Mutual Exclusion

Companion slides for
The Art of Multiprocessor Programming
by Maurice Herlihy & Nir Shavit

Mutual Exclusion

- Today we will try to formalize our understanding of mutual exclusion
- We will also use the opportunity to show you how to argue about and prove various properties in an asynchronous concurrent setting

Mutual Exclusion

- Formal problem definitions
- Solutions for 2 threads
- Solutions for *n* threads
- Fair solutions
- Inherent costs

Warning

- You will never use these protocols
  - Get over it
- You are advised to understand them
  - The same issues show up everywhere
  - Except hidden and more complex

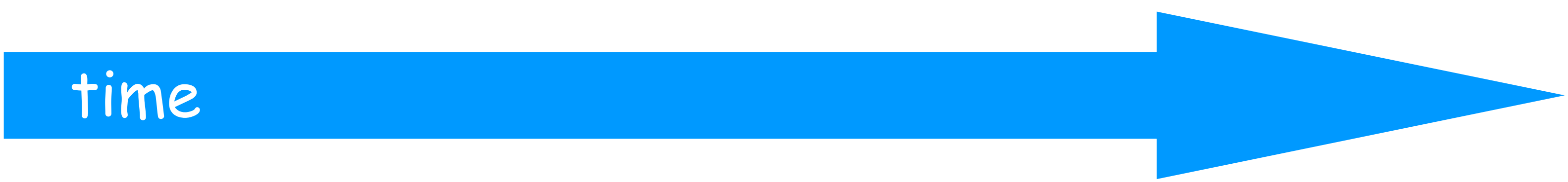
Why is Concurrent Programming so Hard?

- Try preparing a seven-course banquet
  - By yourself
  - With one friend
  - With twenty-seven friends …
- Before we can talk about programs
  - Need a language
  - Describing time and concurrency

Time

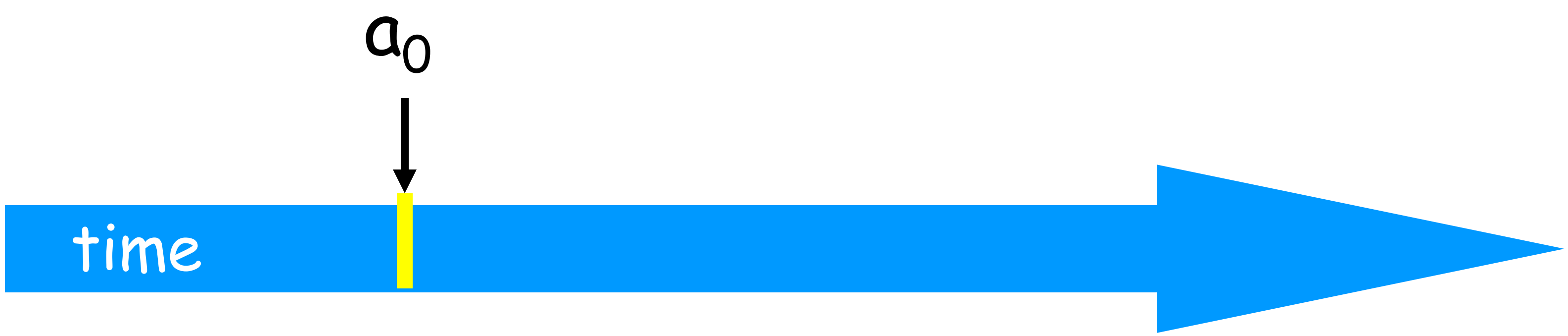
- "Absolute, true and mathematical time, of itself and from its own nature, flows equably without relation to anything external." (I. Newton, 1689)

- "Time is, like, Nature's way of making sure that everything doesn't happen all at once." (Anonymous, circa 1968)



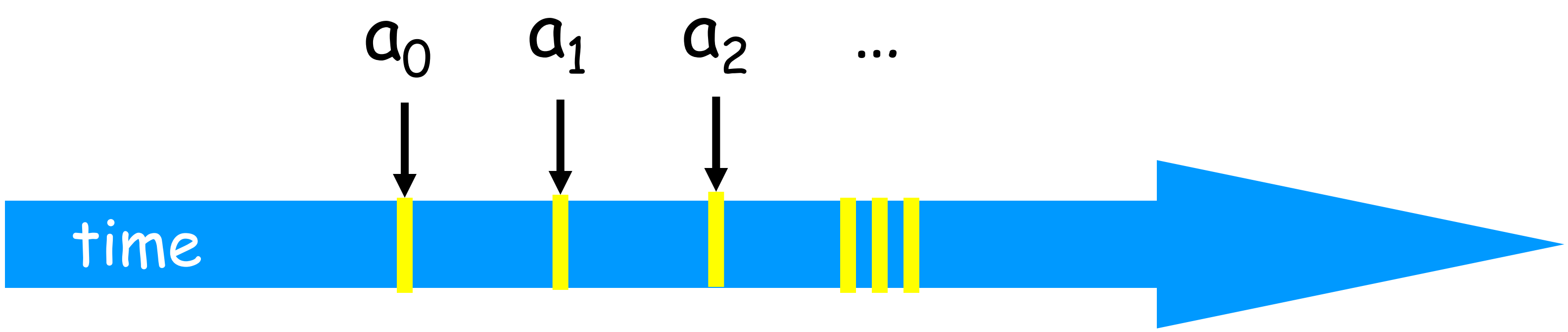
Events

- An *event* a_0 of thread *A* is
  - Instantaneous
  - No simultaneous events (break ties)



Threads

- A *thread* A is (formally) a sequence $a_0, a_1, \ldots$ of events
  - "Trace" model
  - Notation: $a_0 \rightarrow a_1$ indicates order

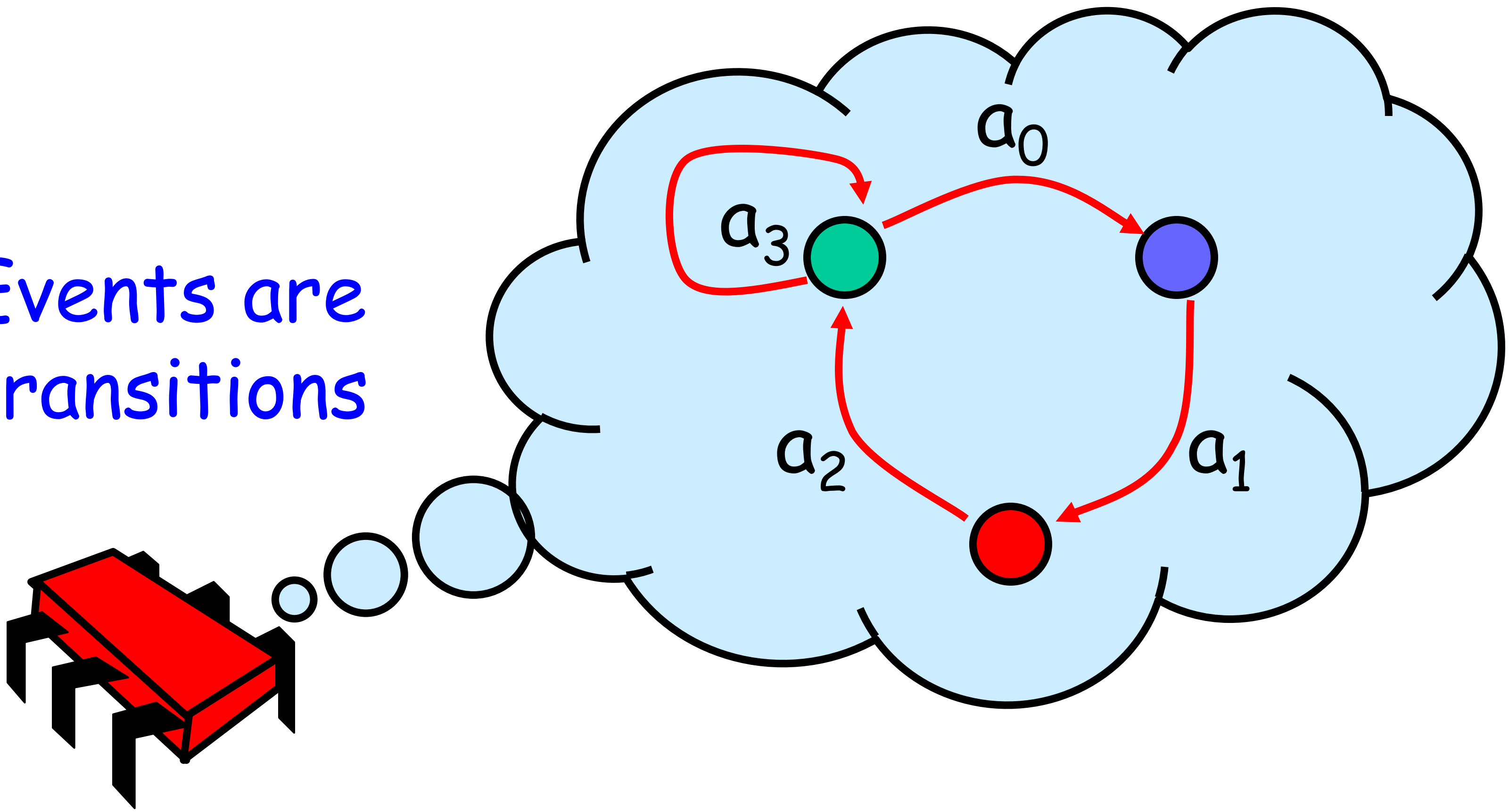


Example Thread Events

- Assign to shared variable
- Assign to local variable
- Invoke method
- Return from method
- Lots of other things …

Threads are State Machines

Events are transitions



States

- Thread State
  - Program counter
  - Local variables
- System state
  - Object fields (shared variables)
  - Union of thread states

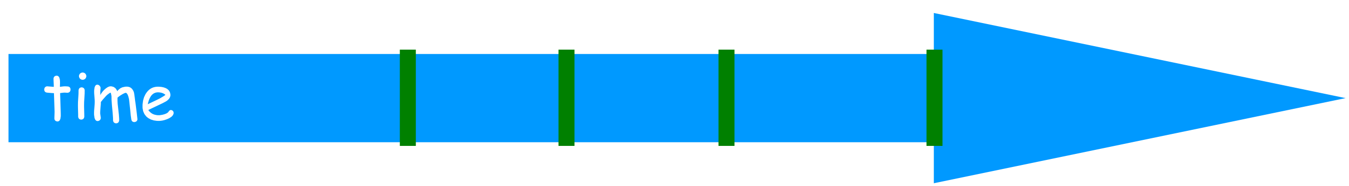
Concurrency

- Thread *A*

Concurrency

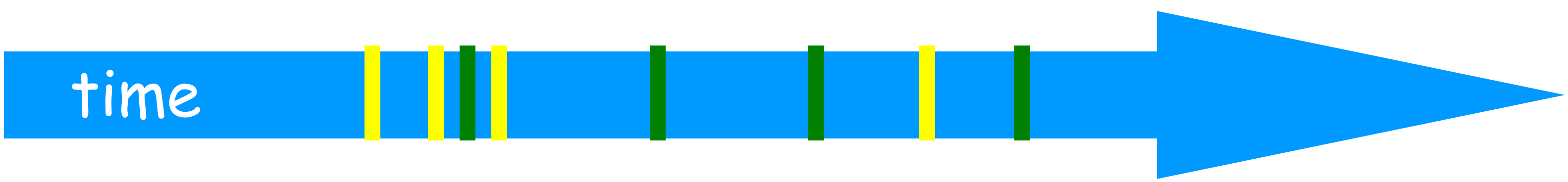
- Thread A

- Thread B



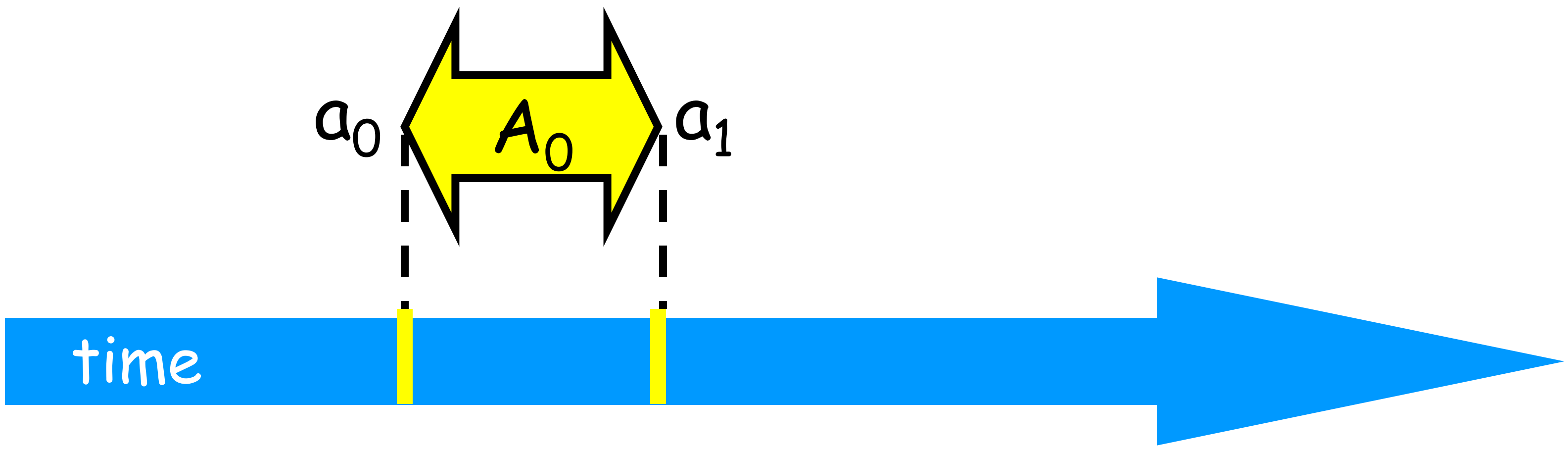
Interleavings

- Events of two or more threads
  - Interleaved
  - Not necessarily independent (why?)

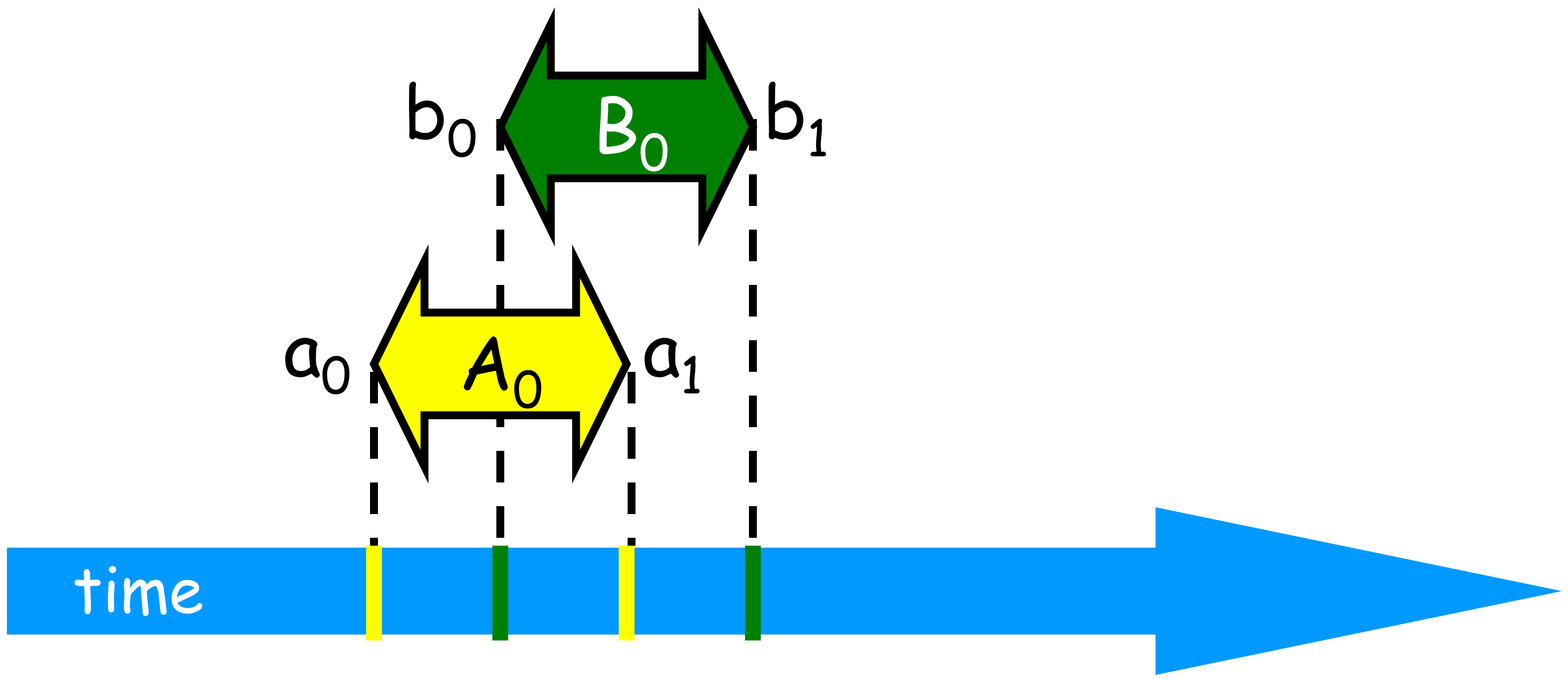


Intervals

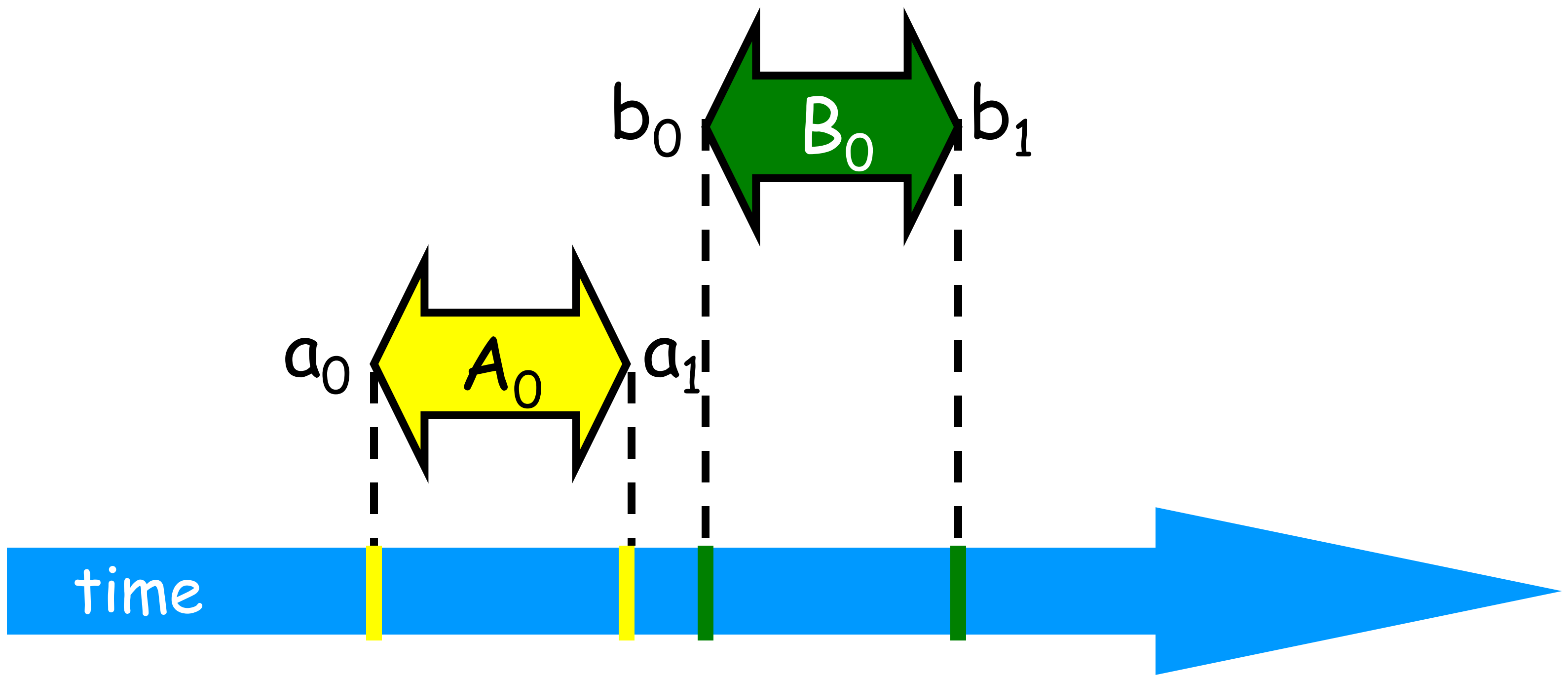
- An *interval* $A_0 = (a_0, a_1)$ is
  - Time between events a_0 and a_1



Intervals may Overlap

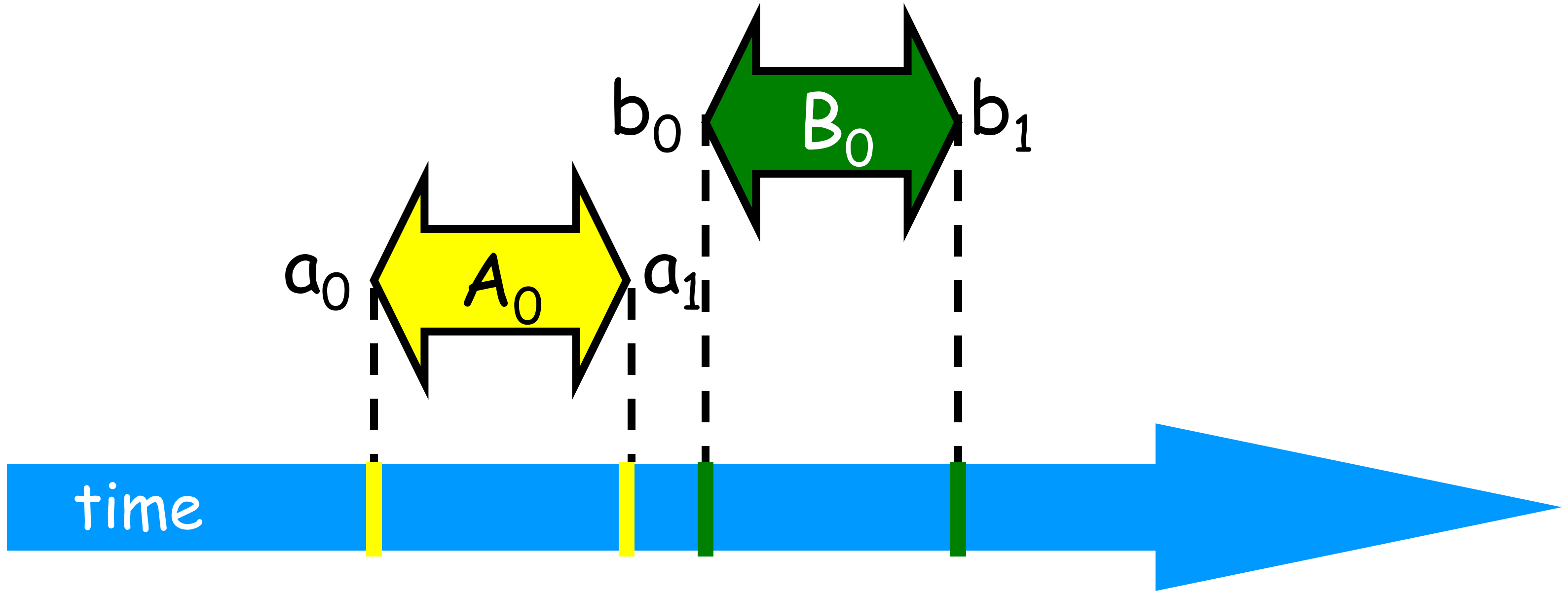


Intervals may be Disjoint

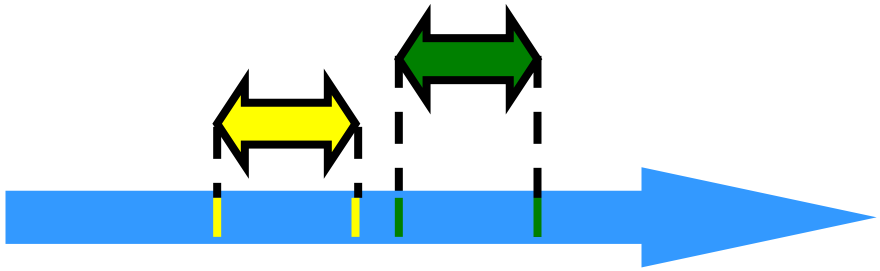


Precedence

Interval A_0 precedes interval B_0

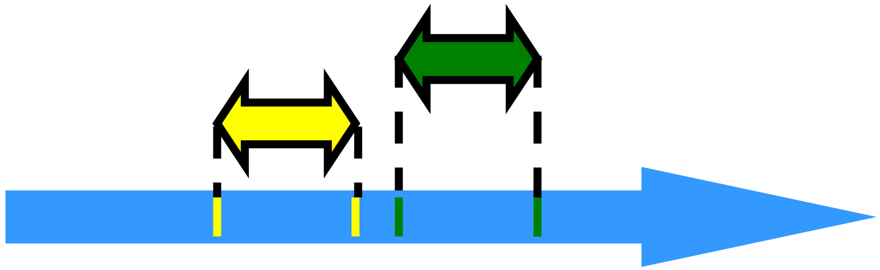


Precedence



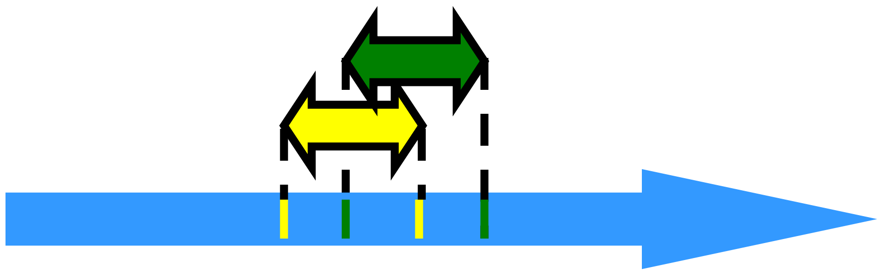
- Notation: $A_0 \rightarrow B_0$
- Formally,
  - End event of A_0 before start event of B_0
  - Also called "happens before" or "precedes"

Precedence Ordering



- Remark: $A_0 \rightarrow B_0$ is just like saying
  - 1066 AD → 1492 AD,
  - Middle Ages → Renaissance,
- Oh wait,
  - what about this week vs this month?

Precedence Ordering



- Never true that A → A
- If A →B then not true that B →A
- If A →B & B →C then A →C
- Funny thing: A →B & B →A might both be false!

