P)v = (\lambda + \bar{\lambda}) v^* P v$ where $v^* Q v > 0$ and $v^* P v > 0 \Rightarrow \lambda(A)$ in LHP (i.e., A is Hurwitz).

CT-LTV Lyapunov Theorem

Consider linear time-varying system

$$\dot{x} = A(t)x, \quad x(t_o)$$

with $x = 0$ equilibrium. Then, $x(t) = \Phi(t, t_o)x(t_o)$, where $\Phi(t, t_o)$ is the state transition matrix (e.g., $\Phi(t, t_o) = e^{A(t-t_o)}$ for LTI system).

Ex) Consider $\dot{x} = A(t)x$ with $A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5 \sin t \cos t & -1 + 1.5 \sin^2 t \end{bmatrix}$.

Then, $\text{eig}(A(t)) = -0.25 \pm 0.25\sqrt{7}j$. Yet, $\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$

Th. 5.5V: Suppose $\exists$ smooth bounded $P(t) \succ 0$ s.t.

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -Q(t)$$

with $Q(t) \succ 0$. Then, $x = 0$ is AS. Also, if $A(t)$ is continuous and bounded and $x \to 0$ (i.e., AS), for **any** $Q(t) \succ 0$, $\exists P(t) \succ 0$ satisfying Lyapunov equation.

- ($\Rightarrow$) Lyapunov analysis with $V = \frac{1}{2}x^T P(t) x \to \dot{V} = -x^T Q(t) x < 0$ unless $V(t) = 0 \to x \to 0$ (i.e., AS, in fact, ES).
- ($\Leftarrow$) Choose $V(x,t) = x^T P(t) x$. Also, given $Q(t)$, $P(t) = \int_t^\infty \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) d\tau$.

CT-LTV Lyapunov Analysis

Th. 8.2 (R): Consider CT-LTV system $\dot{x} = A(t)x$. Denote, at each t, the largest and smallest eigenvalues of $A(t) + A^T(t)$ by $\lambda_{\max}(t)$ and $\lambda_{\min(t)}$. Then,

$$||x_o||e^{\frac{1}{2}\int_{t_o}^{t}\lambda_{\min}(\sigma)d\sigma} \leq ||x(t)|| \leq ||x_o||e^{\frac{1}{2}\int_{t_o}^{t}\lambda_{\max}(\sigma)d\sigma}, \quad \forall t \geq t_o$$

where $||\cdot||$ is the vetor (i.e., Euclidean) 2-norm.

- Define $V(t) = ||x(t)||^2$. Then,

$$\lambda_{\min}(t)||x(t)||^2 \leq \frac{dV}{dt} = x^T(t)(A^T(t) + A(t))x(t) \leq \lambda_{\max}(t)||x(t)||^2$$

  i.e., $V(t_o)e^{\int_{t_o}^{t}\lambda_{\min}(\sigma)d\sigma} \leq V(t) \leq V(t_o)e^{\int_{t_o}^{t}\lambda_{\max}(\sigma)d\sigma}$, for all $t \geq t_o$.

- **Cor. 8.2-1:** CT-LTV system is stable if

$$\int_{\tau}^{t} \lambda_{\max}(\sigma)d\sigma \leq \gamma, \quad \forall t, \tau, \text{ s.t., } t \geq \tau$$

- **Cor. 8.2-2:** CT-LTV system is ES if

$$\int_{\tau}^{t} \lambda_{\max}(\sigma)d\sigma \leq -\lambda(t-\tau) + \gamma, \quad \forall t, \tau, \text{ s.t., } t \geq \tau$$

CT-LTV Stability w/ Perturbation -I

Th. 8.6 (R): Suppose CT-LTV system $\dot{x} = A(t)x$ is ES with continuous/bounded $A(t)$. Then, $\exists$ a small enough $\beta > 0$ s.t.,

$$\dot{z} = [A(t) + F(t)]z$$

is also ES if $||F(t)|| \leq \beta$, where $||\cdot||$ is matrix 2-norm (other norms also work).