Partial Orders
(you may know this already)

- Irreflexive:
  - Never true that A → A
- Antisymmetric:
  - If A → B then not true that B → A
- Transitive:
  - If A → B & B → C then A → C

Total Orders
(you may know this already)

- Also
  - Irreflexive
  - Antisymmetric
  - Transitive
- Except that for every distinct A, B,
  - Either A → B or B → A

Repeated Events

```
while (mumble) {
   a0; a1;
}
```

k-th occurrence of event a_0

a_0^k

k-th occurrence of interval $A_0 = (a_0, a_1)$

A_0^k

Implementing a Counter

```
public class Counter {
  private long value;

  public long getAndIncrement() {
    temp  = value;
    value = temp + 1;
    return temp;
  }
}
```

Make these steps *indivisible* using locks

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

acquire lock

Locks (Mutual Exclusion)

```
public interface Lock {

 public void lock();

 public void unlock();
}
```

acquire lock

release lock

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

acquire Lock

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Release lock
(no matter what)

Using Locks

```
public class Counter {
  private long value;
  private Lock lock;
  public long getAndIncrement() {
   lock.lock();
   try {
    int temp = value;
    value = value + 1;
   } finally {
     lock.unlock();
   }
   return temp;
  }}
```

Critical section

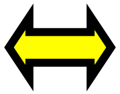
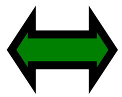
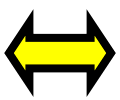
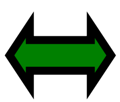
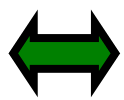
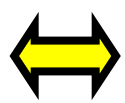
Mutual Exclusion

- Let $CS_i^k \Leftrightarrow$ be thread i's k-th critical section execution

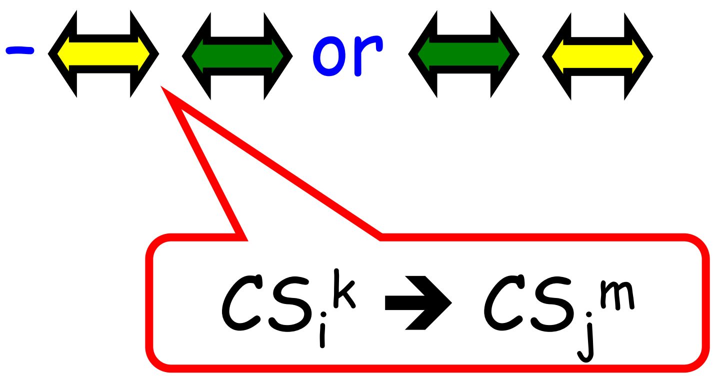
Mutual Exclusion

- Let $CS_i^k \Leftrightarrow$ be thread i's k-th critical section execution
- And $CS_j^m \Leftrightarrow$ be thread j's m-th critical section execution

Mutual Exclusion

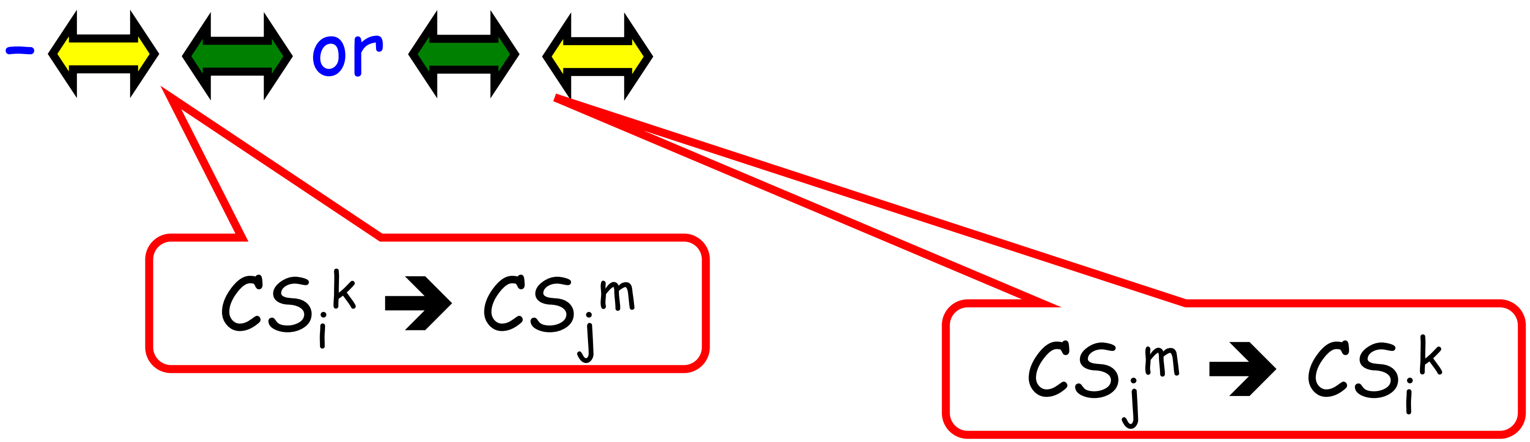
- Let CS_i^k  be thread i's k-th critical section execution
- And CS_j^m  be j's m-th execution
- Then either
  -   or  

Mutual Exclusion

- Let CS_i^k ⟺ be thread i's k-th critical section execution
- And CS_j^m ⟺ be j's m-th execution
- Then either
  - 

Mutual Exclusion

- Let CS_i^k ⟺ be thread i's k-th critical section execution
- And CS_j^m ⟺ be j's m-th execution
- Then either



Deadlock-Free

- If some thread calls **lock()**
  - And never returns
  - Then other threads must complete **lock()** and **unlock()** calls infinitely often
- System as a whole makes progress
  - Even if individuals starve

Starvation-Free

- If some thread calls lock()
  - It will eventually return
- Individual threads make progress

Two-Thread vs *n* -Thread Solutions

- Two-thread solutions first
  - Illustrate most basic ideas
  - Fits on one slide
- Then n-Thread solutions

Two-Thread Conventions

```
class … implements Lock {
  …
  // thread-local index, 0 or 1
  public void lock() {
    int i = ThreadID.get();
    int j = 1 - i;
  …
  }
}
```

Two-Thread Conventions

```
class … implements Lock {
  …
  // thread-local index, 0 or 1
  public void lock() {
    int i = ThreadID.get();
    int j = 1 - i;
  …
  }
}
```

Henceforth: i is current thread, j is other thread

LockOne

```
class LockOne implements Lock {
private volatile boolean[] flag =
                              new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

LockOne

```
class LockOne implements Lock {
private volatile boolean[] flag =
                         new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

Set my flag

LockOne

```
class LockOne implements Lock {
private volatile boolean[] flag =
                        new boolean[2];
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

Set my flag

Wait for other flag to go false

LockOne Satisfies Mutual Exclusion

- Assume CS_A^j overlaps CS_B^k
- Consider each thread's last (j-th and k-th) read and write in the lock() method before entering
- Derive a contradiction

From the Code

- $\text{write}_A(\text{flag}[A]=\text{true}) \rightarrow \text{read}_A(\text{flag}[B]==\text{false}) \rightarrow CS_A$

- $\text{write}_B(\text{flag}[B]=\text{true}) \rightarrow \text{read}_B(\text{flag}[A]==\text{false}) \rightarrow CS_B$

```
class LockOne implements Lock {
…
public void lock() {
  flag[i] = true;
  while (flag[j]) {}
 }
```

From the Assumption

- **$read_A$(flag[B]==false) → $write_B$(flag[B]=true)**

- **$read_B$(flag[A]==false) → $write_A$(flag[B]=true)**

Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
  - $read_B(flag[A]==false) \rightarrow write_A(flag[A]=true)$
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
  - $write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$

Combining

- Assumptions:
  - $read_A(flag[B]==false) \rightarrow write_B(flag[B]=true)$
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Combining

- Assumptions:
  - $\text{read}_A(\text{flag[B]==false}) \rightarrow \text{write}_B(\text{flag[B]=true})$
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Combining

- Assumptions:
  - read_A(flag[B]==false) → write_B(flag[B]=true)
  - read_B(flag[A]==false) → write_A(flag[A]=true)
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  - write_A(flag[A]=true) → read_A(flag[B]==false)
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  - $read_A$(flag[B]==false) → $write_B$(flag[B]=true)
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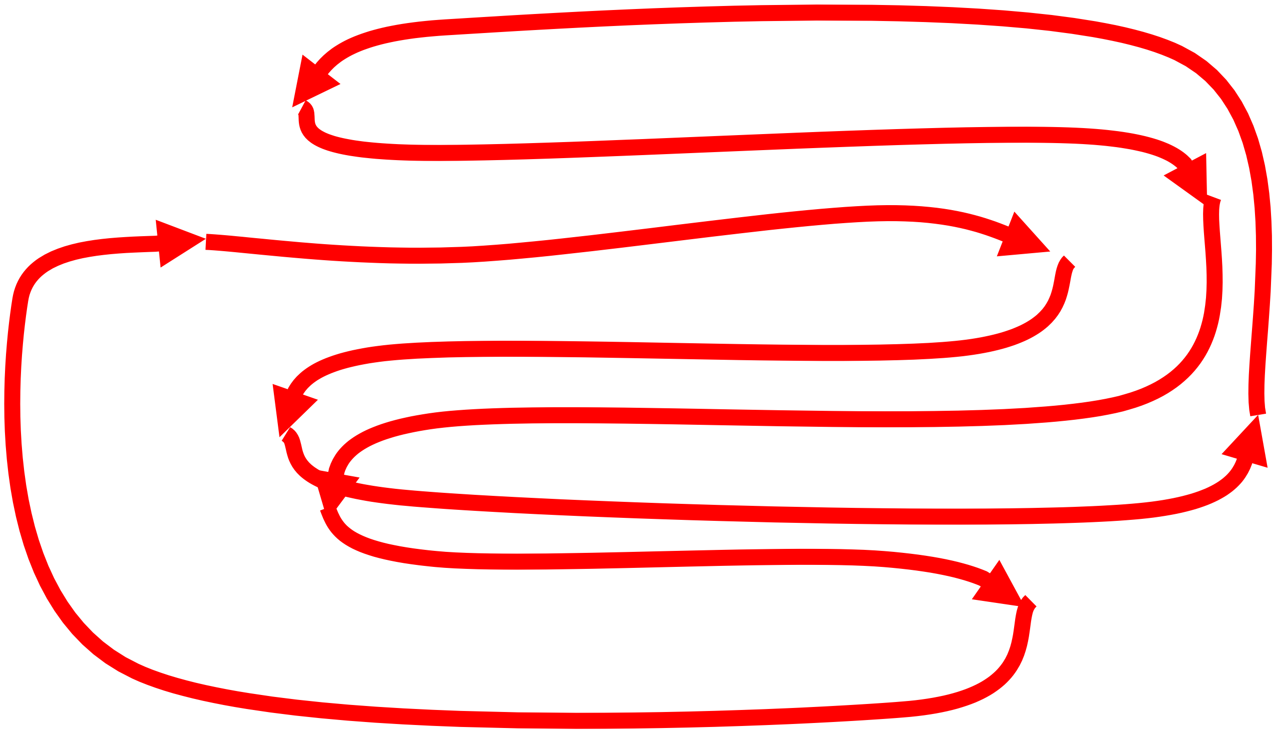
- From the code
  - $write_A(flag[A]=true) \rightarrow read_A(flag[B]==false)$
  - $write_B(flag[B]=true) \rightarrow read_B(flag[A]==false)$

Cycle!



Deadlock Freedom

- LockOne Fails deadlock-freedom
  - Concurrent execution can deadlock

```
flag[i] = true;      flag[j] = true;
while (flag[j]){}    while (flag[i]){}
```

  - Sequential executions OK

LockTwo

```
public class LockTwo implements Lock {
 private volatile int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

LockTwo

```
public class LockTwo implements Lock {
 private volatile int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Let other go first

LockTwo

```
public class LockTwo implements Lock {
 private volatile int victim;
 public void lock() {
 victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Wait for permission

LockTwo

```
public class Lock2 implements Lock {
 private volatile int victim;
 public void lock() {
  victim = i;
  while (victim == i) {};
 }

 public void unlock() {}
}
```

Nothing to do

LockTwo Claims

- Satisfies mutual exclusion
  - If thread `i` in CS
  - Then `victim == j`
  - Cannot be both 0 and 1
- Not deadlock free
  - Sequential execution deadlocks
  - Concurrent execution does not

```
public void LockTwo() {
  victim = i;
  while (victim == i) {};
 }
```

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Defer to other

Peterson's Algorithm

```
public void lock() {
 flag[i] = true;
 victim  = i;
 while (flag[j] && victim == i) {};
}
public void unlock() {
 flag[i] = false;
}
```

Announce I'm interested

Defer to other

Wait while other interested & I'm the victim

Peterson's Algorithm

```
public void lock() {
  flag[i] = true;
  victim  = i;
  while (flag[j] && victim == i) {};
 }
public void unlock() {
  flag[i] = false;
}
```

Announce I'm interested

Defer to other

Wait while other interested & I'm the victim

No longer interested

Mutual Exclusion

```
public void lock() {
  flag[i] = true;
  victim  = i;
  while (flag[j] && victim == i) {};
```

- If thread 0 in critical section,
  - flag[0]=true,
  - victim = 1
- If thread 1 in critical section,
  - flag[1]=true,
  - victim = 0

Cannot both be true

Deadlock Free

```
public void lock() {
  …
  while (flag[j] && victim == i) {};
```

- Thread blocked
  - only at `while` loop
  - only if it is the victim
- One or the other must not be the victim

Starvation Free

- Thread `i` blocked only if `j` repeatedly re-enters so that

  `flag[j] == true` and `victim == i`
- When `j` re-enters
  - it sets `victim` to `j`.
  - So `i` gets in

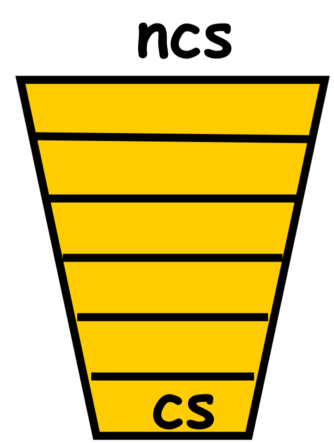
```
public void lock() {
  flag[i] = true;
  victim   = i;
  while (flag[j] && victim == i) {};
}

public void unlock() {
  flag[i] = false;
}
```

The Filter Algorithm for *n* Threads

There are ***n-1*** "waiting rooms" called levels

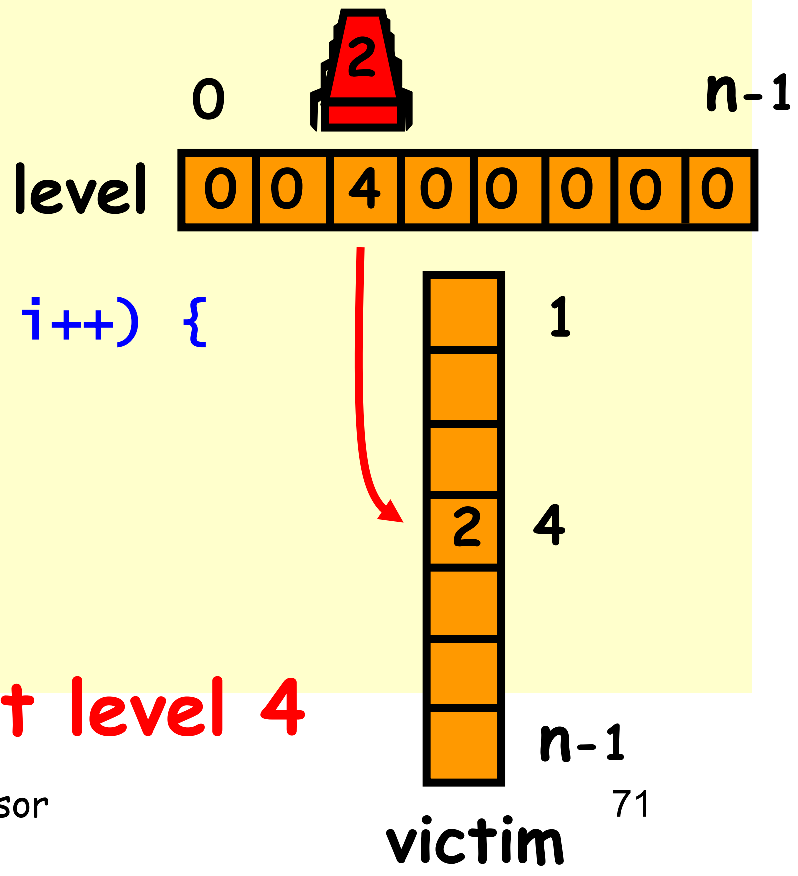
- At each level
  - At least one enters level
  - At least one blocked if many try
- Only one thread makes it through



Filter

```
class Filter implements Lock {
  volatile int[] level;  // level[i] for thread i
  volatile int[] victim; // victim[L] for level L

  public Filter(int n) {
      level  = new int[n];
      victim = new int[n];
      for (int i = 1; i < n; i++) {
          level[i] = 0;
      }}
   …
}
```



Thread 2 at level 4

Filter

```
class Filter implements Lock {
  …

  public void lock(){
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i level[k] >= L) &&
             victim[L] == i );
    }}
  public void unlock() {
    level[i] = 0;
  }}
```

Filter

```
class Filter implements Lock {
  …

  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;

      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i);
    }}
  public void release(int i) {
    level[i] = 0;
  }}
```

One level at a time

Filter

```
class Filter implements Lock {
  …

  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i);
    }}
  public void release(int i)
    level[i] = 0;
  }}
```

Announce intention to enter level L

Filter

```
class Filter implements Lock {
  int level[n];
  int victim[n];
  public void lock() {
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i);
    }}
  public void release(int i)
    level[i] = 0;
  }}
```

Give priority to anyone but me

Filter

Wait as long as someone else is at same or higher level, and I'm designated victim

```
public void lock() {
  for (int L = 1; L < n; L++) {
    level[i]  = L;
    victim[L] = i;
    while ((∃ k != i) level[k] >= L) &&
           victim[L] == i);
  }}
public void release(int i) {
  level[i] = 0;
}}
```

Filter

```
class Filter implements Lock {
  int level[n];
  int victim[n];
  public void lock() {
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      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i) level[k] >= L) &&
             victim[L] == i);
    }}
```
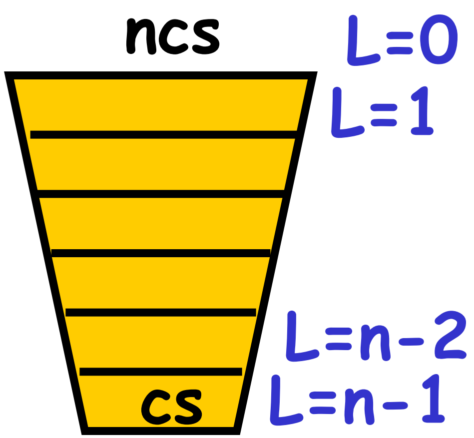

Thread enters level L when it completes the loop

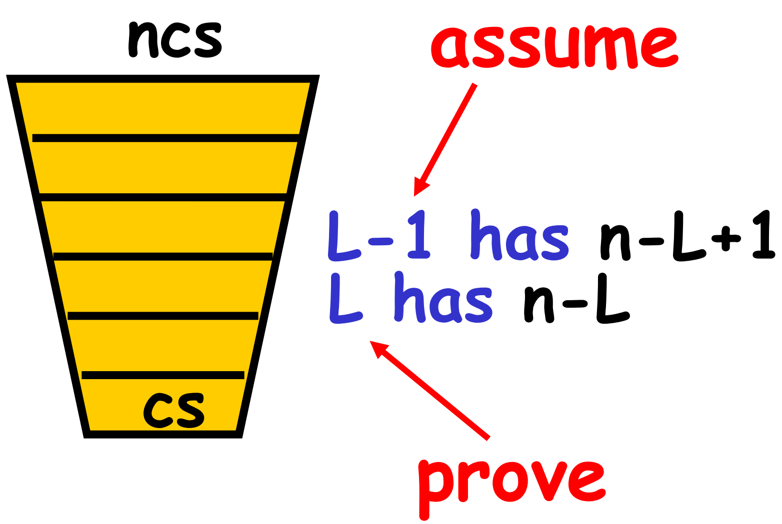
Claim

- Start at level L=0
- At most n-L threads enter level L
- Mutual exclusion at level L=n-1

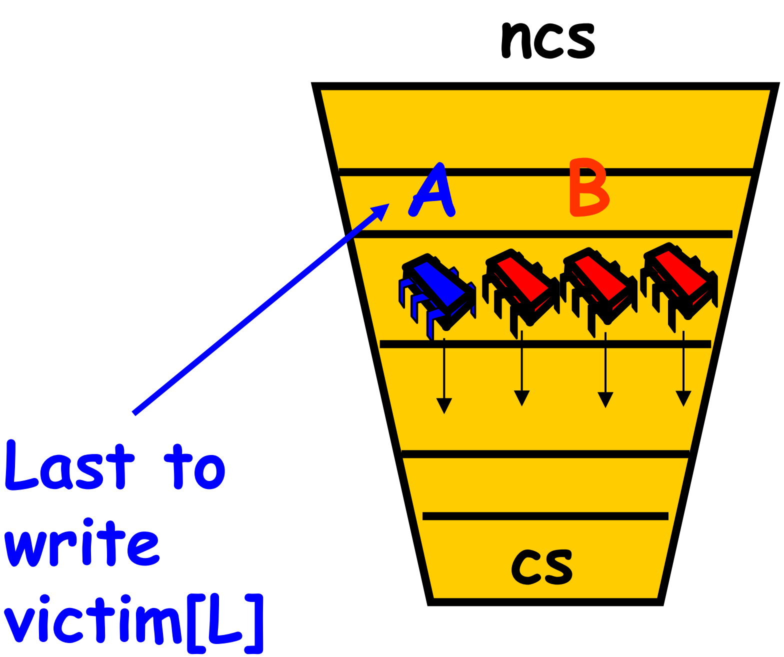


Induction Hypothesis

- No more than n-L+1 at level L-1
- Induction step: by contradiction
- Assume all at level L-1 enter level L
- A last to write victim[L]
- B is any other thread at level L



Proof Structure



Assumed to enter L-1

n-L+1 = 4

n-L+1 = 4

By way of contradiction
all enter L

Show that A must have seen
B at level L and since victim[L] == A
could not have entered

From the Code

(1) $write_B(level[B]=L) \rightarrow write_B(victim[L]=B)$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i] = L;
   victim[L]  = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

From the Code

(2) $\text{write}_A(\text{victim}[L]=A) \rightarrow \text{read}_A(\text{level}[B])$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i] = L;
   victim[L]  = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

By Assumption

(3) $write_B(victim[L]=B) \rightarrow write_A(victim[L]=A)$

By assumption, *A* is the last thread to write **victim[L]**

Combining Observations

(1) $write_B(level[B]=L) \rightarrow write_B(victim[L]=B)$

(3) $write_B(victim[L]=B) \rightarrow write_A(victim[L]=A)$

(2) $write_A(victim[L]=A) \rightarrow read_A(level[B])$

Combining Observations

(1) $write_B(level[B]=L)$ ➔

(3) $write_B(victim[L]=B)$ ➔ $write_A(victim[L]=A)$

(2) ➔ $read_A(level[B])$

```
public void lock() {
 for (int L = 1; L < n; L++) {
   level[i]  = L;
   victim[L] = i;
   while ((∃ k != i) level[k] >= L)
          && victim[L] == i) {};
   }}
```

Combining Observations

(1) $\text{write}_B(\text{level}[B]=L) \rightarrow$

(3) $\text{write}_B(\text{victim}[L]=B) \rightarrow \text{write}_A(\text{victim}[L]=A)$

(2)

$\rightarrow \text{read}_A(\text{level}[B])$

Thus, A read level[B] ≥ L,
A was last to write victim[L],
so it could not have entered level L!

No Starvation

- Filter Lock satisfies properties:
  - Just like Peterson Alg at any level
  - So no one starves
- But what about fairness?
  - Threads can be overtaken by others

Bounded Waiting

- Want stronger fairness guarantees
- Thread not "overtaken" too much
- Need to adjust definitions ....

Bounded Waiting

- Divide `lock()` method into 2 parts:
  - Doorway interval:
    - Written D_A
    - always finishes in finite steps
  - Waiting interval:
    - Written W_A
    - may take unbounded steps

r-Bounded Waiting

- For threads A and B:
  - If $D_A^k \rightarrow D_B^j$
    - A's k-th doorway precedes B's j-th doorway
  - Then $CS_A^k \rightarrow CS_B^{j+r}$
    - A's k-th critical section precedes B's (j+r)-th critical section
    - B cannot overtake A by more than r times
- First-come-first-served means r = 0.

Fairness Again

- Filter Lock satisfies properties:
  - No one starves
  - But very weak fairness
    - Not **r**-bounded for any r!
  - That's pretty lame…

Bakery Algorithm

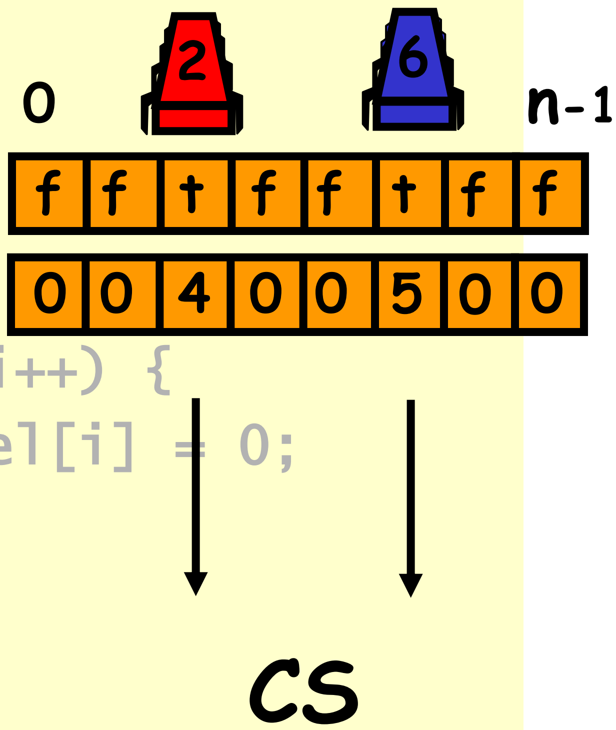
- Provides First-Come-First-Served
- How?
  - Take a "number"
  - Wait until lower numbers have been served
- Lexicographic order
  - (a,i) > (b,j)
    - If a > b, or a = b and i > j

Bakery Algorithm

```
class Bakery implements Lock {
  volatile boolean[] flag;
  volatile Label[] label;
  public Bakery (int n) {
    flag  = new boolean[n];
    label = new Label[n];
    for (int i = 0; i < n; i++) {
       flag[i] = false; label[i] = 0;
    }
  }
 …
```

Bakery Algorithm

```
class Bakery implements Lock {
  volatile boolean[] flag;
  volatile Label[] label;
  public Bakery (int n) {
    flag  = new boolean[n];
    label = new Label[n];
    for (int i = 0; i < n; i++) {
       flag[i] = false; label[i] = 0;
    }
  }
 …
```



Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Doorway

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

I'm interested

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Take increasing label (read labels in some arbitrary order)

Bakery Algorithm

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
            && (label[i],i) > (label[k],k));
 }
```

Someone is interested

Bakery Algorithm

```
class Bakery implements Lock {
   boolean flag[n];
   int label[n];

 public void lock() {
   flag[i]  = true;
   label[i] = max(label[0], …,label[n-1])+1;
   while (∃k flag[k]
              && (label[i],i) > (label[k],k));
 }
```

Someone is interested

With lower (label,i) in lexicographic order

Bakery Algorithm

```
class Bakery implements Lock {

    …

 public void unlock() {
   flag[i] = false;
 }
}
```

Bakery Algorithm

```
class Bakery implements Lock {

    …

  public void unlock() {
    flag[i] = false;
  }
}
```

No longer interested

labels are always increasing

No Deadlock

- There is always one thread with earliest label
- Ties are impossible (why?)

First-Come-First-Served

- If $D_A \rightarrow D_B$ then A's label is earlier
  - $write_A(label[A]) \rightarrow read_B(label[A]) \rightarrow write_B(label[B]) \rightarrow read_B(flag[A])$
- So B is locked out while flag[A] is true

```
class Bakery implements Lock {

public void lock() {
  flag[i]  = true;
  label[i] = max(label[0],
               …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) >
  (label[k],k));
 }
```

Mutual Exclusion

- Suppose A and B in CS together
- Suppose A has earlier label
- When B entered, it must have seen
  - flag[A] is *false*, or
  - label[A] > label[B]

```
class Bakery implements Lock {

public void lock() {
  flag[i]  = true;
  label[i] = max(label[0],
                 …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) >
  (label[k],k));
 }
```

Mutual Exclusion

- Labels are strictly increasing so
- B must have seen flag[A] == false

Mutual Exclusion

- Labels are strictly increasing so
- B must have seen flag[A] == false
- $\text{Labeling}_B \rightarrow \text{read}_B(\text{flag}[A]) \rightarrow \text{write}_A(\text{flag}[A]) \rightarrow \text{Labeling}_A$

Mutual Exclusion

- Labels are strictly increasing so
- B must have seen flag[A] == false
- $\text{Labeling}_B \rightarrow \text{read}_B(\text{flag}[A]) \rightarrow \text{write}_A(\text{flag}[A]) \rightarrow \text{Labeling}_A$
- Which contradicts the assumption that A has an earlier label

Bakery Y2^{32}K Bug

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Bakery $Y2^{32}K$ Bug