- From Lyapunov theorem, $\exists\ P(t), Q(t) \succ 0$ s.t.,

$$\dot{P}(t) + A(t)^T P(t) + A(t)P(t) = -Q(t)$$

  where $P(t) := \int_t^\infty \Phi^T(\sigma, t)Q(t)\Phi(t,\sigma)d\sigma$. Then,

$$[A(t)+F(t)]^T P(t) + P(t)[A(t)+F(t)] + \dot{P}(t) = F^T(t)P(t) + P(t)F(t) - Q(t)$$

  where LHS is PD if $||F(t)||$ small enough, since $P(t)$ bounded (from ES).

- ES is robust against bounded perturbation.
- This robustness also true for CT-LTI systems.
- If CT-LTV system is AS, stability is in general fragile.

CT-LTV Stability w/ Perturbation - II

Th. 8.5 (R): Suppose CT-LTV system $\dot{x} = A(t)x$ is stable. Then, $\exists$ a small enough $\beta > 0$ s.t.,

$$\dot{z} = [A(t) + F(t)]z$$

is also stable if $\int_\tau^\infty ||F(\sigma)||d\sigma \leq \beta$, $\forall \tau \geq 0$.

- Using convolution: $z(t) = \Phi(t,t_o)z_o + \int_{t_o}^t \Phi(t,\sigma)F(\sigma)z(\sigma)d\sigma$, where, from stability, $||\Phi(t,\sigma)|| \leq \gamma\ \forall t, \sigma,\ t \geq \sigma$.
- Taking the 2-norm: $||z(t)|| \leq \gamma||z_o|| + \int_{t_o}^t \gamma||F(\sigma)||||z(\sigma)||d\sigma$, i.e., an implicit inequality w.r.t., $||z(t)||$.
- From Gronwall-Bellman inequality: $||z(t)|| \leq \gamma||z_o||e^{\int_{t_o}^t \gamma||F(\sigma)||d\sigma} \leq \gamma||z_o||e^{\gamma\beta}$, $\forall t \geq t_o$, i.e., stable.
- **Lem. 3.2 (Gronwall-Bellman):** For continuous $\phi(t), v(t)$ with $v(t) \geq 0$, $\forall t \geq t_o$,

$$\phi(t) \leq \psi + \int_{t_o}^t v(\sigma)\phi(\sigma) \quad \Rightarrow \quad \phi(t) \leq \psi e^{\int_{t_o}^t v(\sigma)d\sigma}$$

  - $r(t) = \psi + \int_{t_o}^t v(\sigma)d\sigma \Rightarrow \dot{r}(t) = v(t)\phi(t) \leq v(t)r(t) \Rightarrow$ multiplying $e^{-\int_{t_o}^t v(\sigma)d\sigma} \Rightarrow \frac{d}{dt}\left[r(t)e^{-\int_{t_o}^t v(\sigma)d\sigma}\right] \leq 0 \Rightarrow \phi(t) \leq r(t) \leq \psi e^{\int_{t_o}^t v(\sigma)d\sigma}$.
- However small persistent perturbation can destabilize CT-LTV systems.

DT-LTI Lyapunov Theorem

Th. 5-D5: $A \in \Re^{n\times n}$ has $\rho(A) < 1$ (i.e., DT-LTI AS) iff, for <u>**any**</u> $Q \succ 0$, $\exists$ a <u>**unique**</u> $P \succ 0$ s.t.