```
class Bakery implements Lock {
  …
 public void lock() {
  flag[i]  = true;
  label[i] = max(label[0], …,label[n-1])+1;
  while (∃k flag[k]
           && (label[i],i) > (label[k],k));
 }
```

Mutex breaks if `label[i]` overflows

Does Overflow Actually Matter?

- Yes
  - Y2K
  - 18 January 2038 (Unix `time_t` rollover)
  - 16-bit counters
- No
  - 64-bit counters
- Maybe
  - 32-bit counters

Does Overflow Actually Matter?

- 32bit counters
  - Signed integer : $(-2^{31}, 2^{31} - 1)$
    - In seconds, (-78 years, 78 years)
  - Unsigned : $(0, 2^{32})$
    - In seconds, 136 years
- Unix `time_t`
  - Started at Jan 1, 1970
  - On Jan 19, 2038, overflow

Timestamps

- Label variable is really a timestamp
- Need ability to
  - Read others' timestamps
  - Compare them
  - Generate a later timestamp
- Can we do this without overflow?

The Good News

- One can construct a
  - Wait-free (no mutual exclusion)
  - Concurrent
  - Timestamping system
  - That never overflows

The ~~Good~~ Bad News

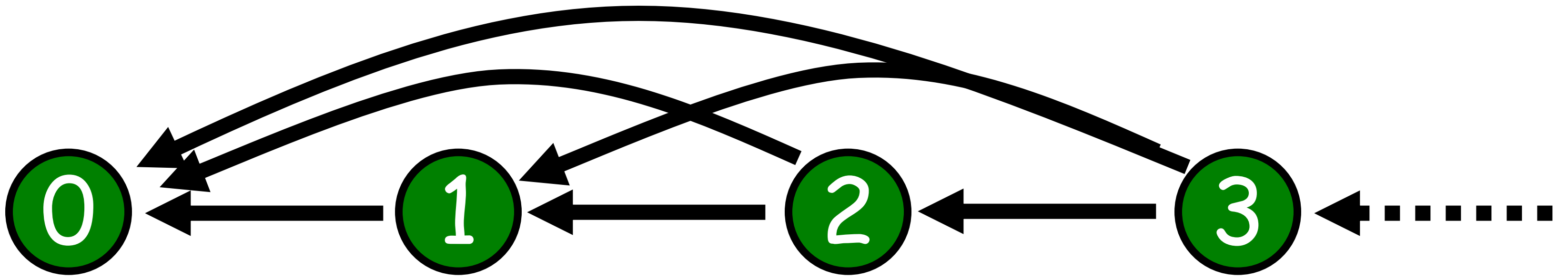
- One can construct a
  - Wait-free (no mutual exclusion)
  - Concurrent
  - Timestamping system
  - That never overflows

This part is hard

Instead …

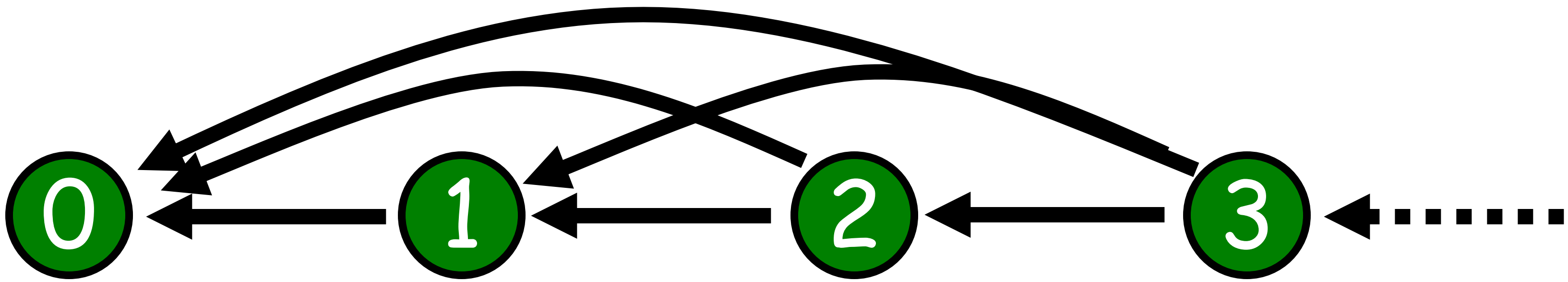
- We construct a Sequential timestamping system
  - Same basic idea
  - But simpler
- Uses mutex to read & write atomically
- No good for building locks
  - But useful anyway

Precedence Graphs



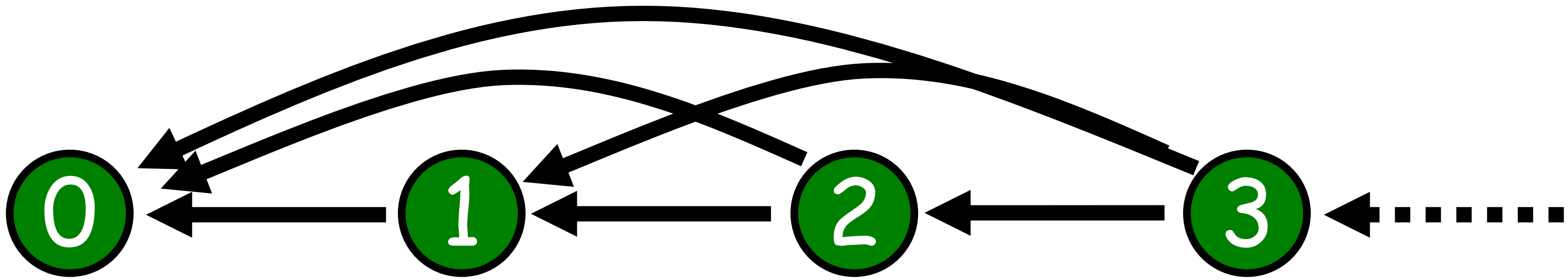
- Timestamps form directed graph
- Edge x to y
  - Means x is later timestamp
  - We say x dominates y

Unbounded Counter Precedence Graph

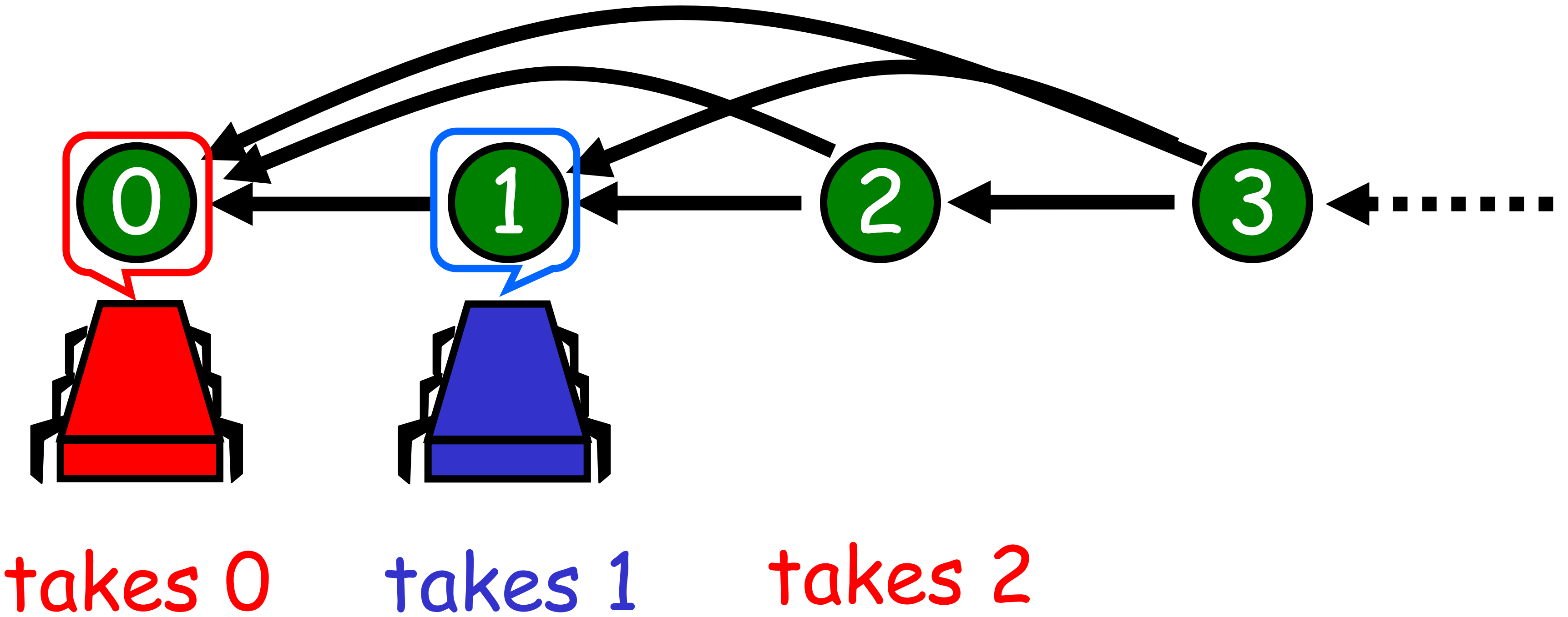


- Timestamping = move tokens on graph
- Atomically
  - read others' tokens
  - move mine
- Ignore tie-breaking for now

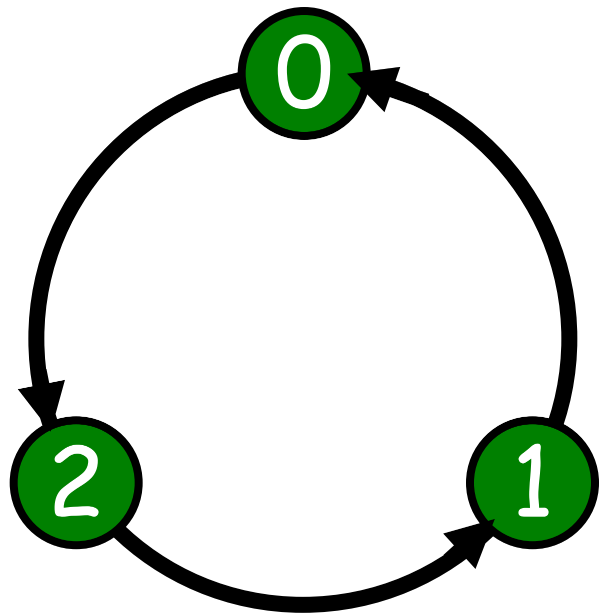
Unbounded Counter Precedence Graph



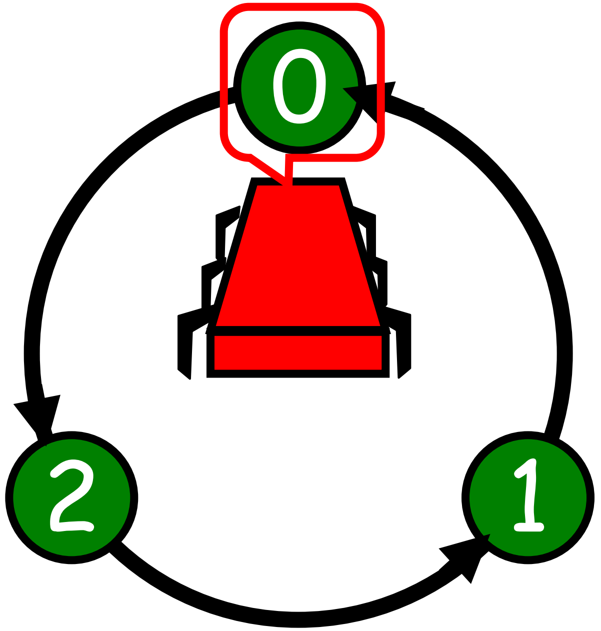
Unbounded Counter Precedence Graph



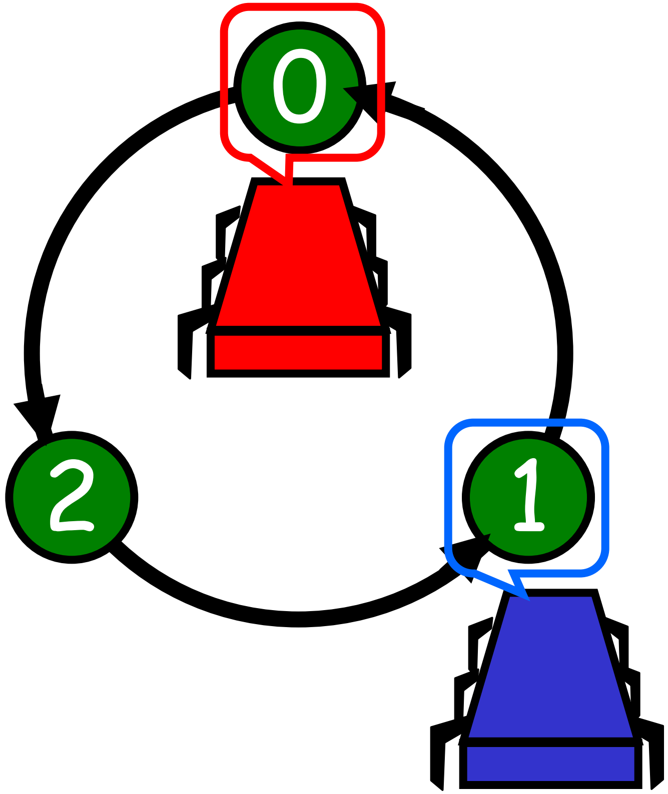
Two-Thread Bounded Precedence Graph



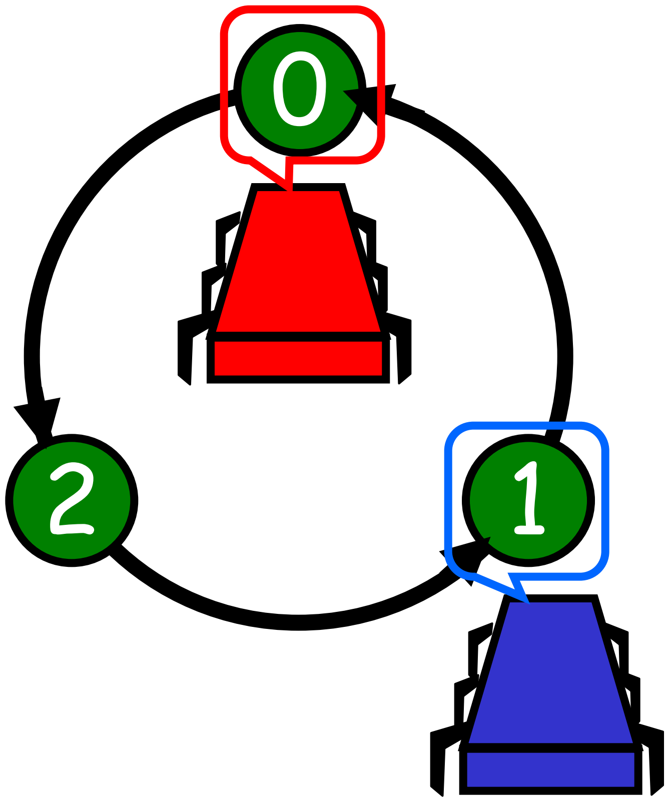
Two-Thread Bounded Precedence Graph



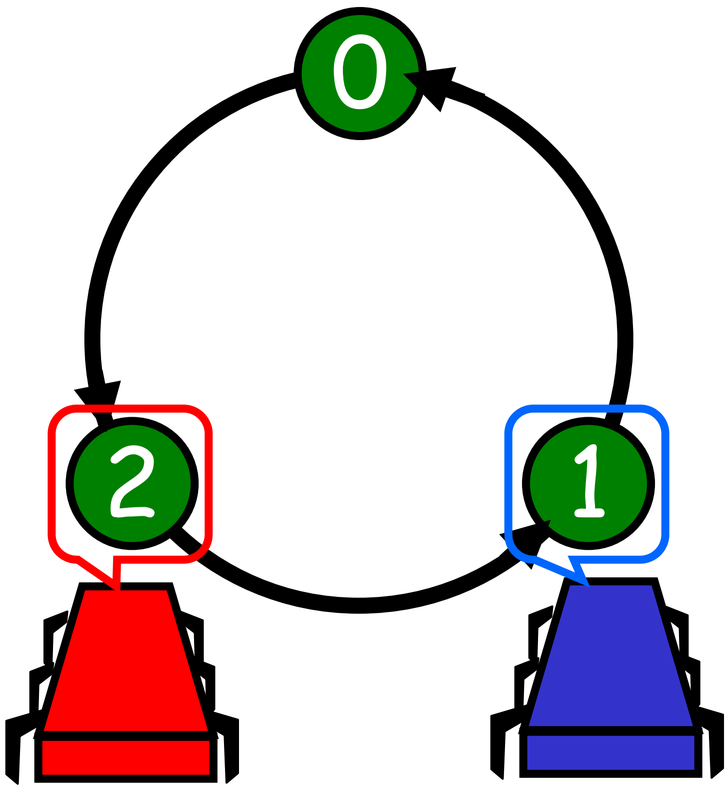
Two-Thread Bounded Precedence Graph



Two-Thread Bounded Precedence Graph

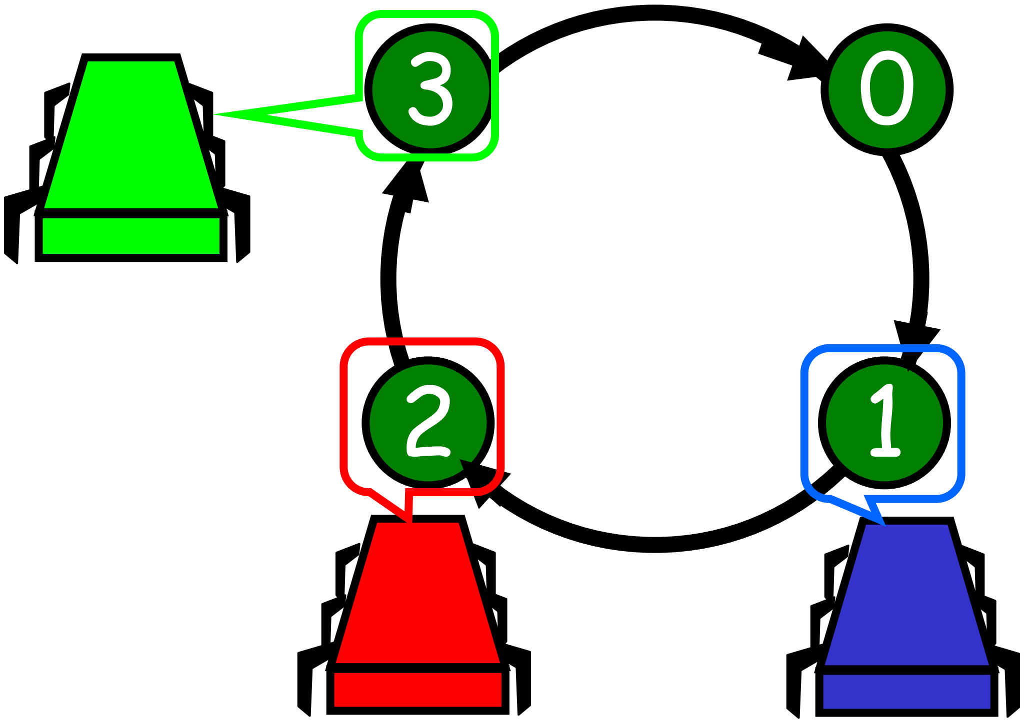


Two-Thread Bounded Precedence Graph T^2

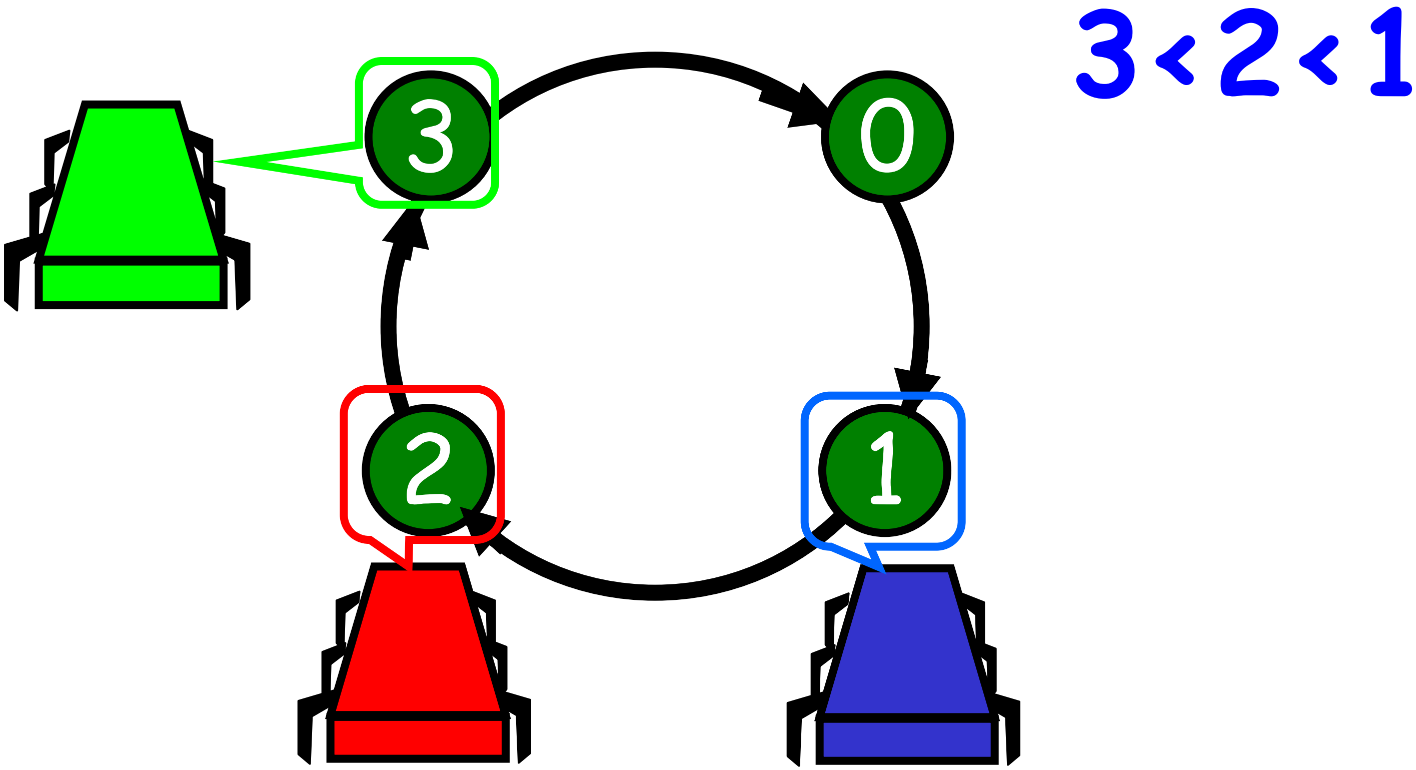


and so on ...

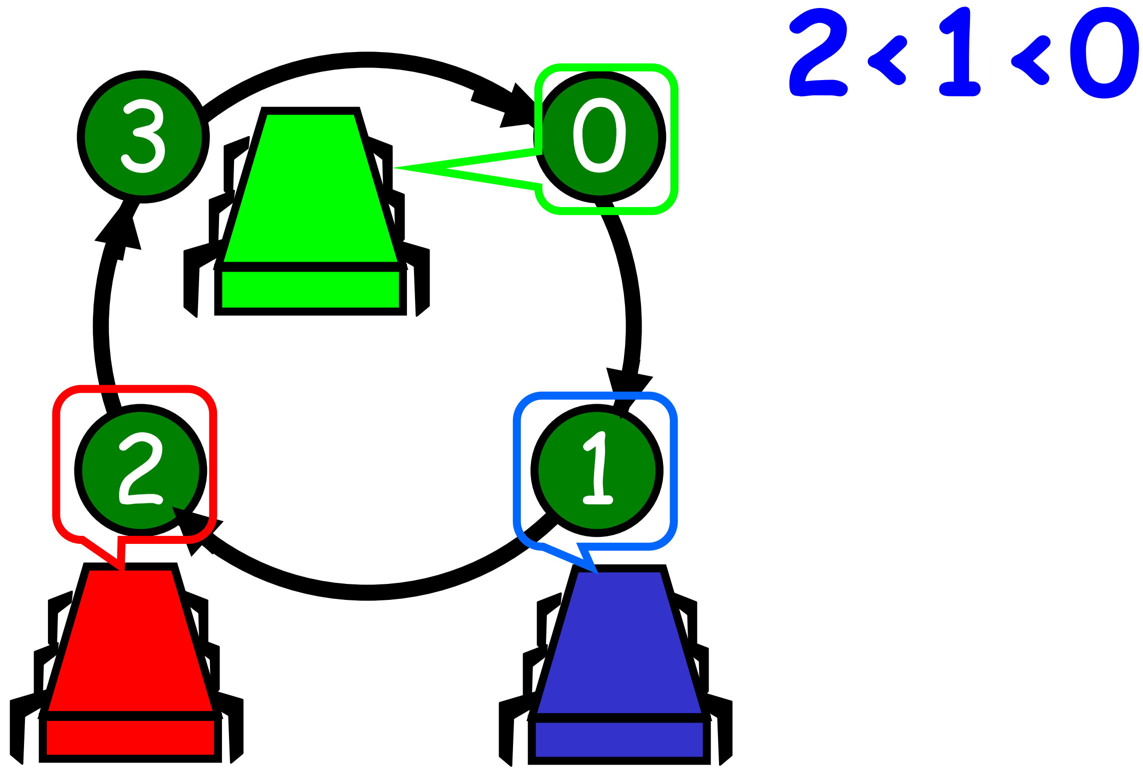
Three-Thread Bounded Precedence Graph?



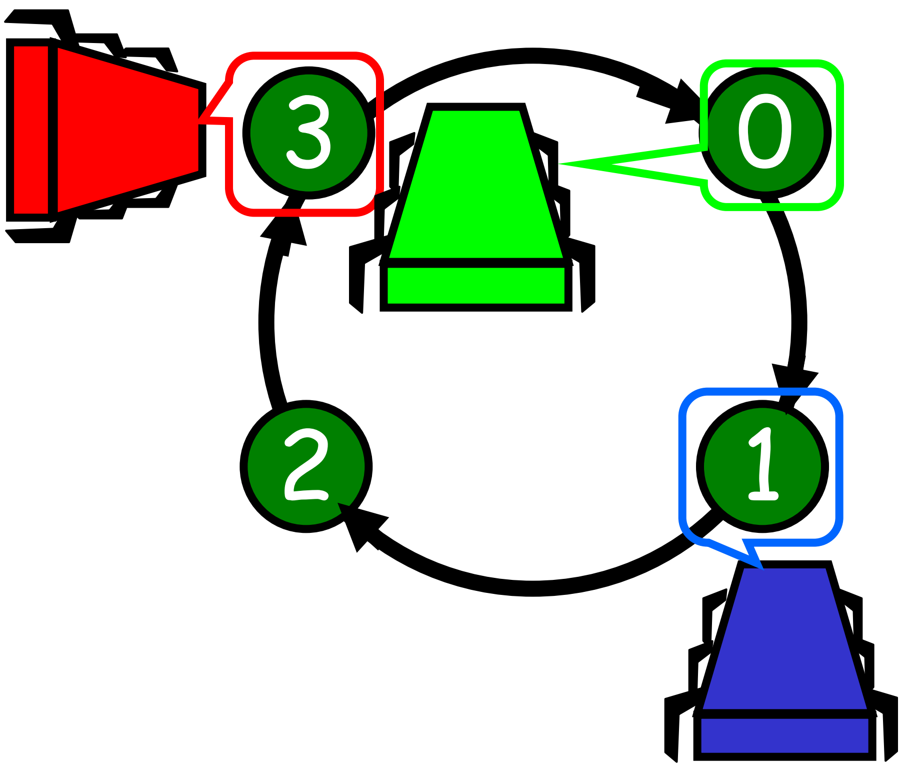
Three-Thread Bounded Precedence Graph?



Three-Thread Bounded Precedence Graph?

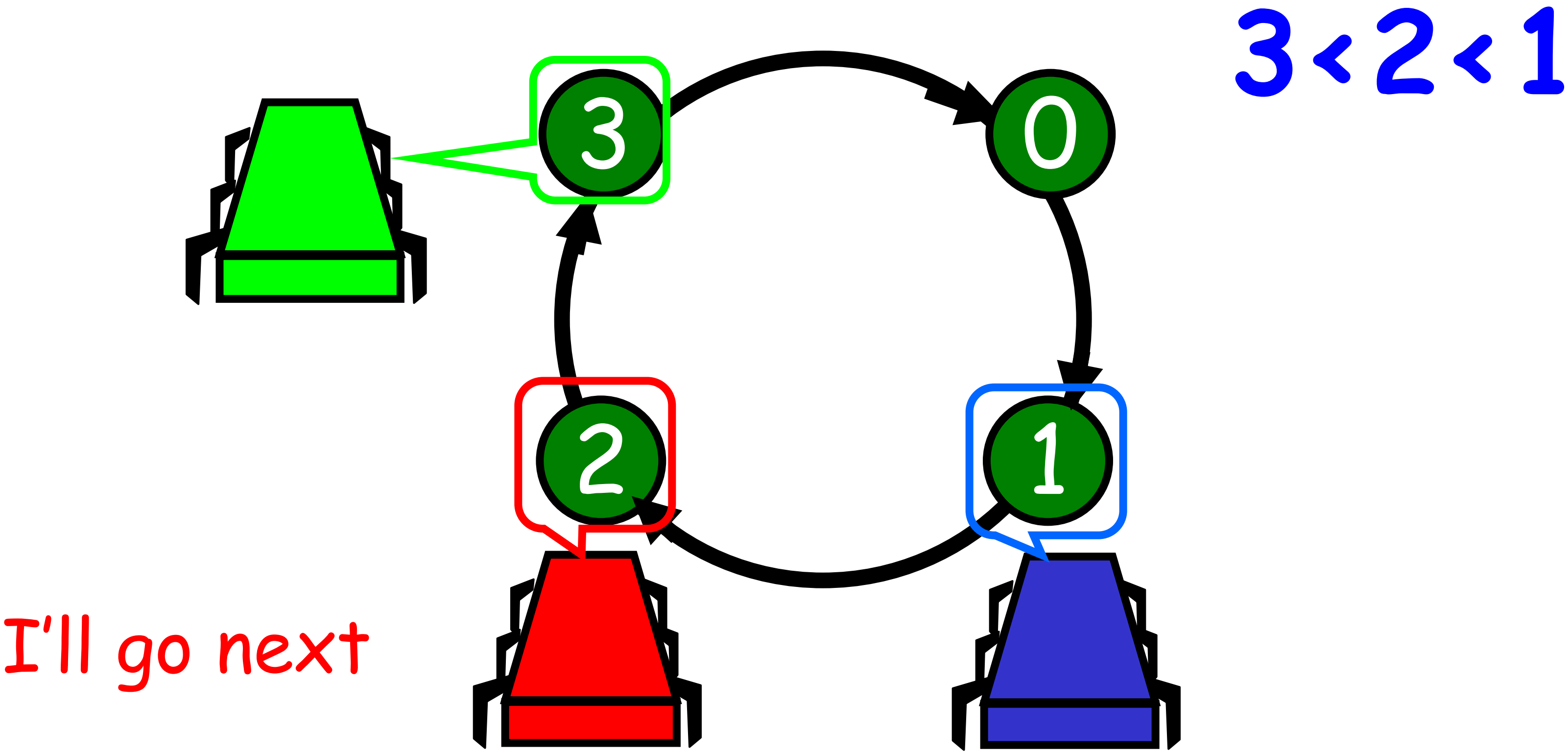


Three-Thread Bounded Precedence Graph?

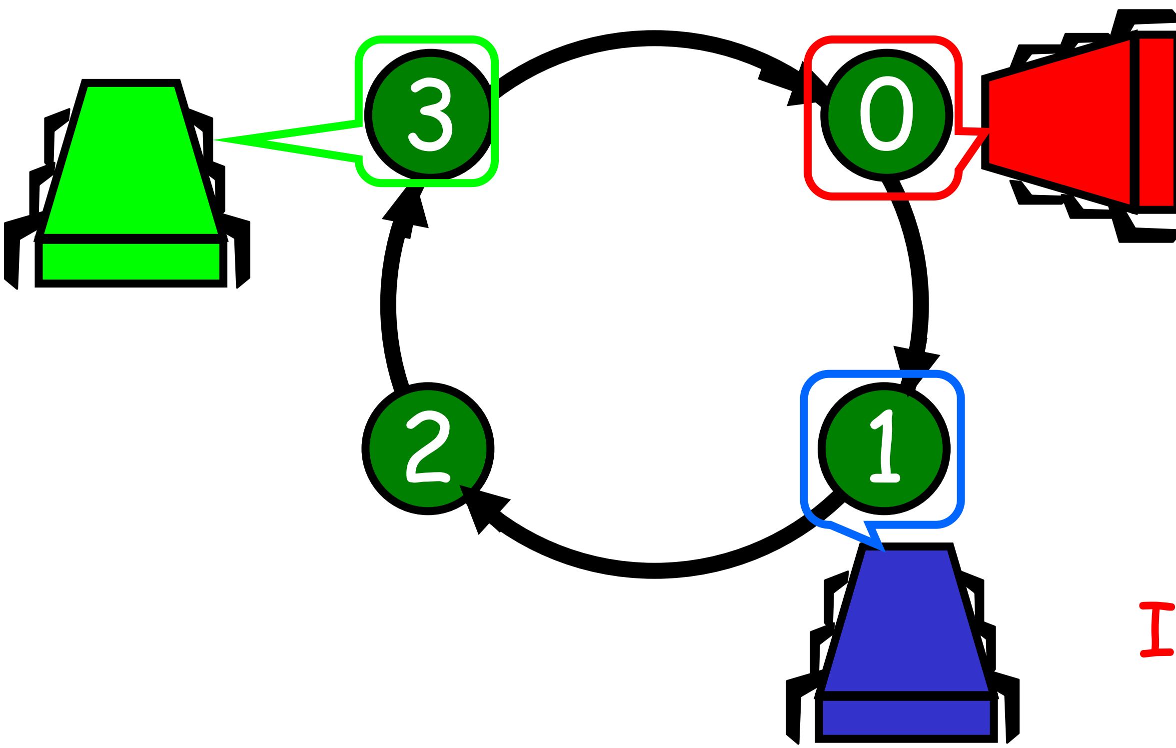


1 < 0 < 3

Three-Thread Bounded Precedence Graph?



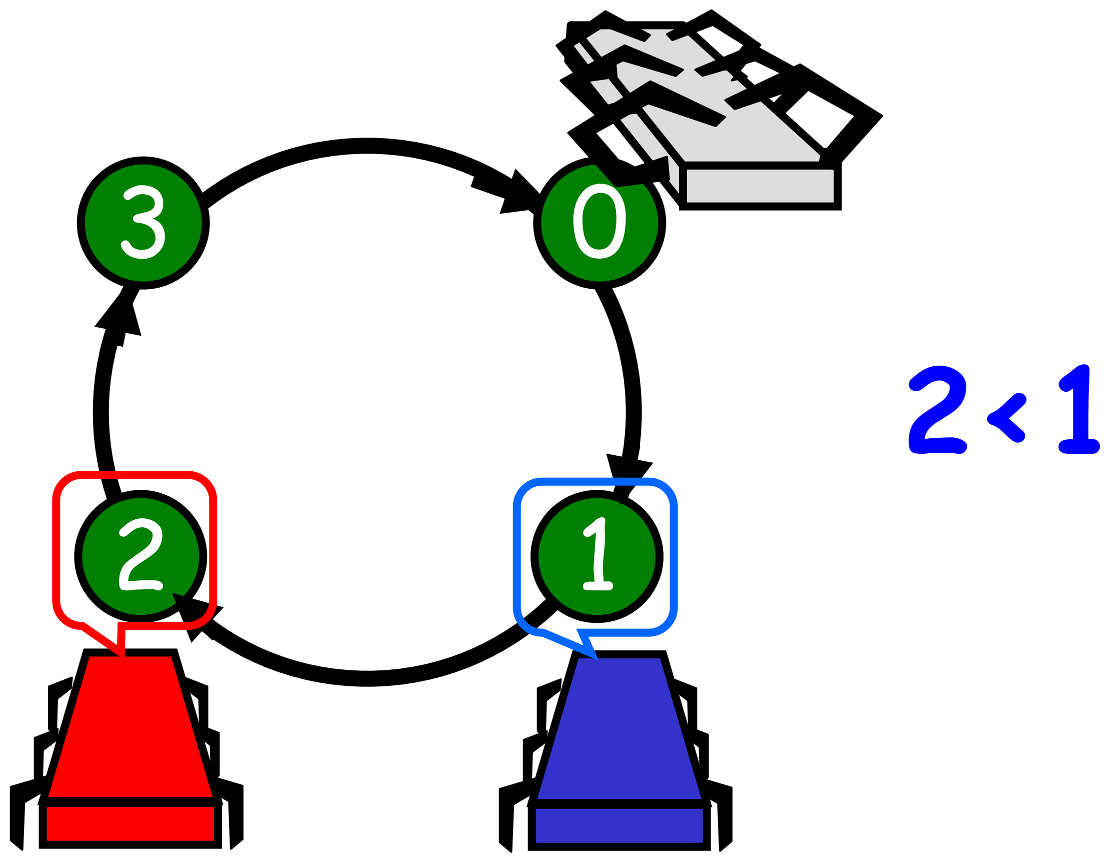
Three-Thread Bounded Precedence Graph?



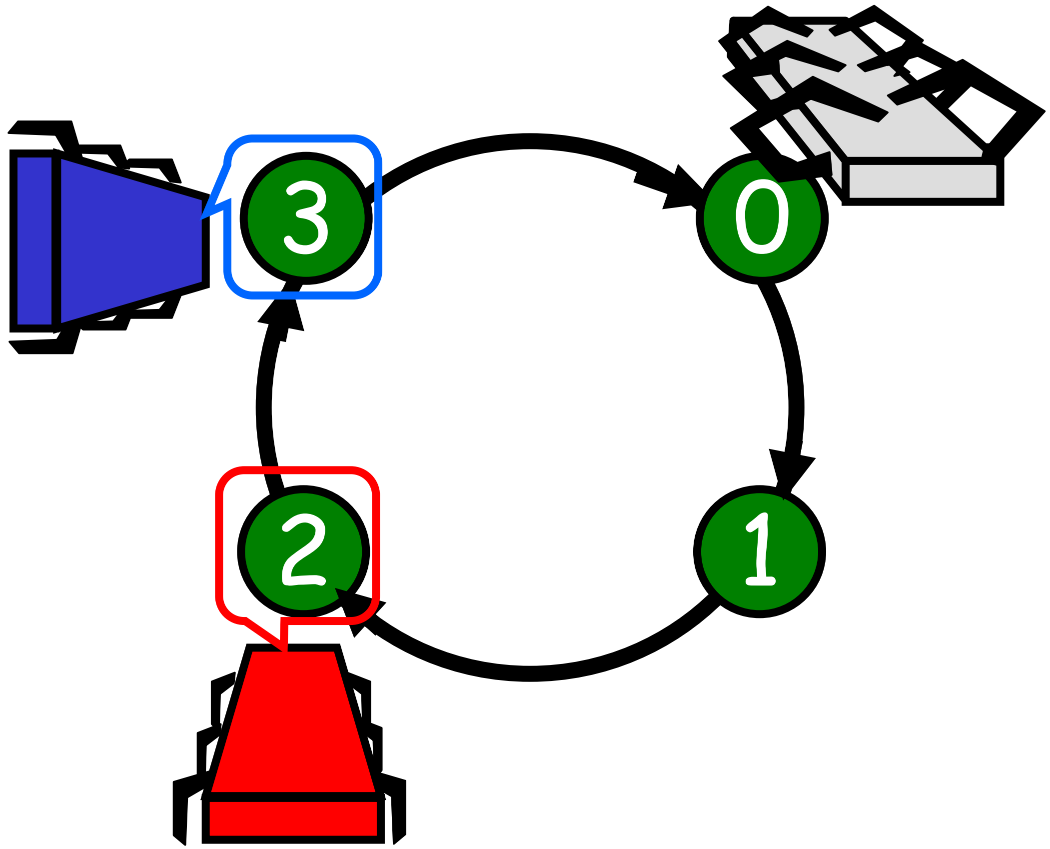
1<0<3

Is it O.K. ?

How about this ?



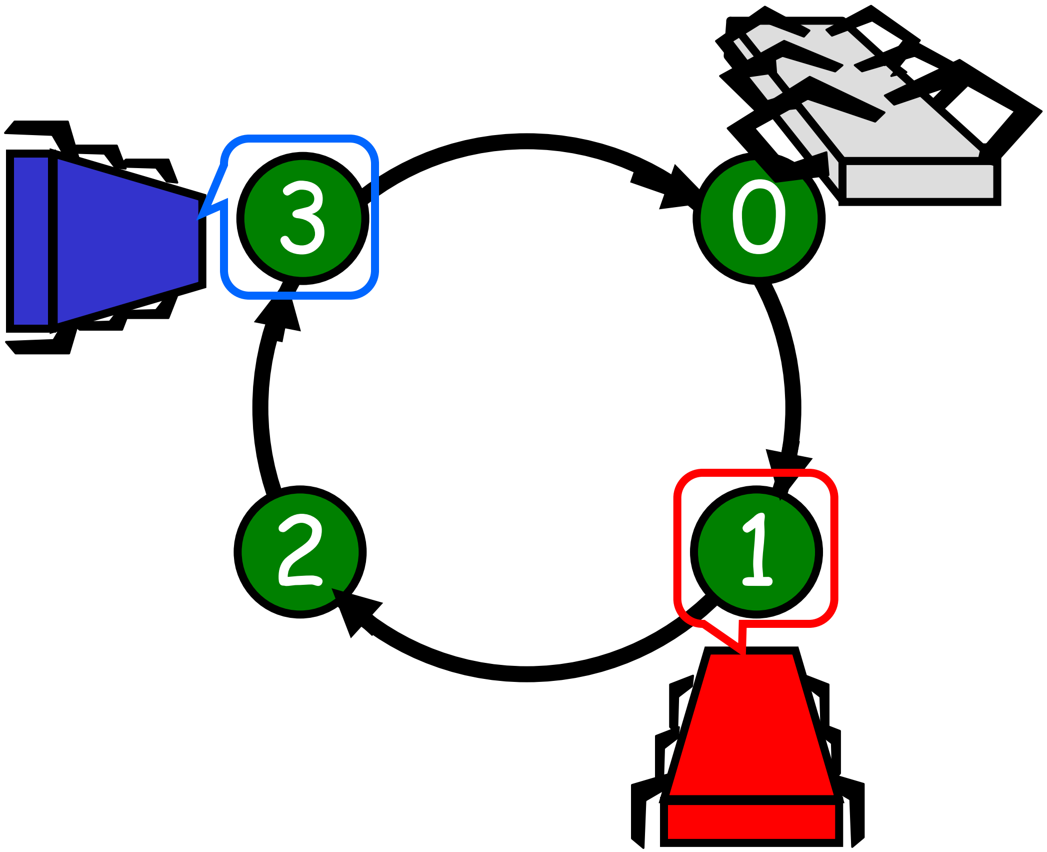
How about this ?



3<2

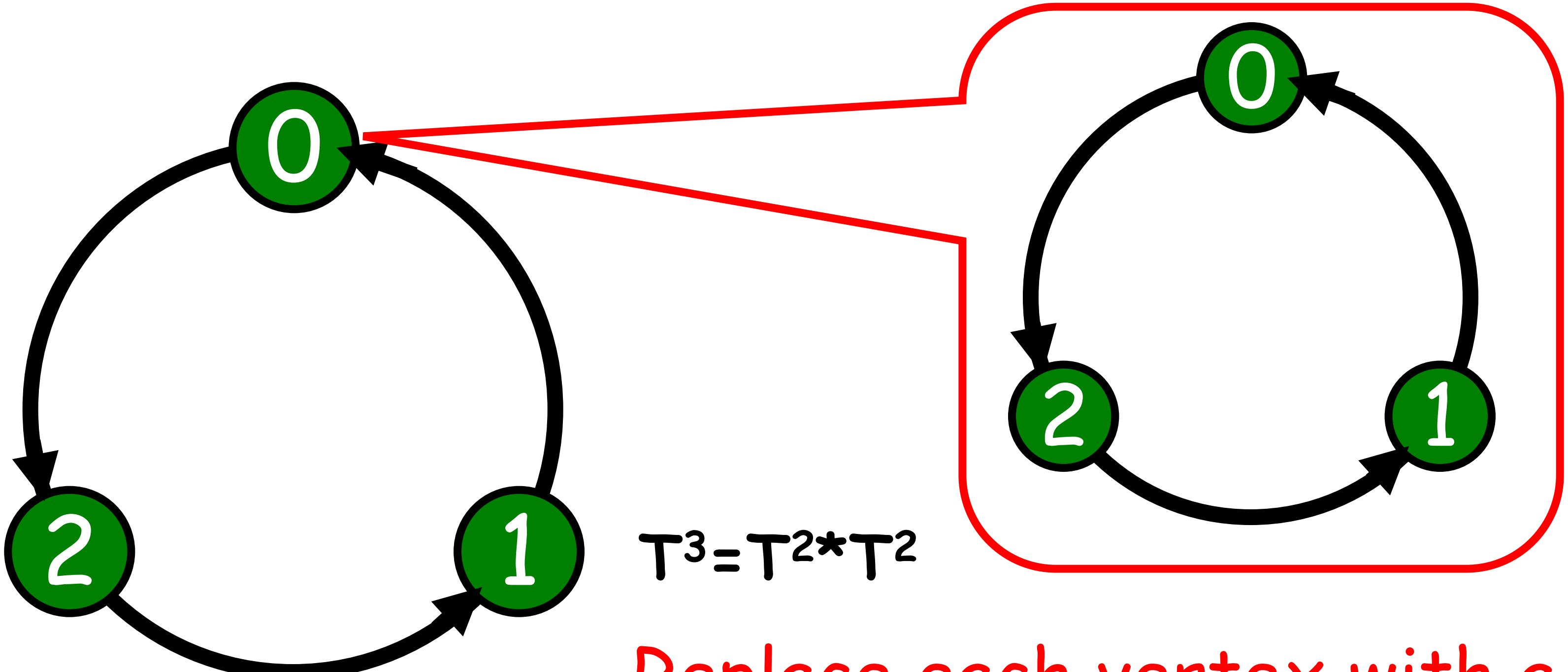
O.K.

How about this ?



3 ? 1

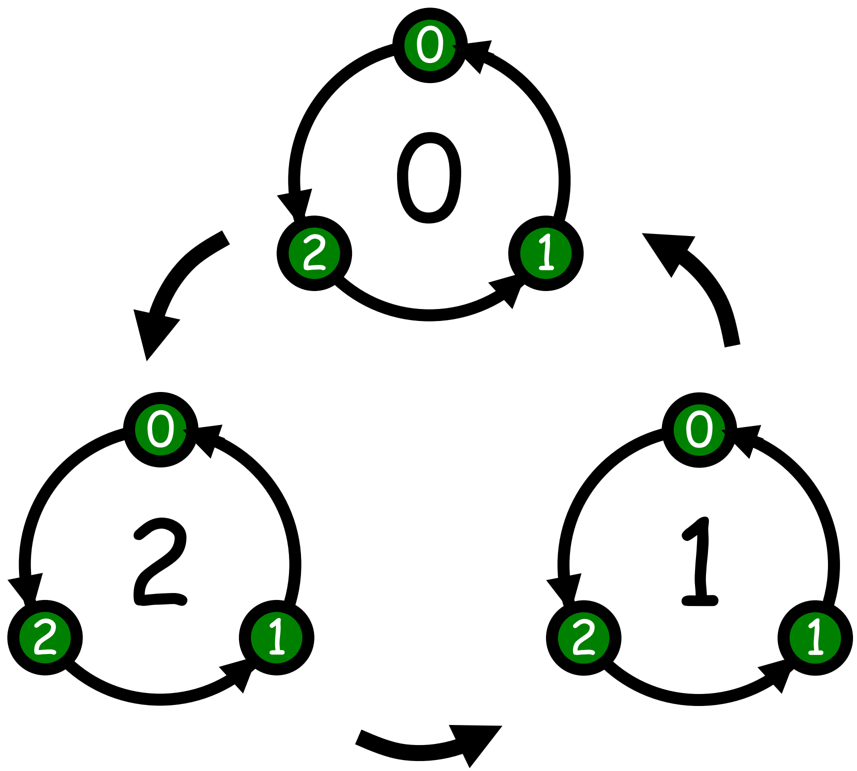
Graph Composition



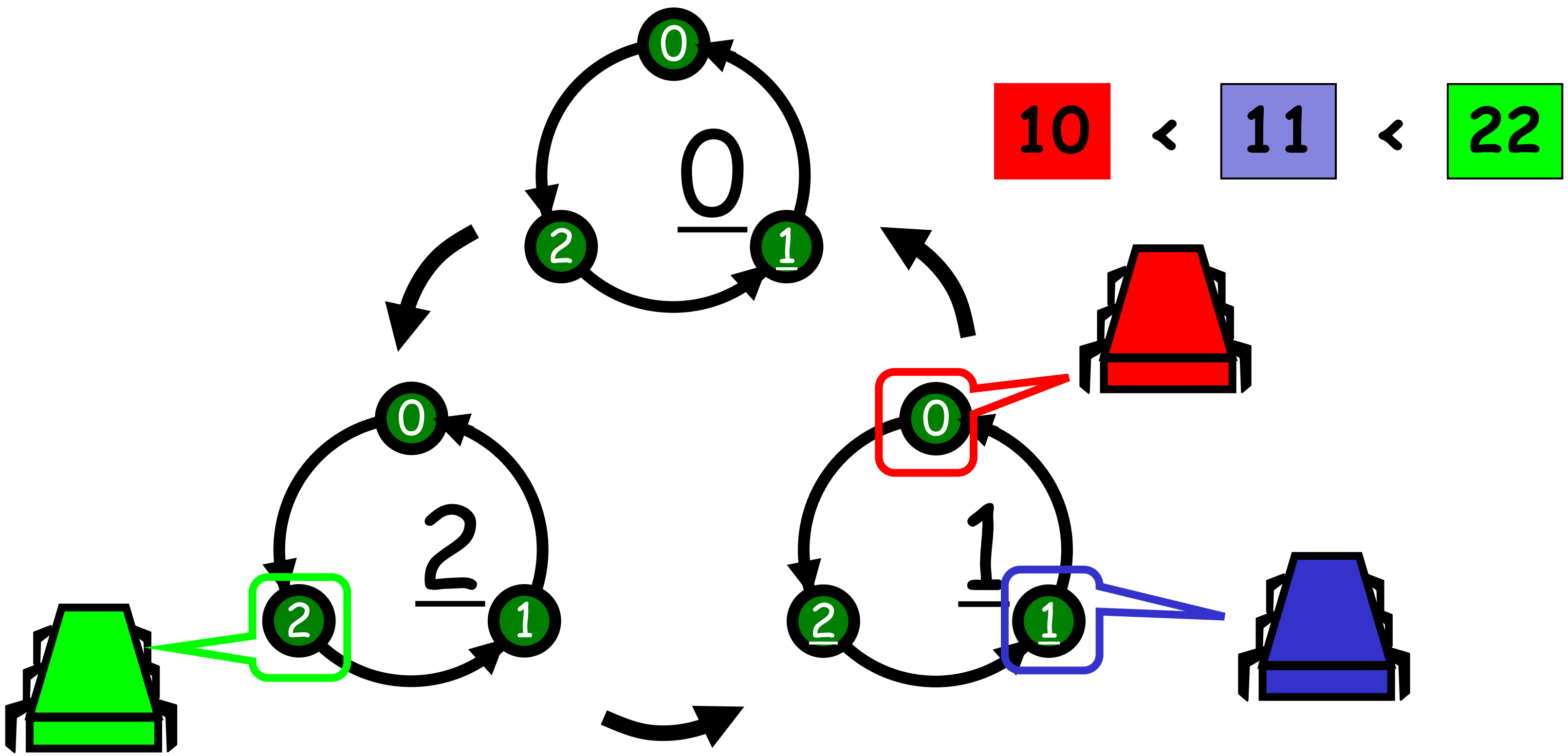
$T^3 = T^2 * T^2$

Replace each vertex with a copy of the graph

Three-Thread Bounded Precedence Graph T^3

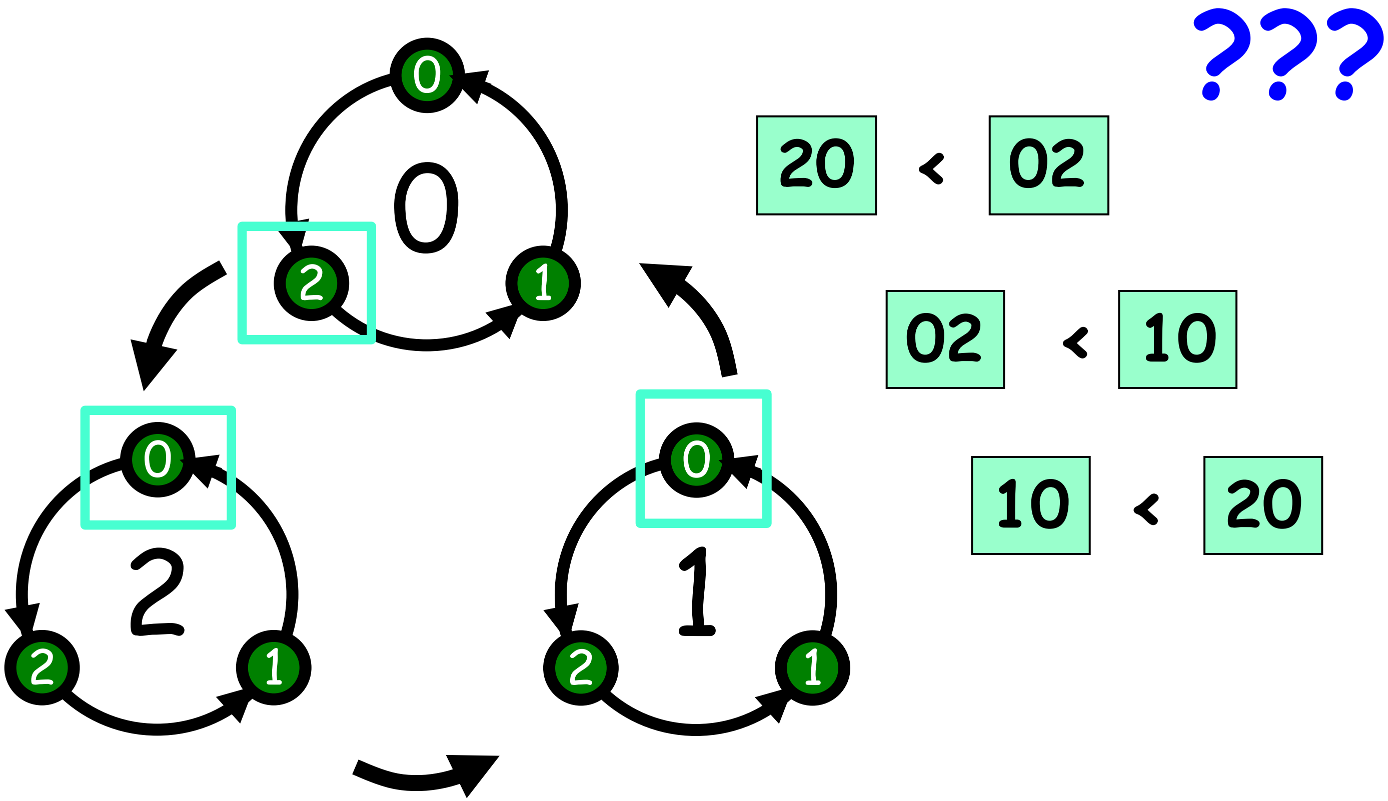


Three-Thread Bounded Precedence Graph T^3



and so on…

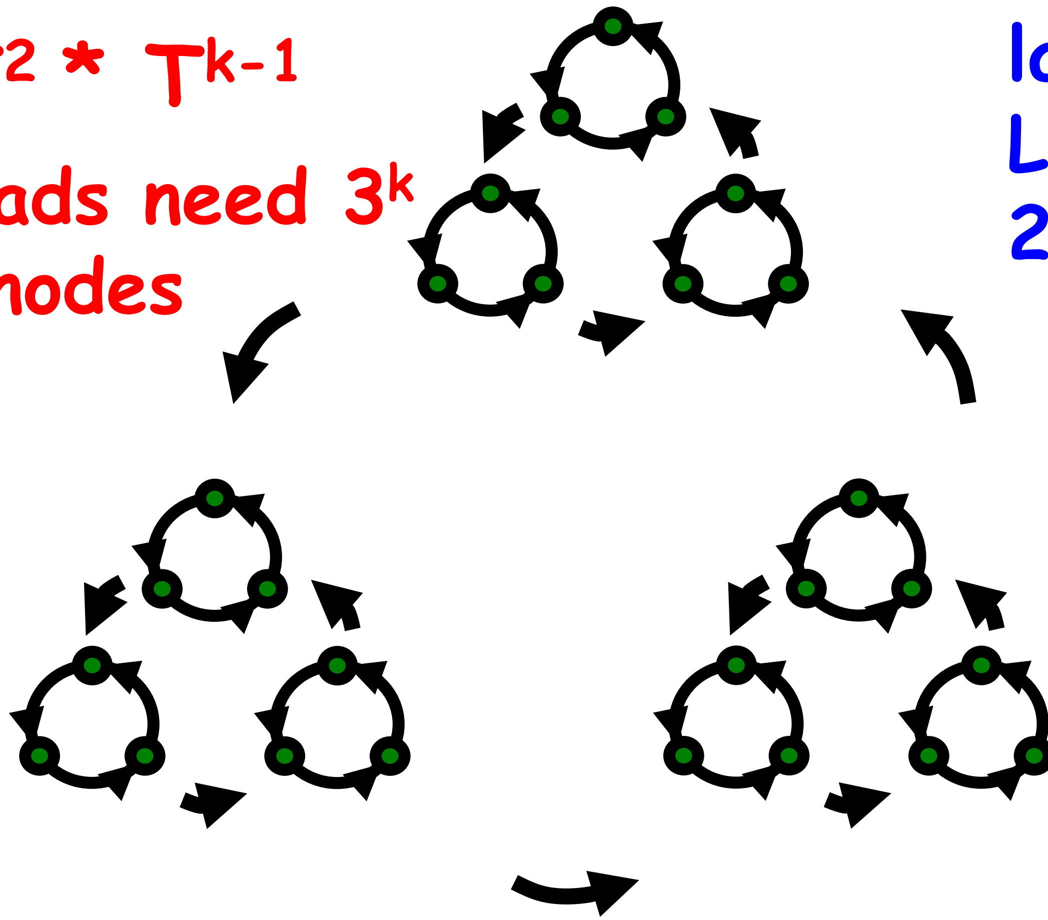
Three-Thread Bounded Precedence Graph T^3



In General

$T^k = T^2 * T^{k-1}$