$$A^T PA - P = -Q \quad \text{(Lyapunov equation)}$$

- Lyapunov analysis: for $x_{k+1} = Ax_k$, define $V_k = \frac{1}{2}x_k^T Px \Rightarrow V_{k+1} - V_k = \frac{1}{2}x_k^T(A^T PA - P)x_k = -\frac{1}{2}x_k^T Qx_k < 0$ unless $V_k = 0 \Rightarrow V_k \to 0$ (AS).
- (Sufficiency $\Rightarrow$): Given $Q \succ 0$, define $P := \sum_{m=0}^\infty (A^T)^m QA^m \succ 0$, which exists (from AS) and PD. Then,

$$\begin{aligned} A^T PA - P &= \sum_{m=0}^\infty A^T(A^T)^m QA^m A - \sum_{m=0}^\infty (A^T)^m QA^m \\ &= \sum_{m=1}^\infty (A^T)^m QA^m - \sum_{m=0}^\infty (A^T)^m QA^m = -Q \end{aligned}$$

- (Necessity $\Leftarrow$): Define λ, v s.t., $Av = \lambda v \Rightarrow v^*A^* = v^*\bar{A}^T = v^*A^T = \bar{\lambda}v^*$. Further,

$$-v^*Qv = v^*(A^T PA - P)v = (|\lambda|^2 - 1)v^*Pv$$

  where $v^*Qv > 0$, $v^*Pv > 0 \Rightarrow |\lambda(A)|^2 < 1$ (i.e., $\rho(A) < 1 \Rightarrow$ DT-LTI AS).

IO-Stability and Internal Stability

- LTI system stability conditions:

$$\int_0^\infty ||Ce^{At}B + \delta(t)D||_{i,\infty} dt \le M \quad \text{(BIBO)} \quad \text{and} \quad \lambda_i(A) \in \text{LHP} \quad \text{(AS)}$$

  - AS implies BIBO, but not vice versa (e.g., pz-cancelation).
  - BIBO excludes marginal stability.

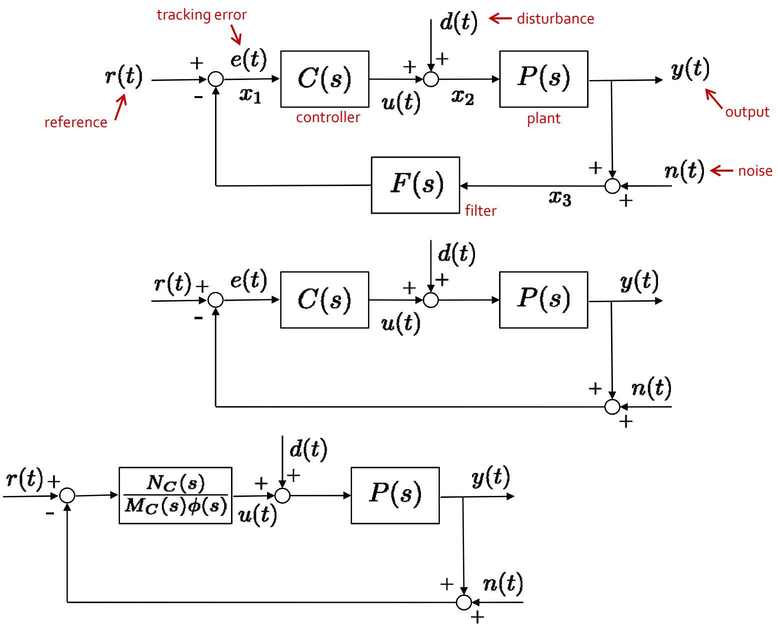
- LTV system stability conditions:

$$\int_0^\infty ||C(t)\Phi(t,\tau)B(\tau) + \delta(t-\tau)D(t)||_{i,\infty} dt \le M \quad \text{(BIBO)}$$

$$||\Phi(t,\tau)||_{i,\infty} \to 0 \quad \text{as } t \to \infty \quad \text{(AS)}$$

  - AS may not even imply BIBO (since only AS, not ES as for LTI).
  - AS implies BIBO, if ES and $C(t), B(t)$ bounded.

Basic Feedback Loop



Basic Feedback Loop