K threads need 3^k nodes



label size = $\log_2(3^k)$ = 2n

Deep Philosophical Question

- The Bakery Algorithm is
  - Succinct,
  - Elegant, and
  - Fair.
- Q: So why isn't it practical?
- A: Well, you have to read N distinct variables

Shared Memory

- Shared read/write memory locations called Registers (historical reasons)
- Come in different flavors
  - Multi-Reader-Single-Writer (**`Flag[]`**)
  - Multi-Reader-Multi-Writer (**Victim[]**)
  - Not interesting: SRMW and SRSW

Theorem

At least N MRSW (multi-reader/single-writer) registers are needed to solve deadlock-free mutual exclusion.

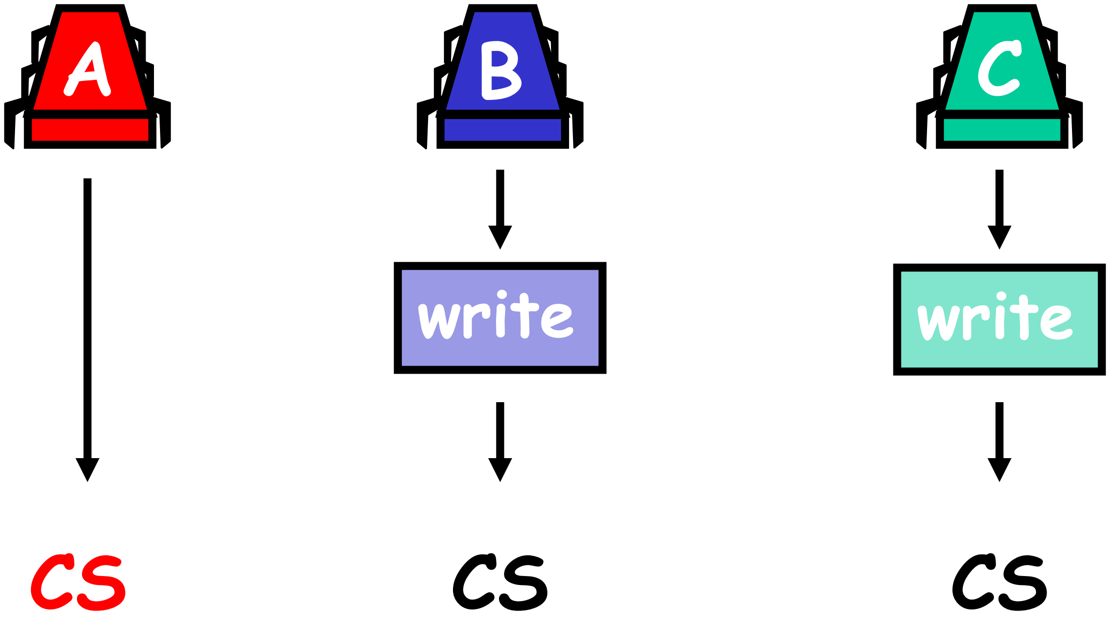
N registers like Flag[]…

Proving Algorithmic Impossibility

- To show no algorithm exists:
  - assume by way of contradiction one does,
  - show a bad execution that violates properties:
  - in our case assume an alg for deadlock free mutual exclusion using < N registers

Proof: Need N-MRSW Registers

Each thread must write to some register



…can't tell whether A is in critical section

Upper Bound

- Bakery algorithm
  - Uses 2N MRSW registers
- So the bound is (pretty) tight
- But what if we use MRMW registers?
  - Like **victim[]** ?

Bad News Theorem

At least N MRMW multi-reader/multi-writer registers are needed to solve deadlock-free mutual exclusion.

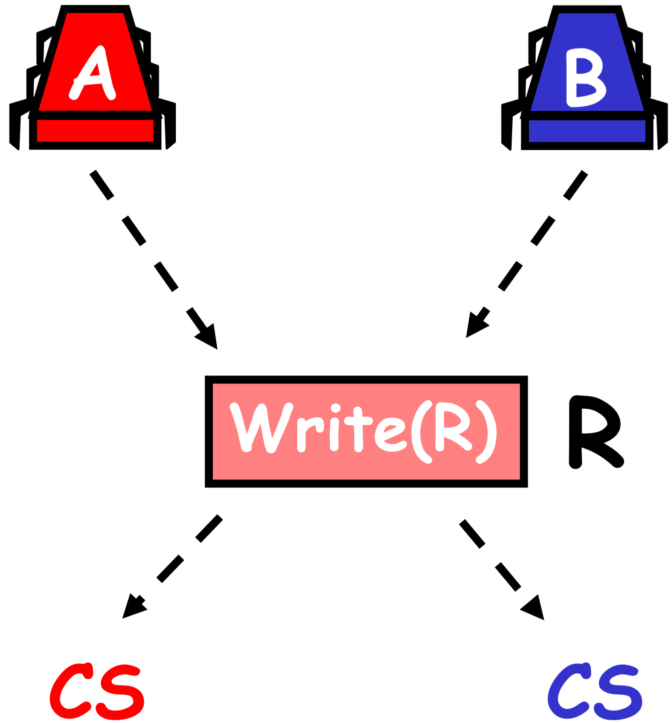
(So multiple writers don't help)

Theorem (First 2-Threads)

Theorem: Deadlock-free mutual exclusion for 2 threads requires at least 2 multi-reader multi-writer registers

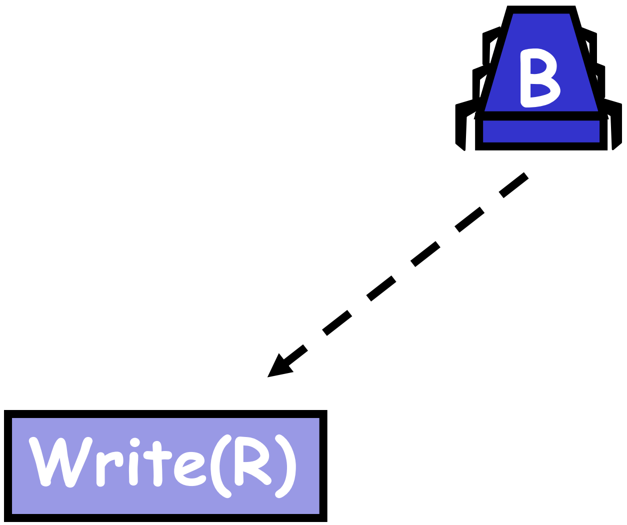
Proof: assume one register suffices and derive a contradiction

Two Thread Execution



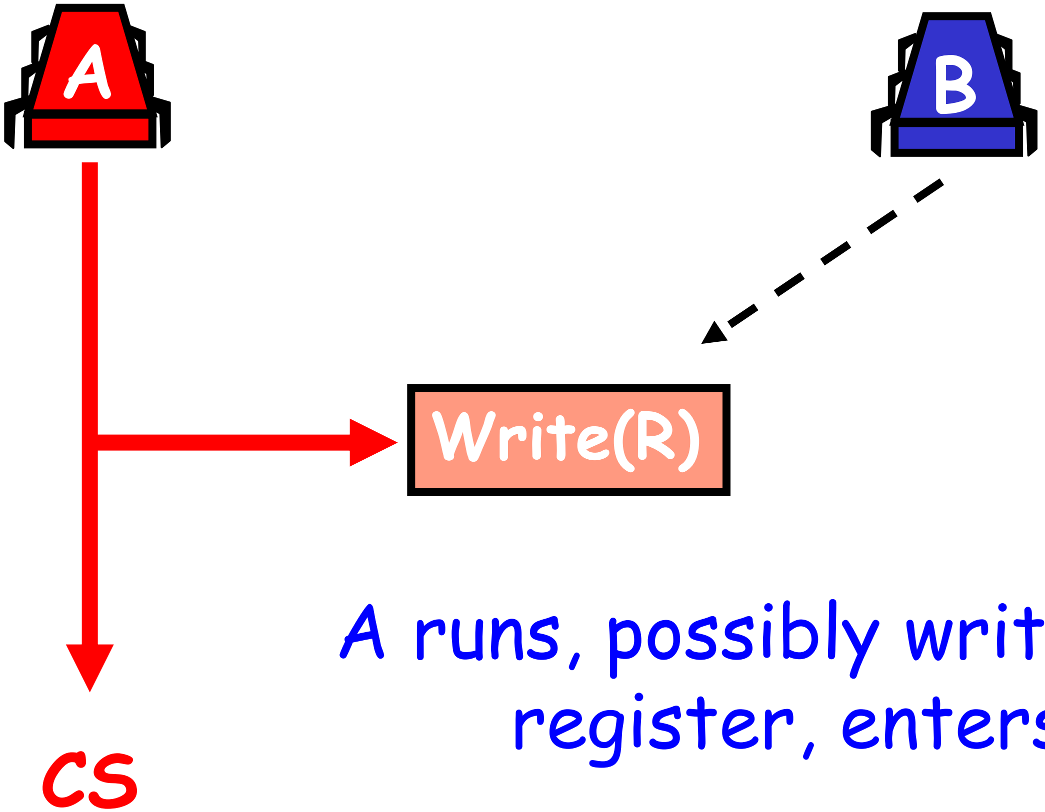
- Threads run, reading and writing R
- Deadlock free so at least one gets in

Covering State for One Register



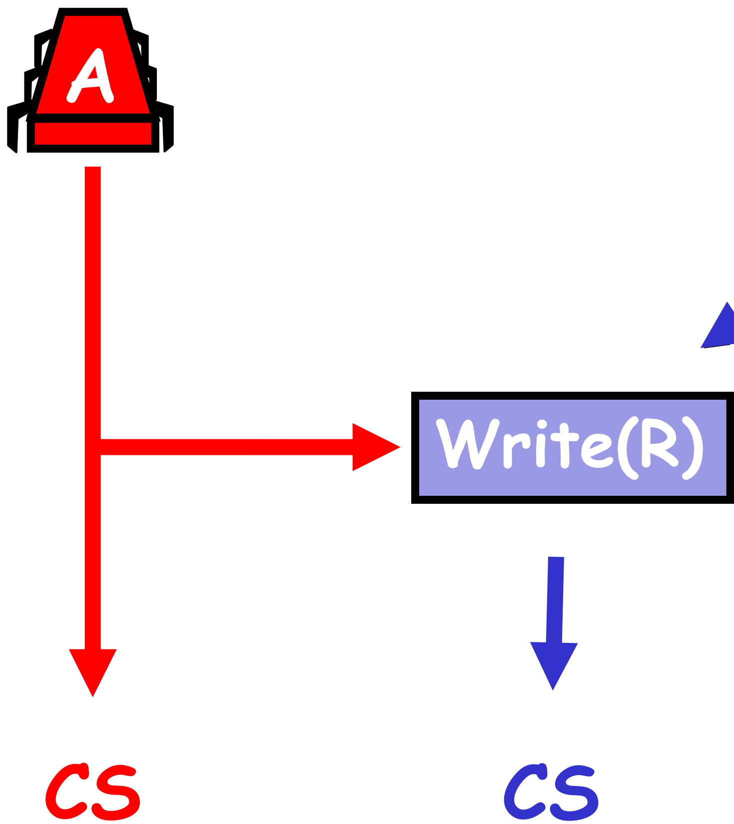
B has to write to the register before entering CS, so stop it just before

Proof: Assume Cover of 1



A runs, possibly writes to the register, enters CS

Proof: Assume Cover of 1

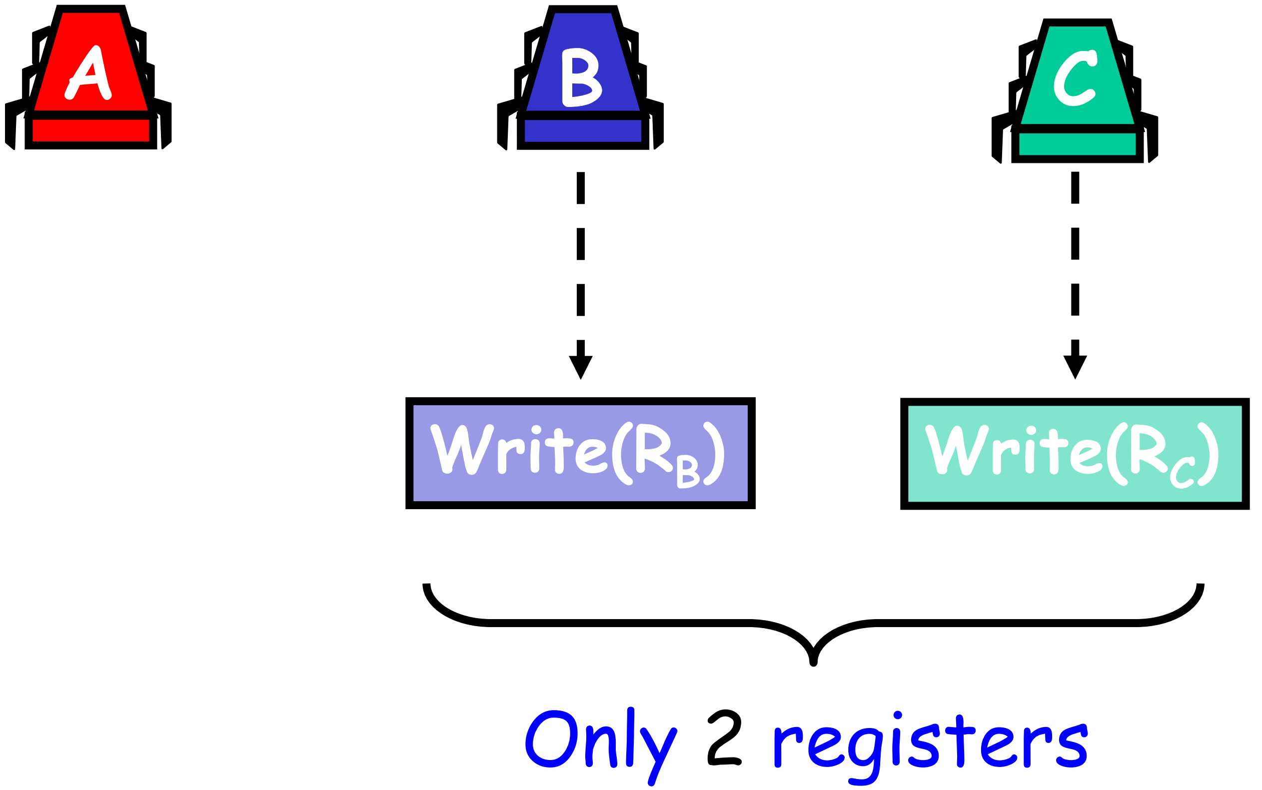


B Runs, first obliterating any trace of A, then also enters the critical section

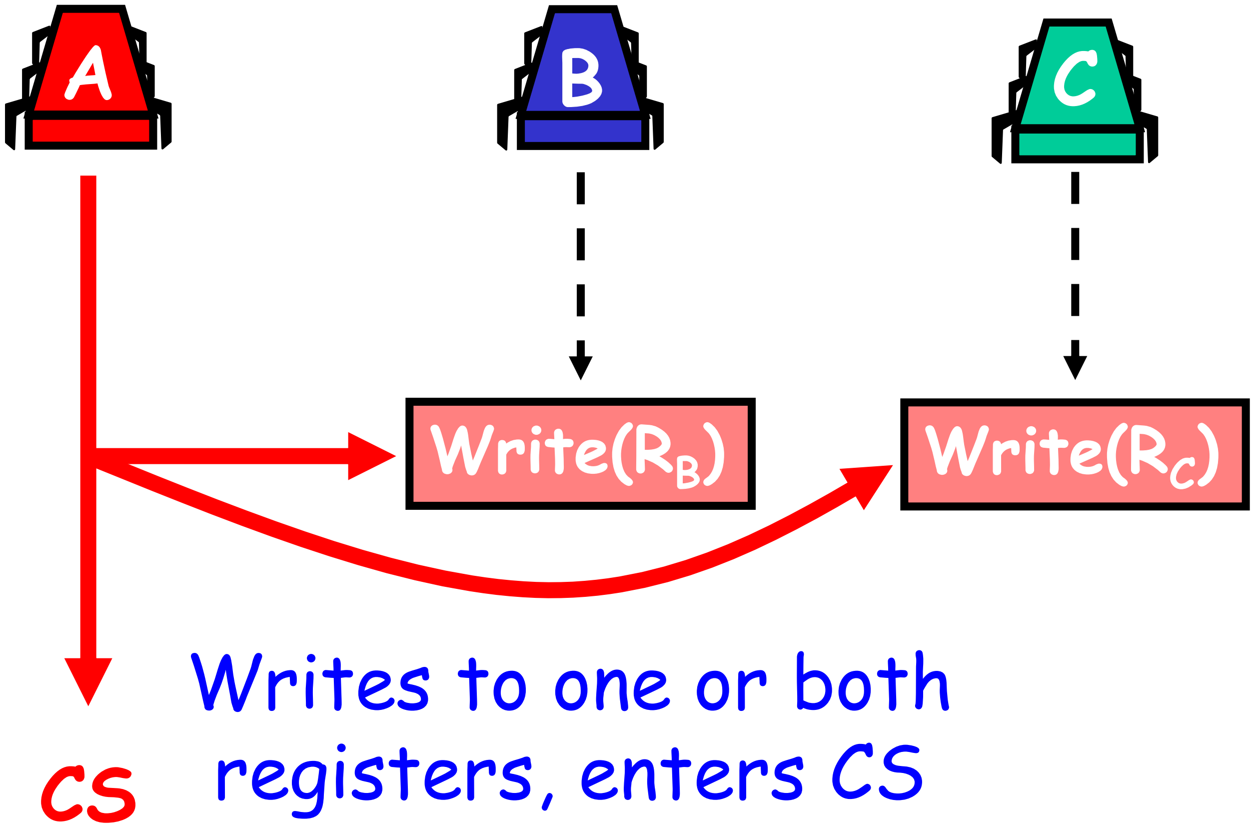
Theorem

Deadlock-free mutual exclusion for 3 threads requires at least 3 multi-reader multi-writer registers

Proof: Assume Cover of 2

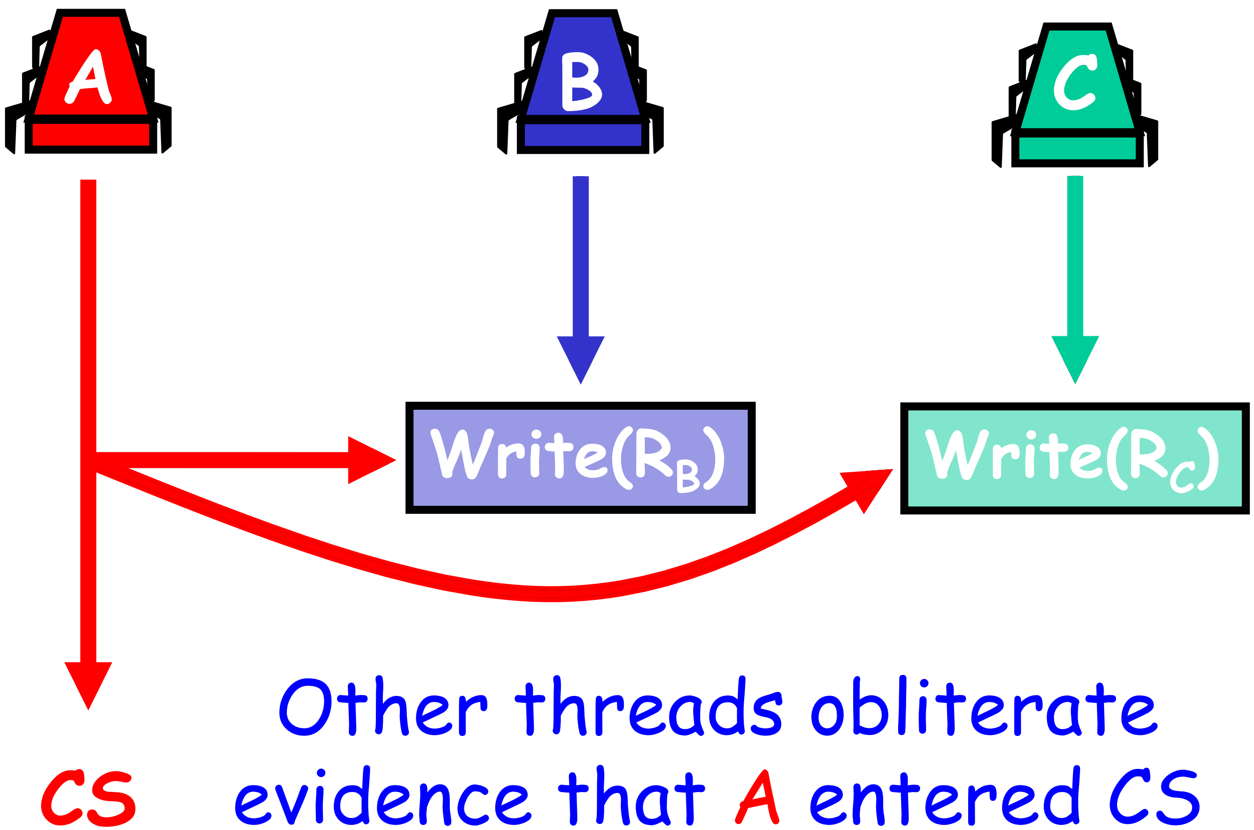


Run A Solo



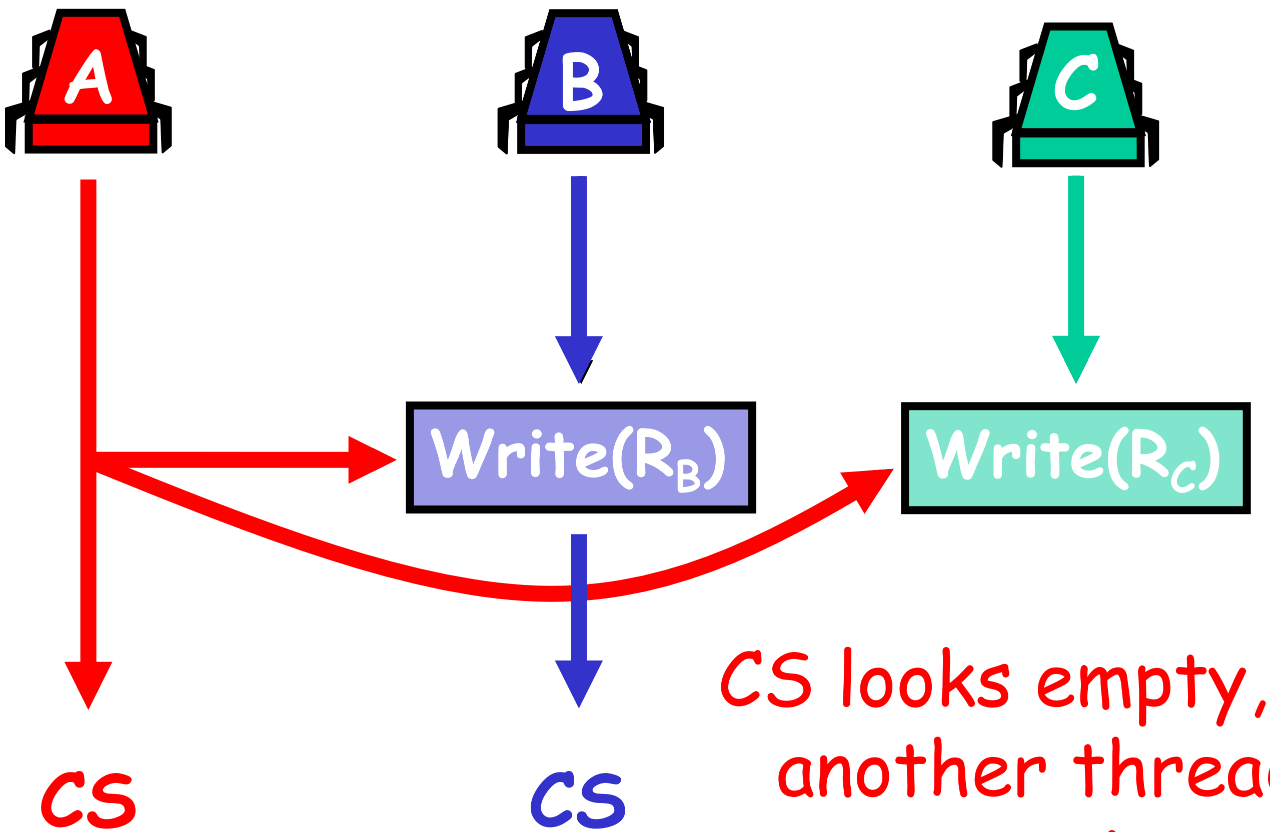
Writes to one or both registers, enters CS

Obliterate Traces of A



Other threads obliterate evidence that A entered CS

Mutual Exclusion Fails

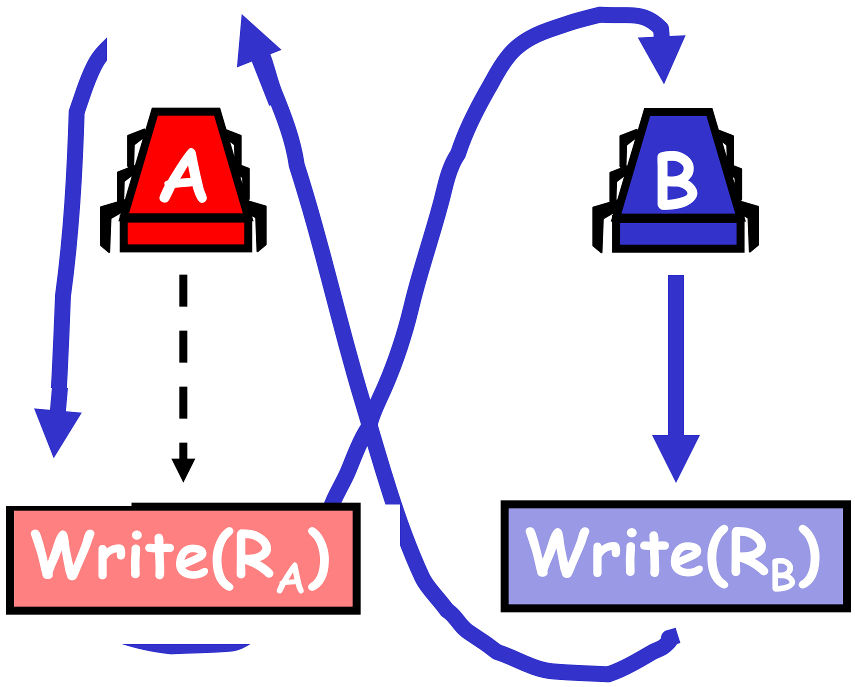


CS looks empty, so another thread gets in

Proof Strategy

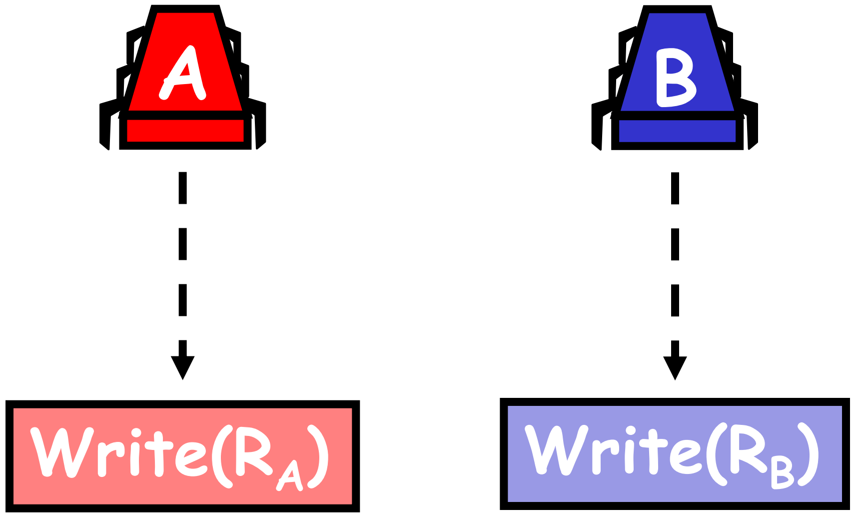
- Proved: a contradiction starting from a covering state for 2 registers
- Claim: a covering state for 2 registers is reachable from any state where CS is empty

Covering State for Two



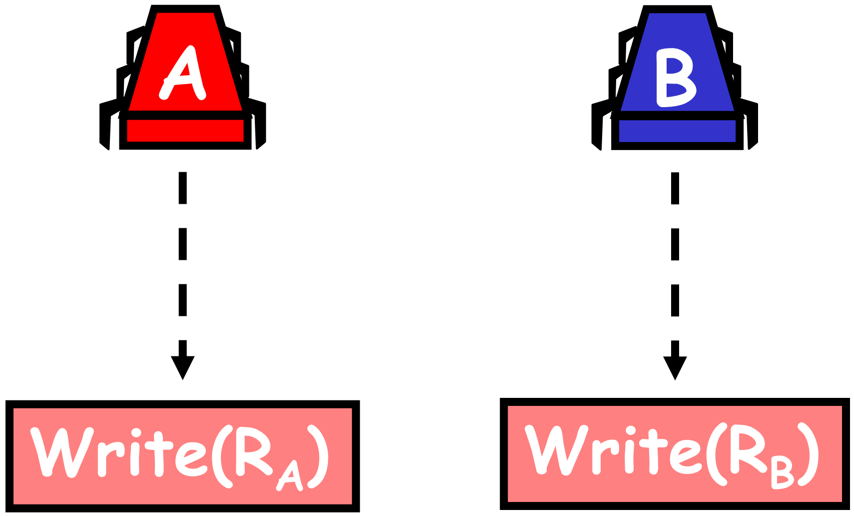
- If we run B through CS 3 times, B must return twice to cover some register, say R_B

Covering State for Two



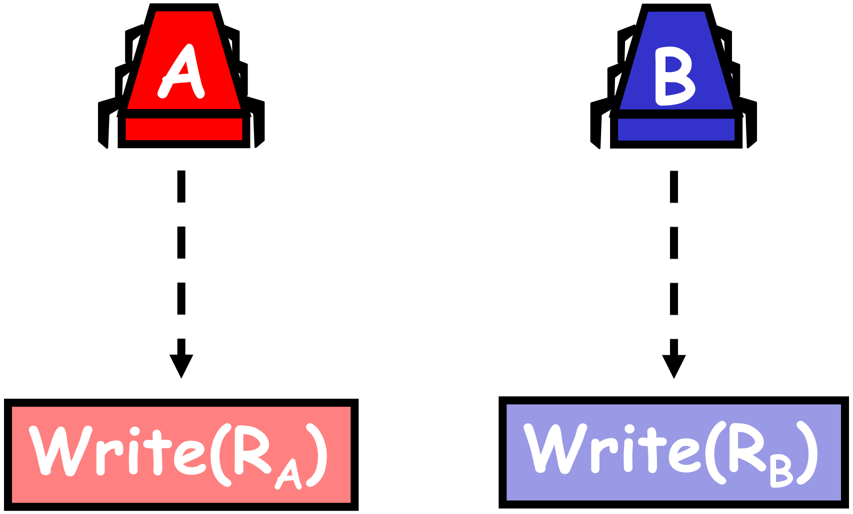
- Start with B covering register R_B for the 1^{st} time
- Run A until it is about to write to uncovered R_A
- Are we done?

Covering State for Two



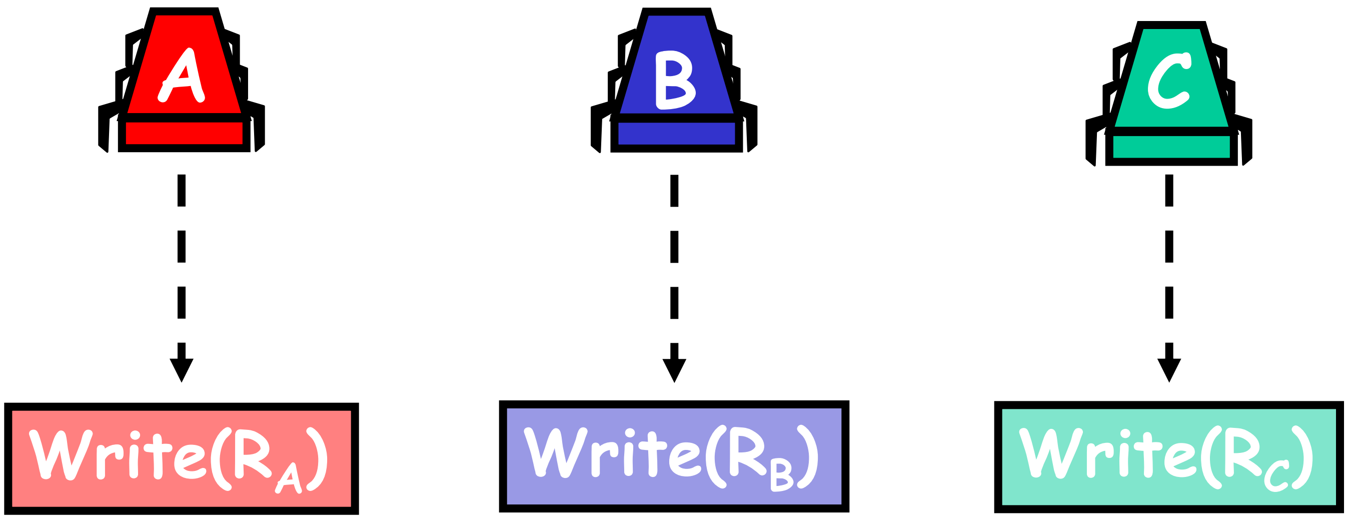
- NO! A could have written to R_B
- So CS no longer looks empty

Covering State for Two



- Run B obliterating traces of A in R_B
- Run B again until it is about to write to R_B
- Now we are done

Inductively We Can Show



- There is a covering state
  - Where k threads not in CS cover k distinct registers
  - Proof follows when k = N-1

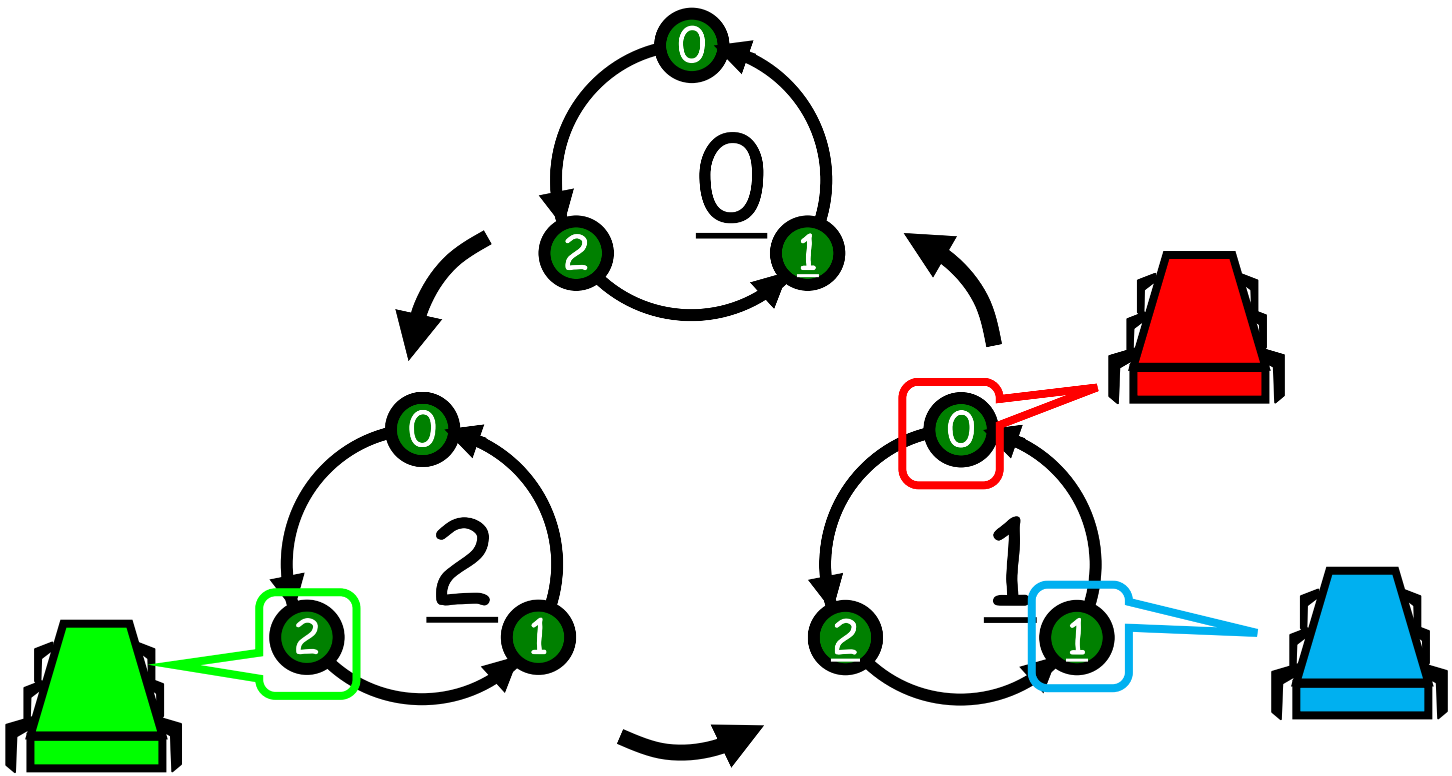
Summary of Lecture

- In the 1960's many incorrect solutions to starvation-free mutual exclusion using RW-registers were published…
- Today we know how to solve FIFO N thread mutual exclusion using 2N RW-Registers

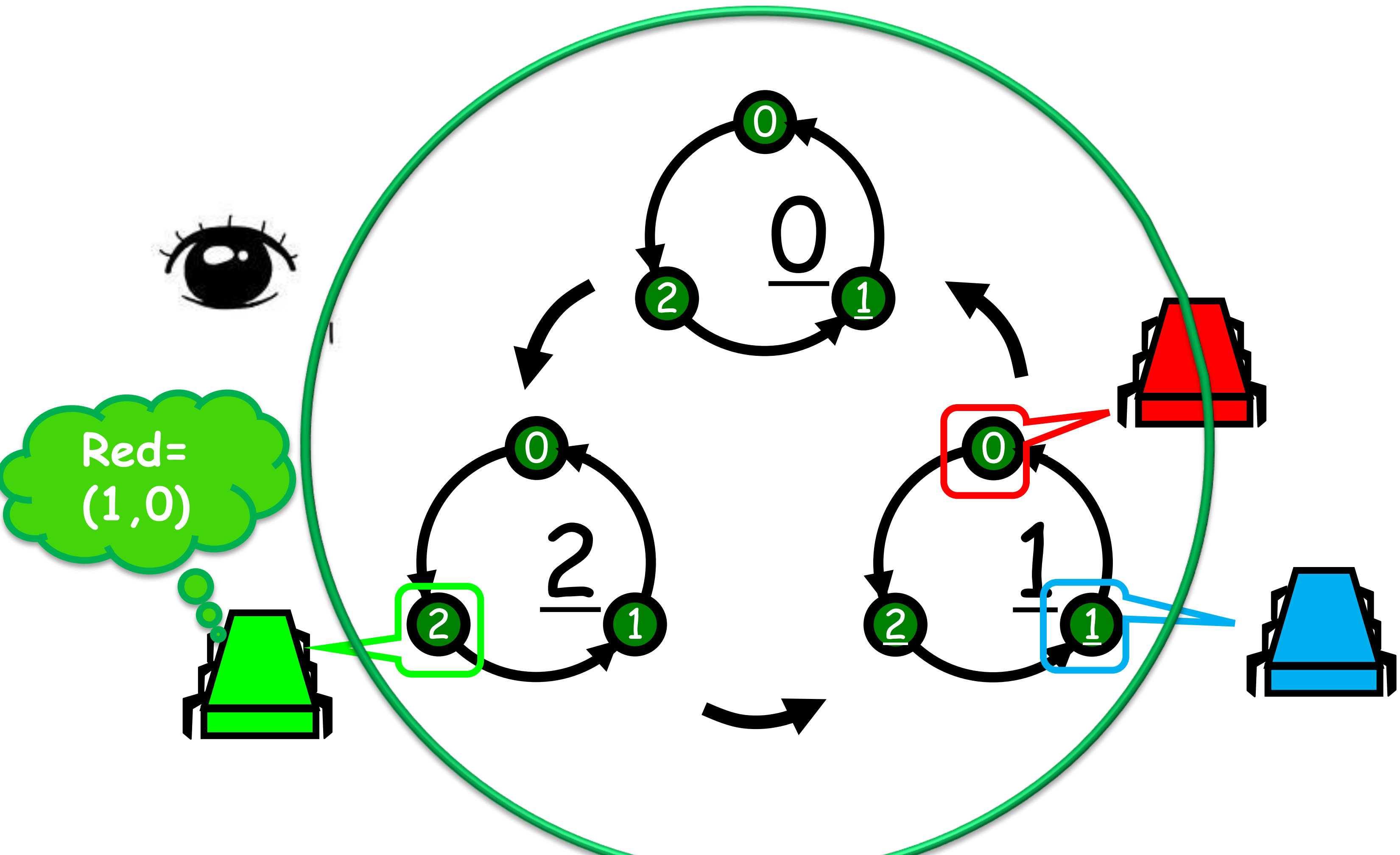
Summary of Lecture

- N RW-Registers inefficient
  - Because writes "cover" older writes
- Need stronger hardware operations
  - do not have the "covering problem"
- In next lectures - understand what these operations are…

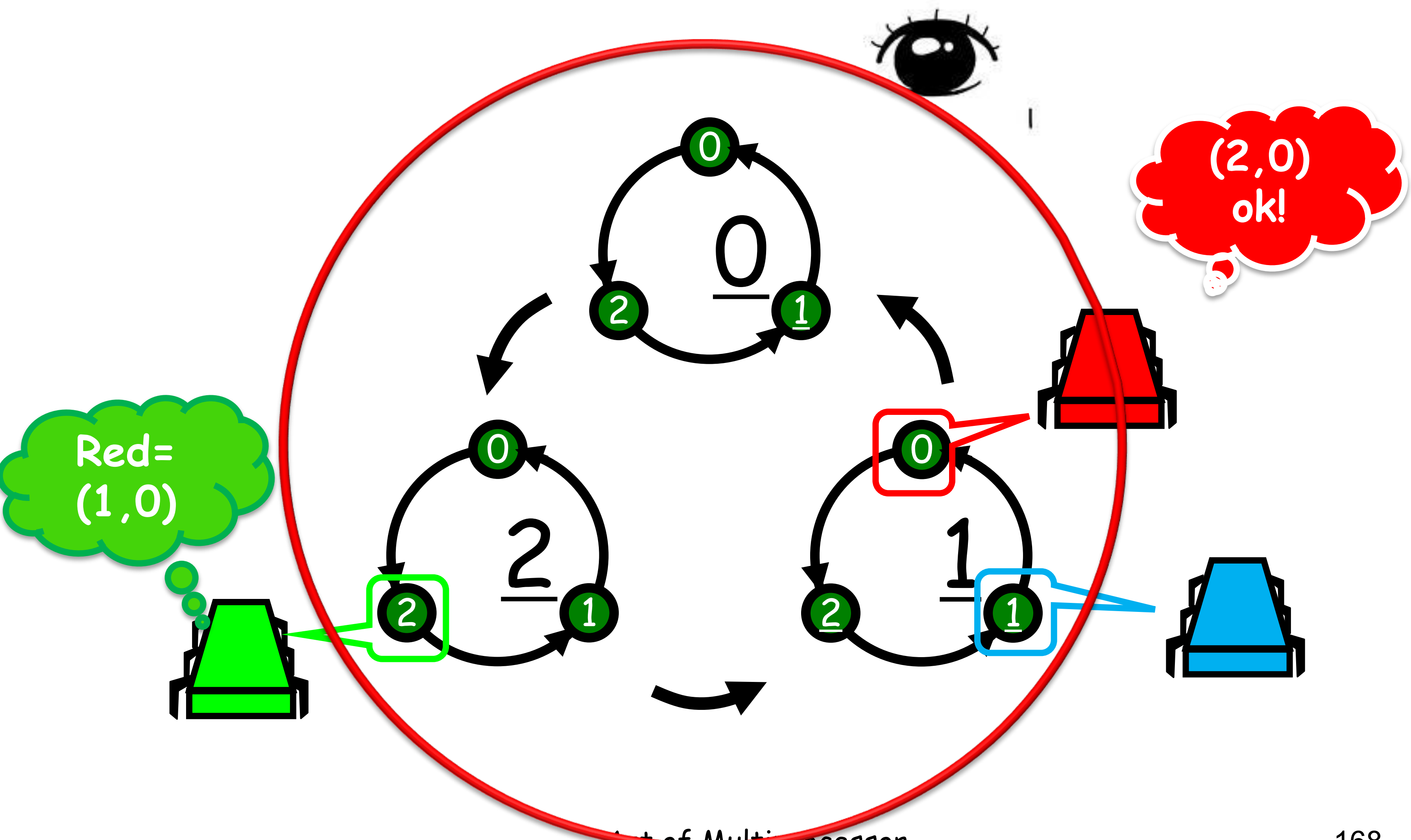
Initial State



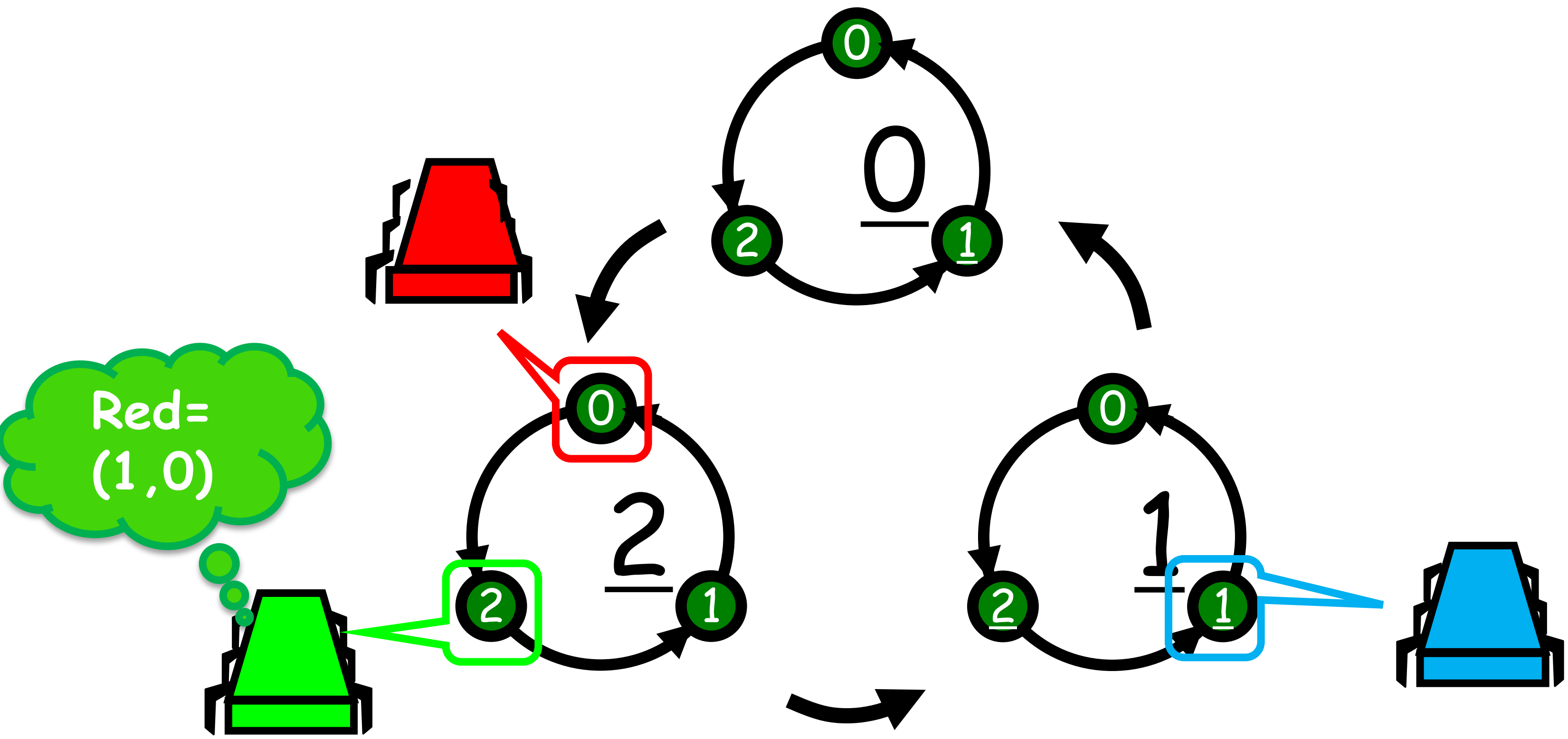
Green reads Red



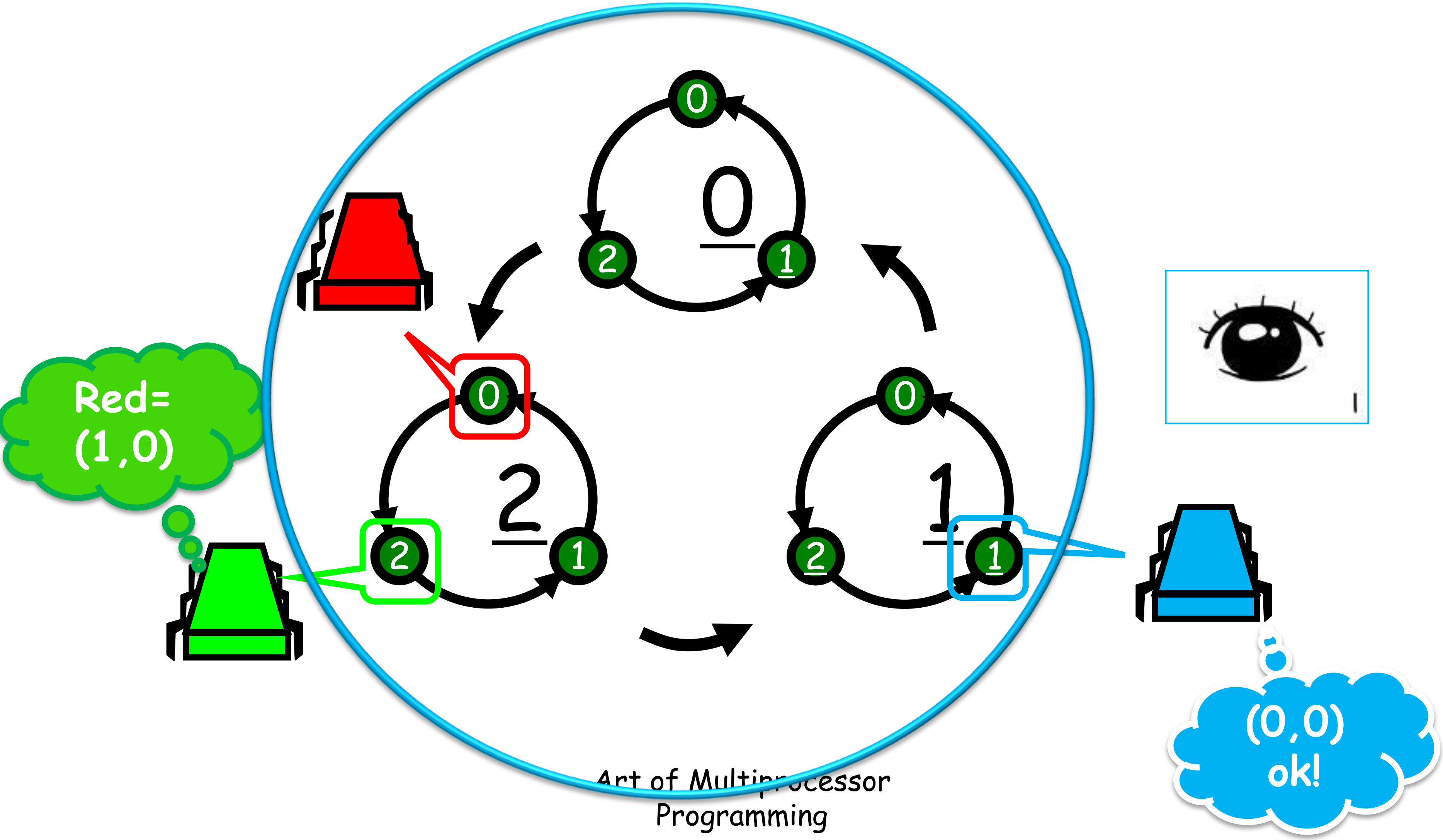
Red looks…



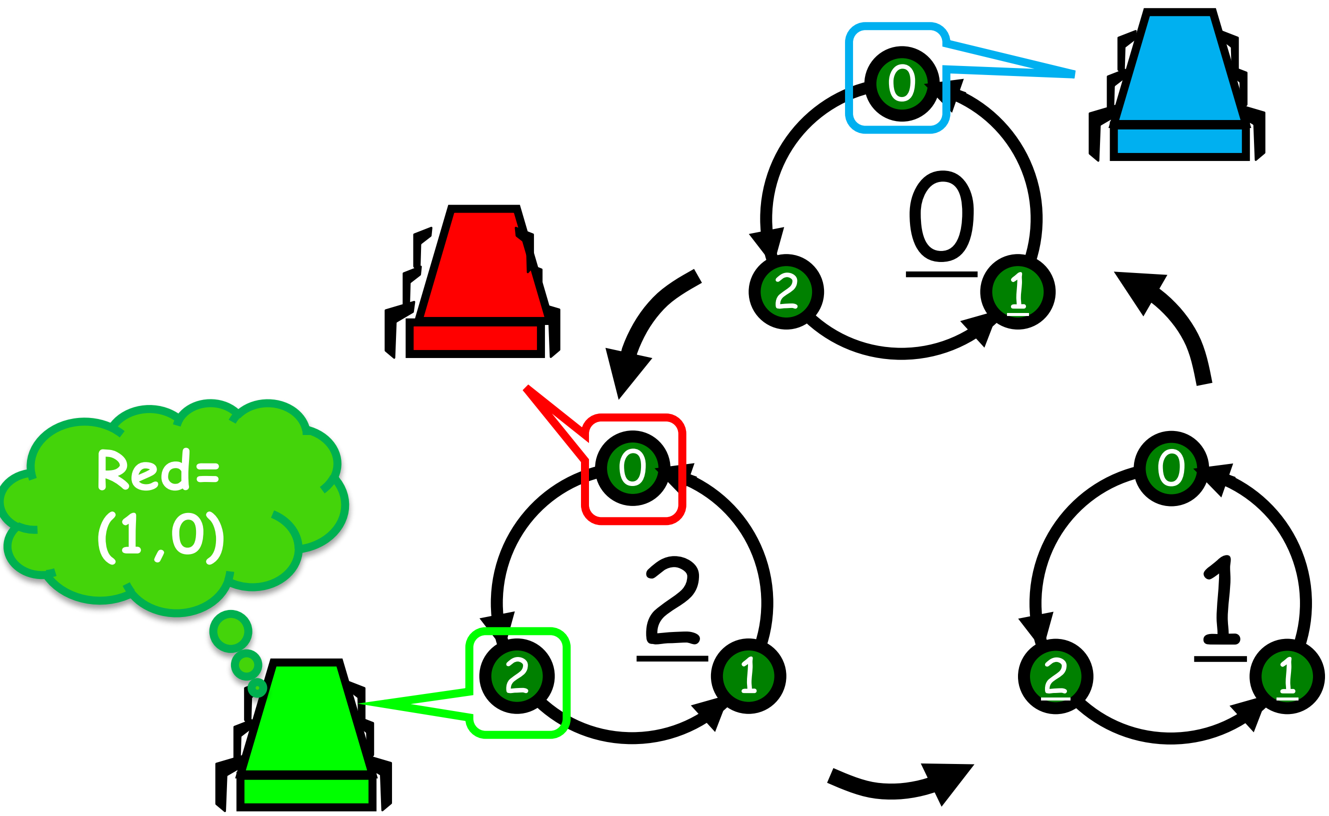
Red moves…



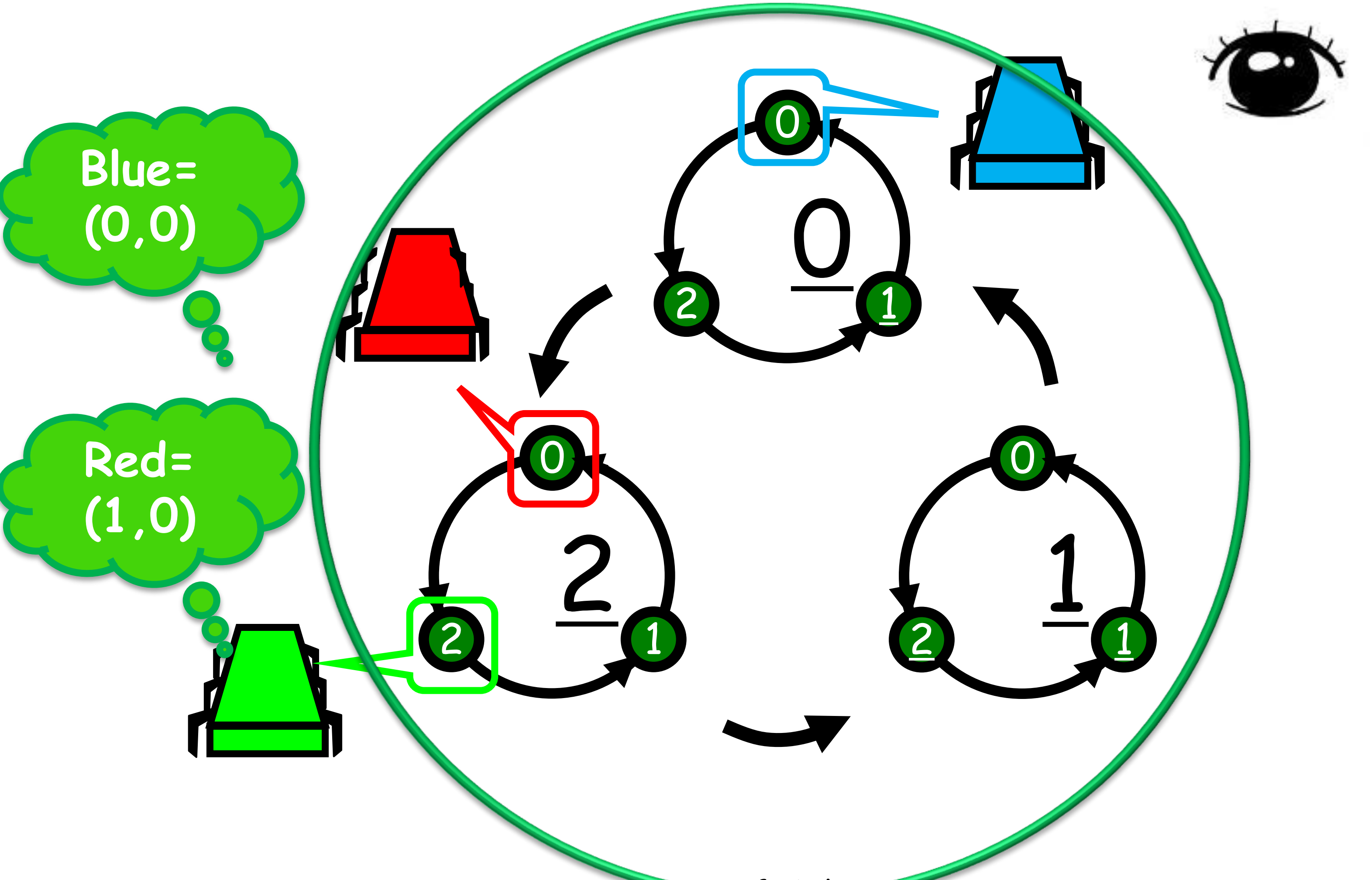
Blue looks



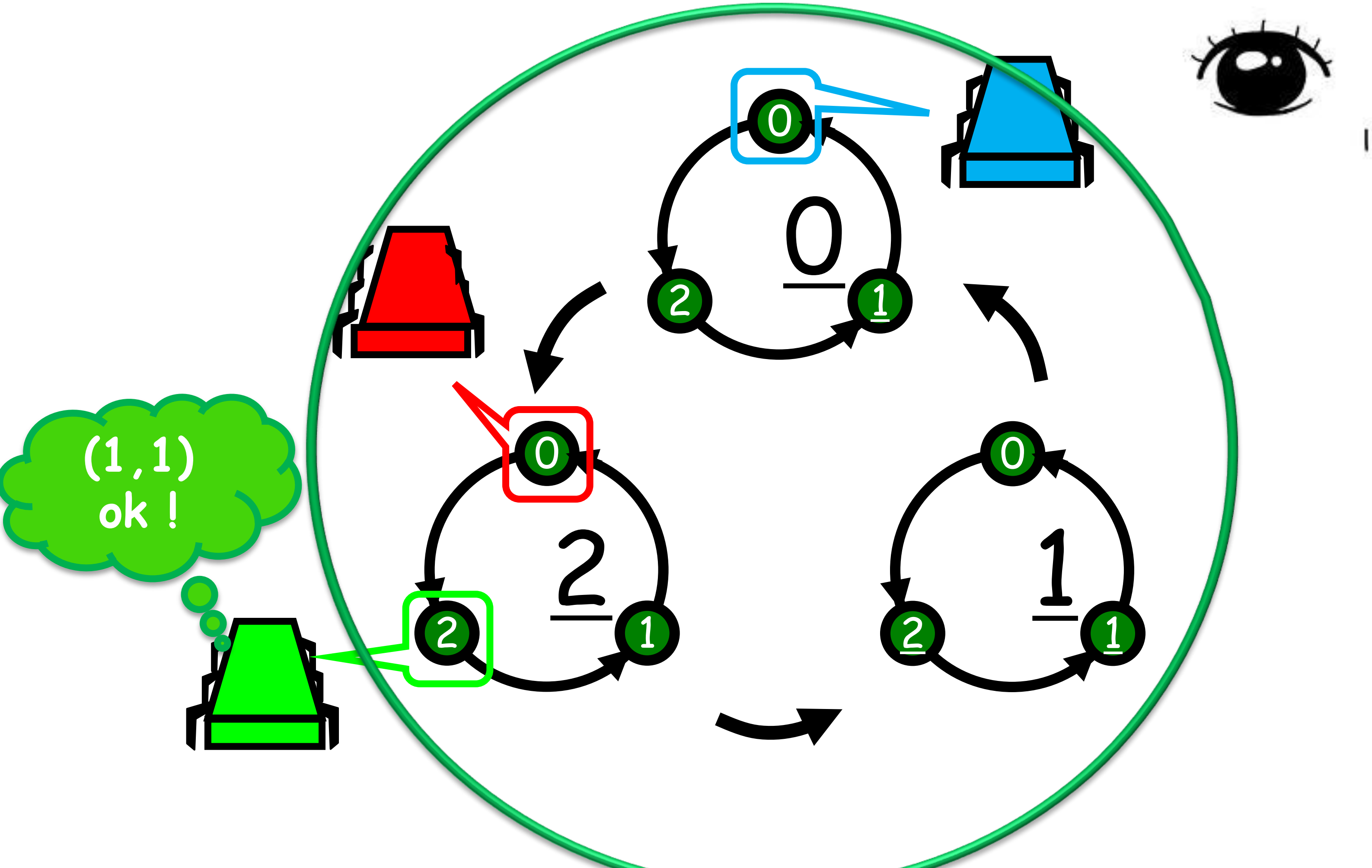
Blue moves



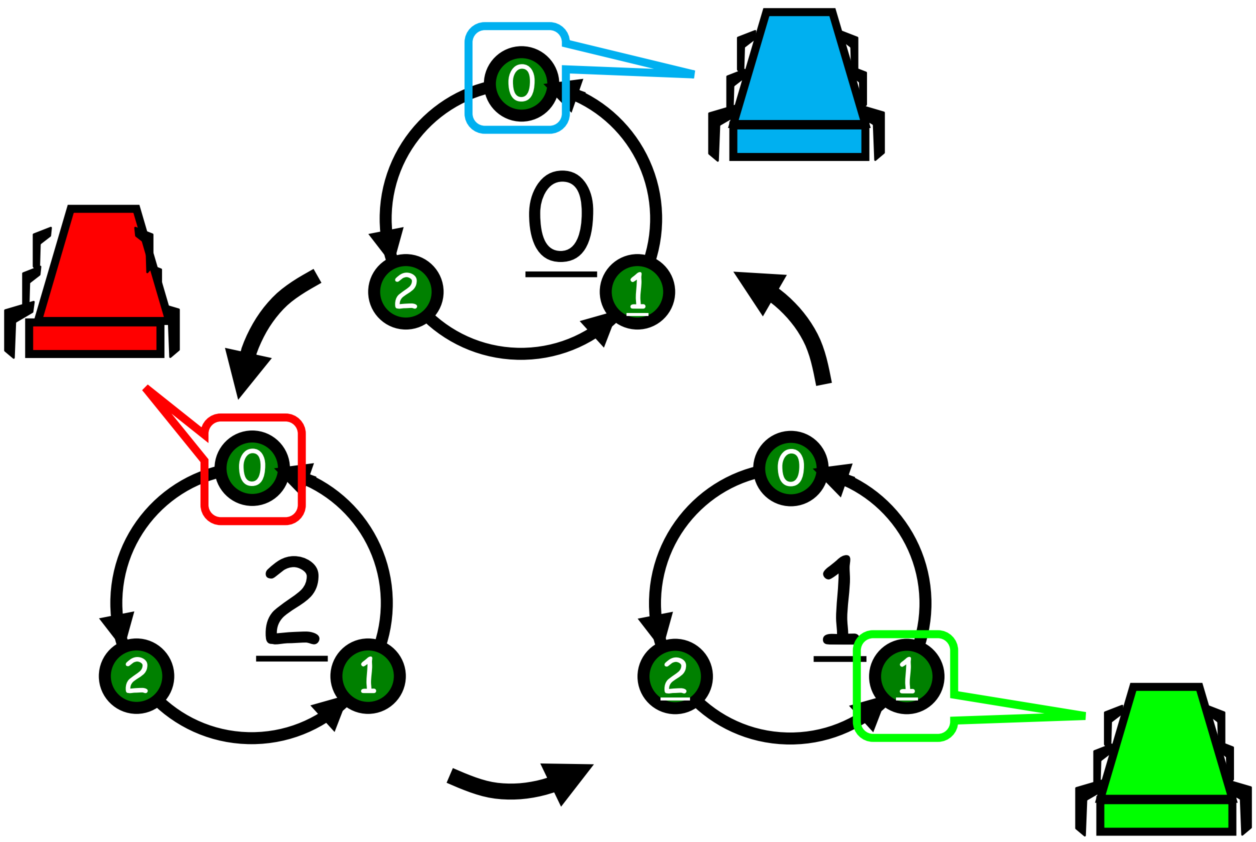
Green reads Blue



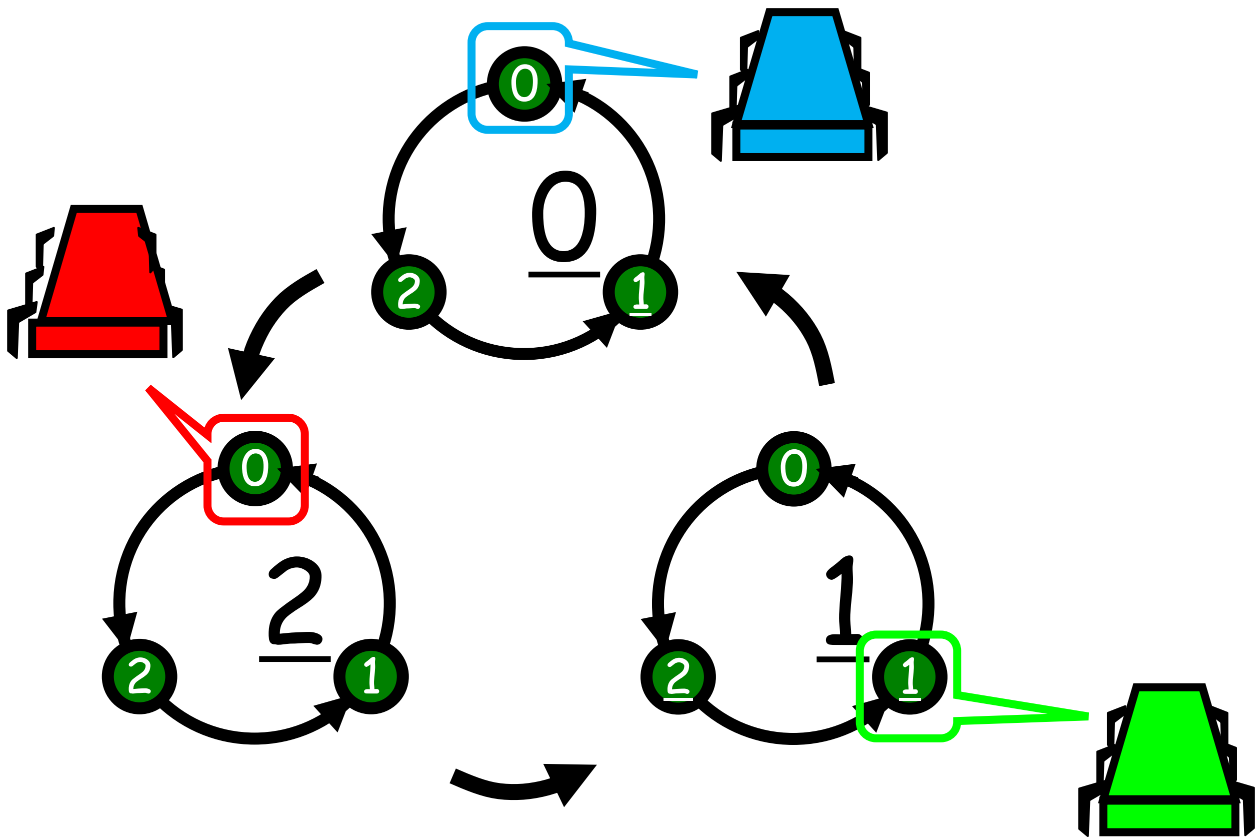
Green decides



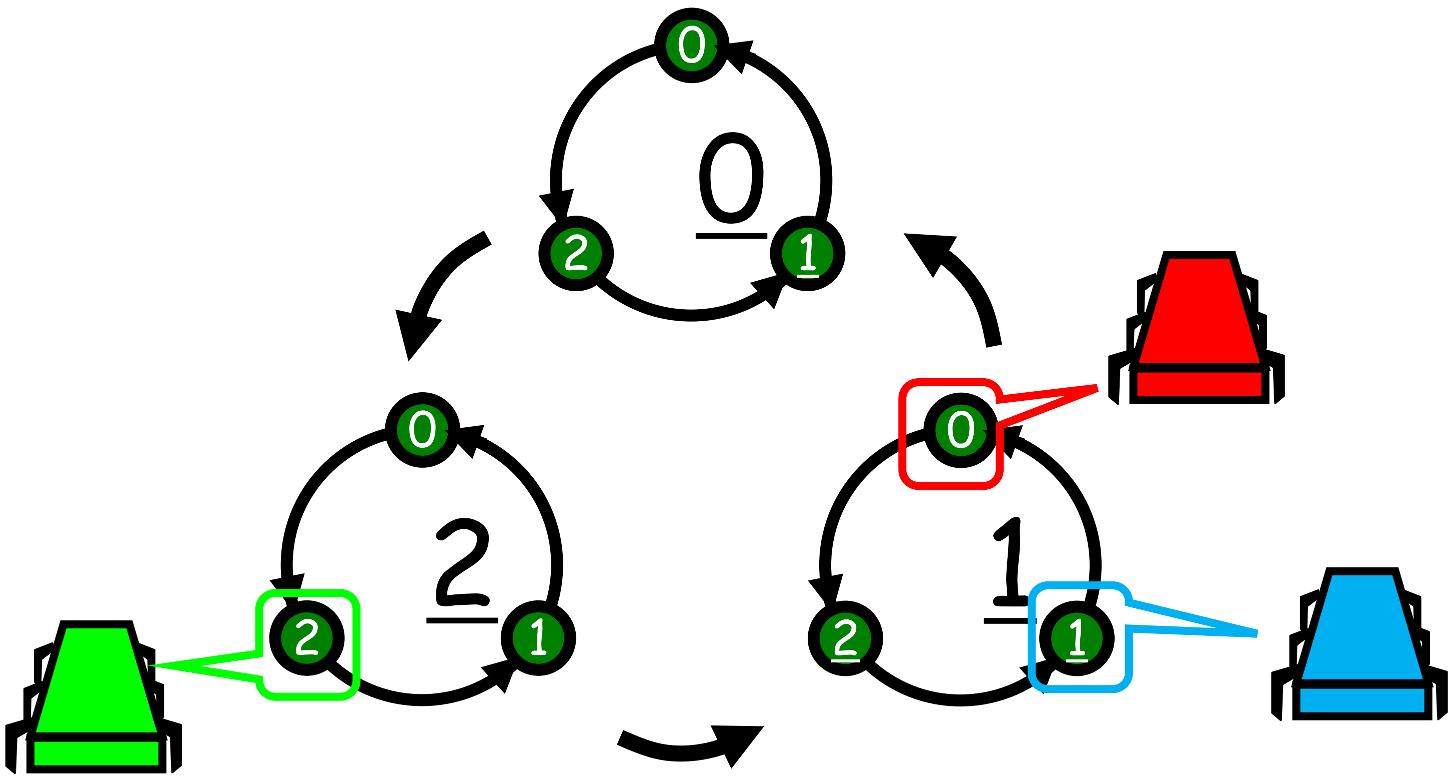
Green moves



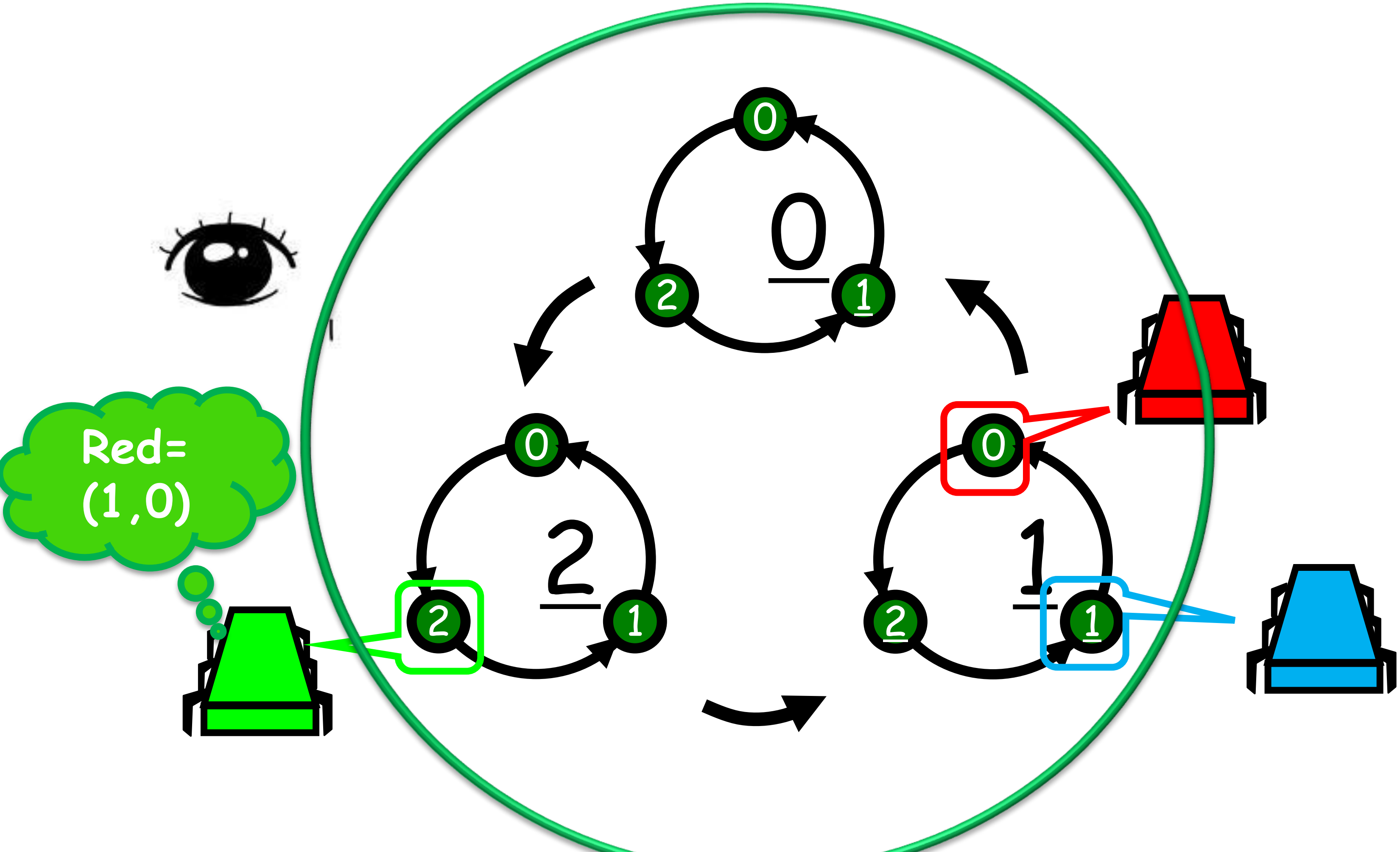
Oh Oh, No precedence !



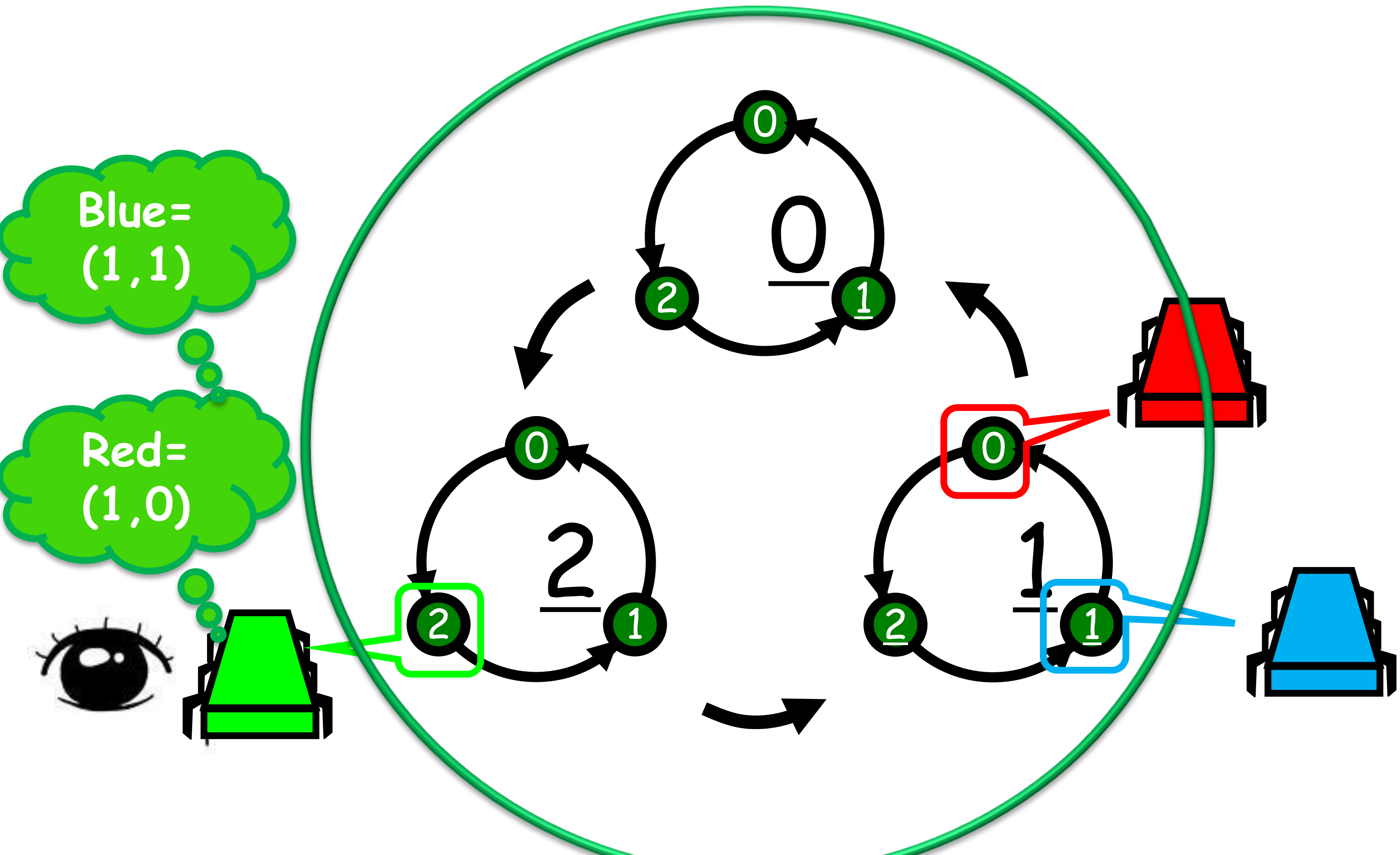
Initial State



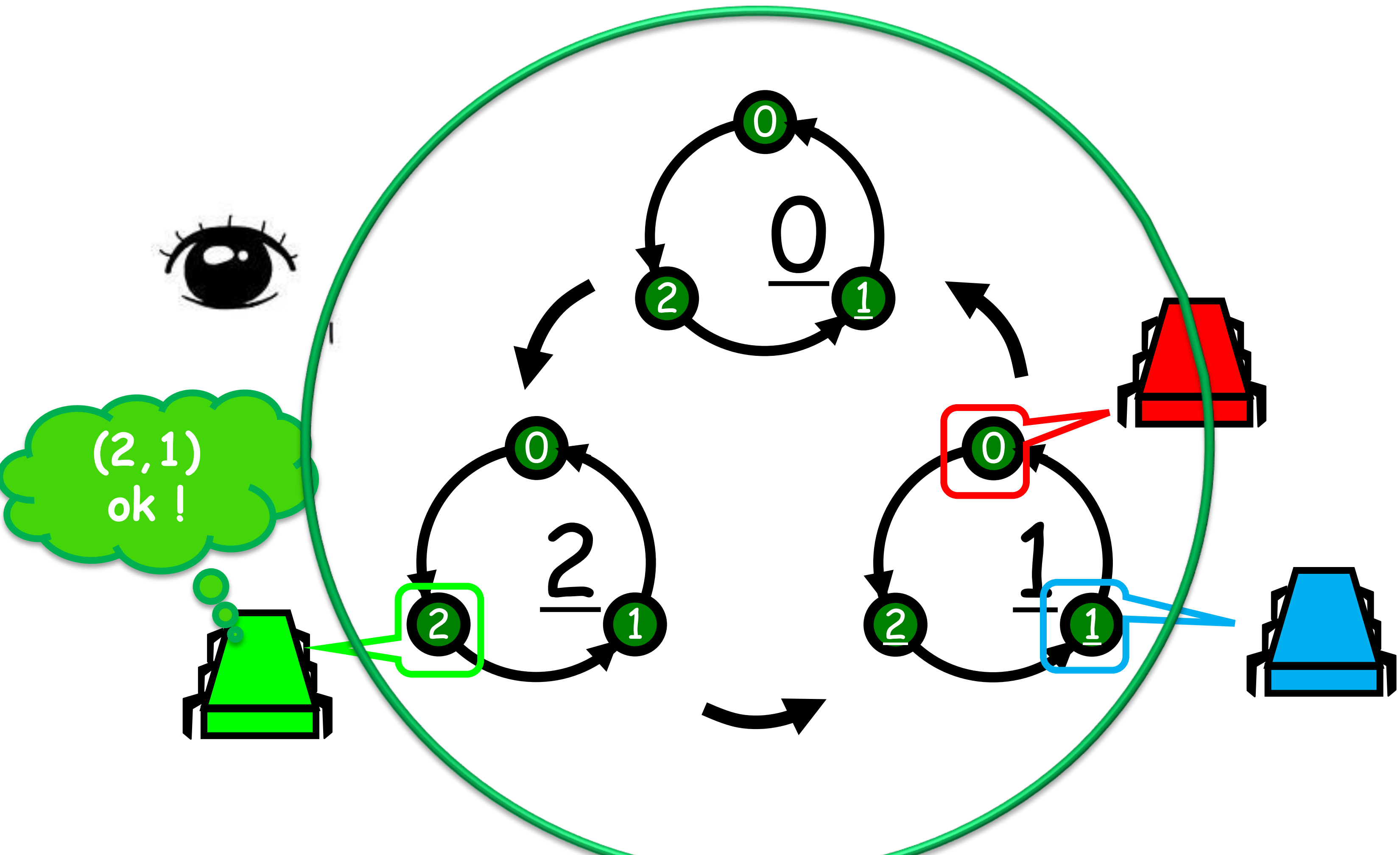
Green reads Red



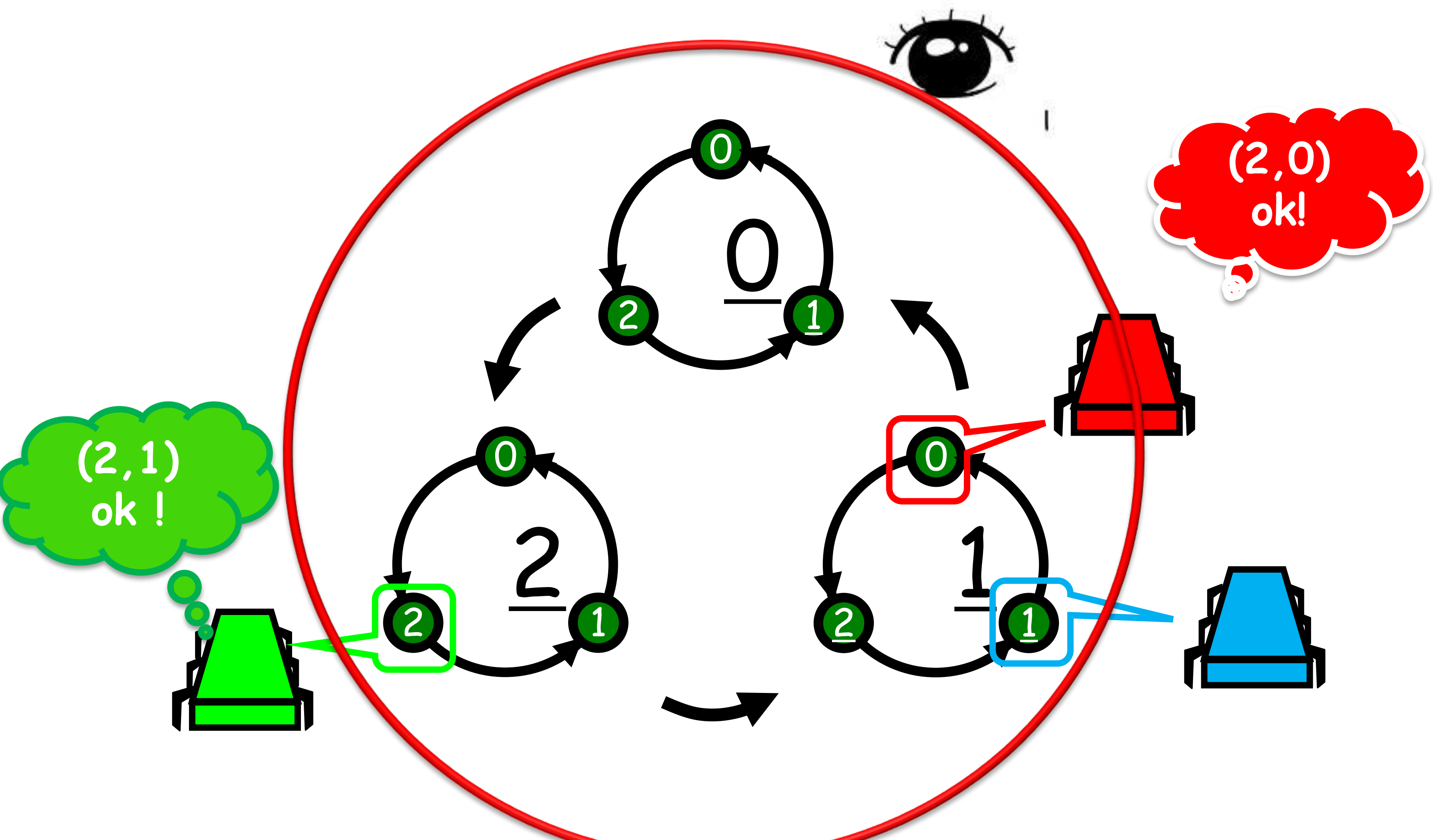
Green reads Red,Blue



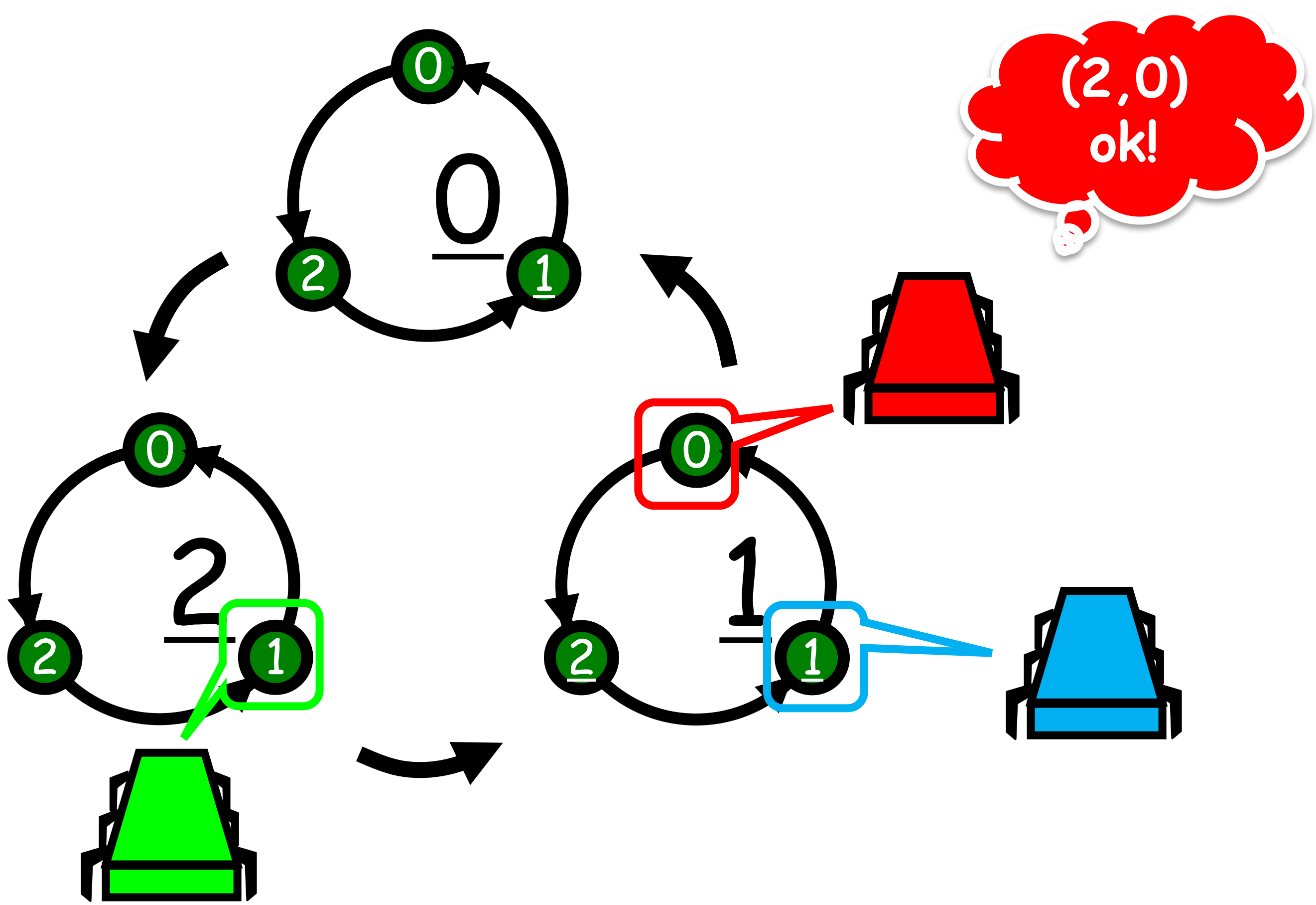
Green decides to move to (2,1)



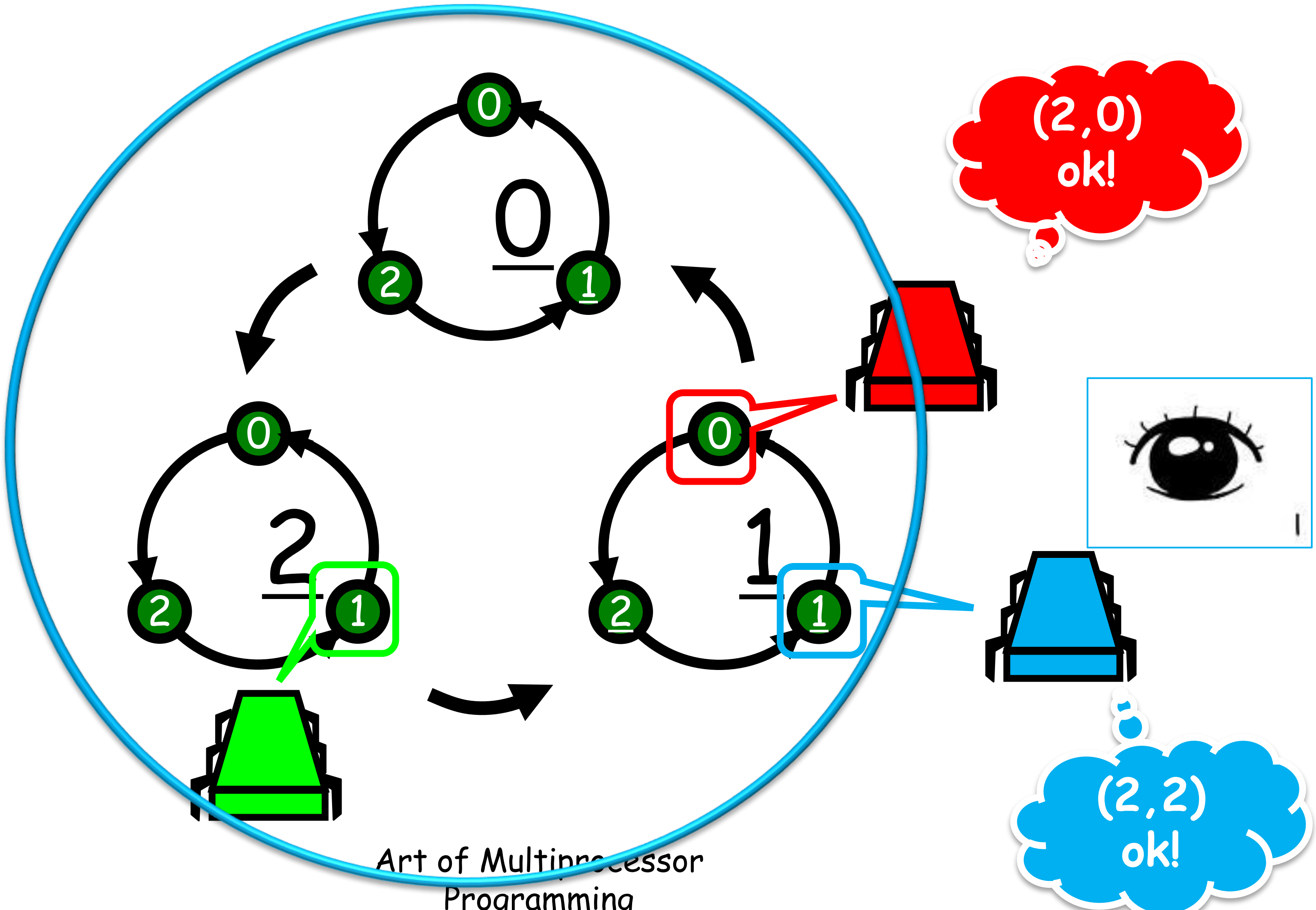
Red looks…



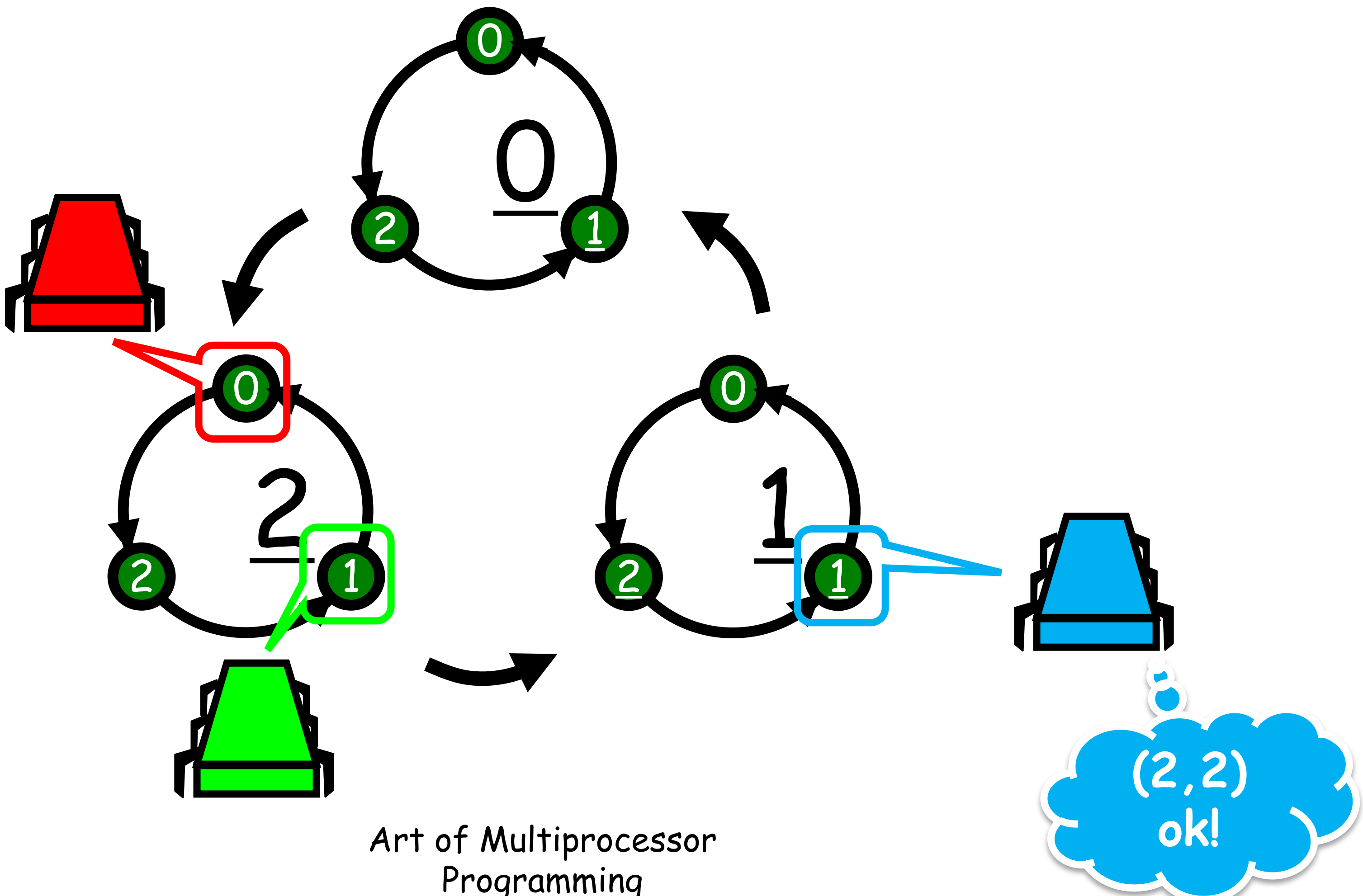
Green moves...



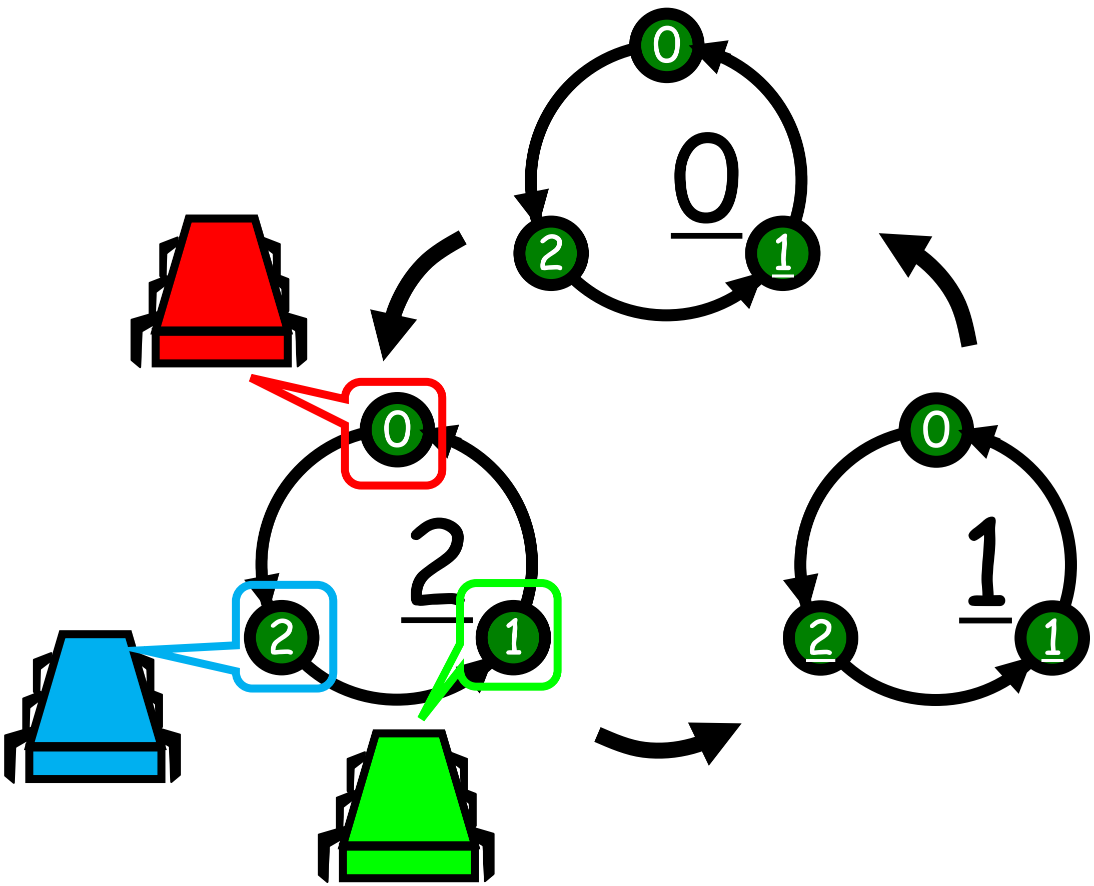
Blue looks



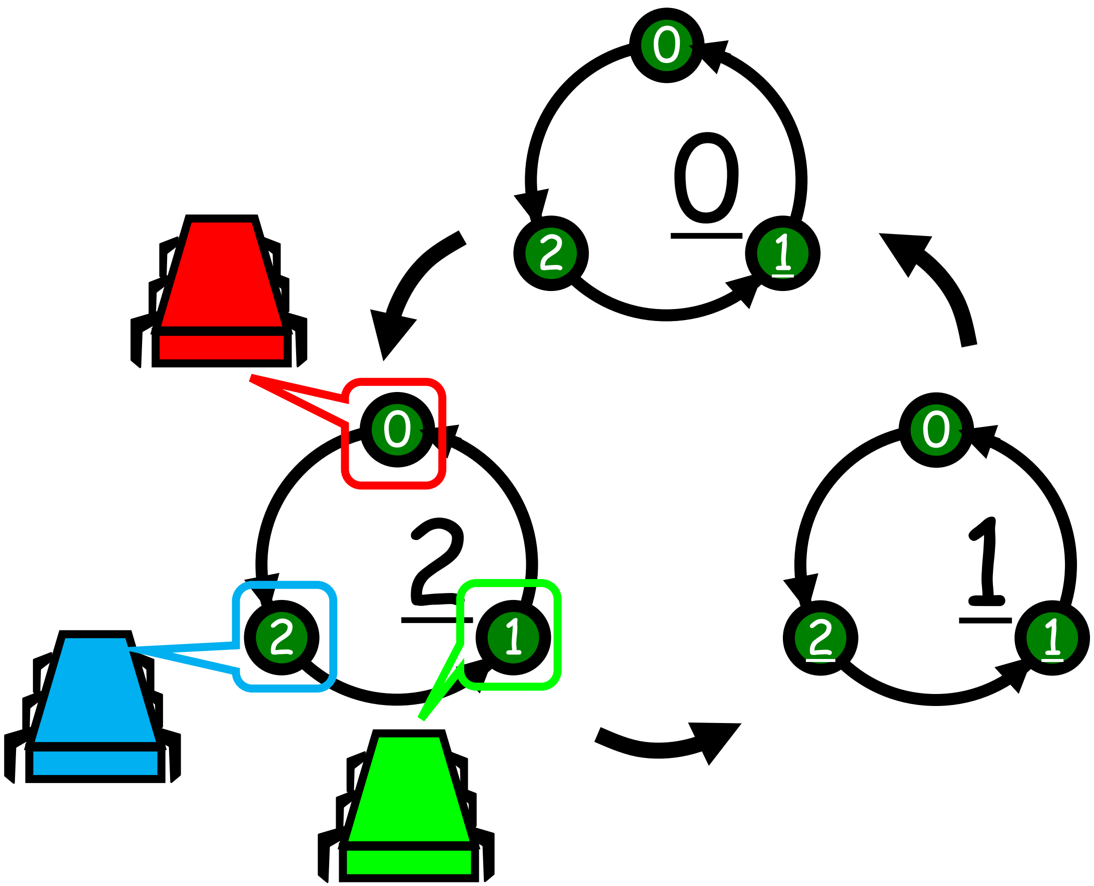
Red moves…



Blue moves…

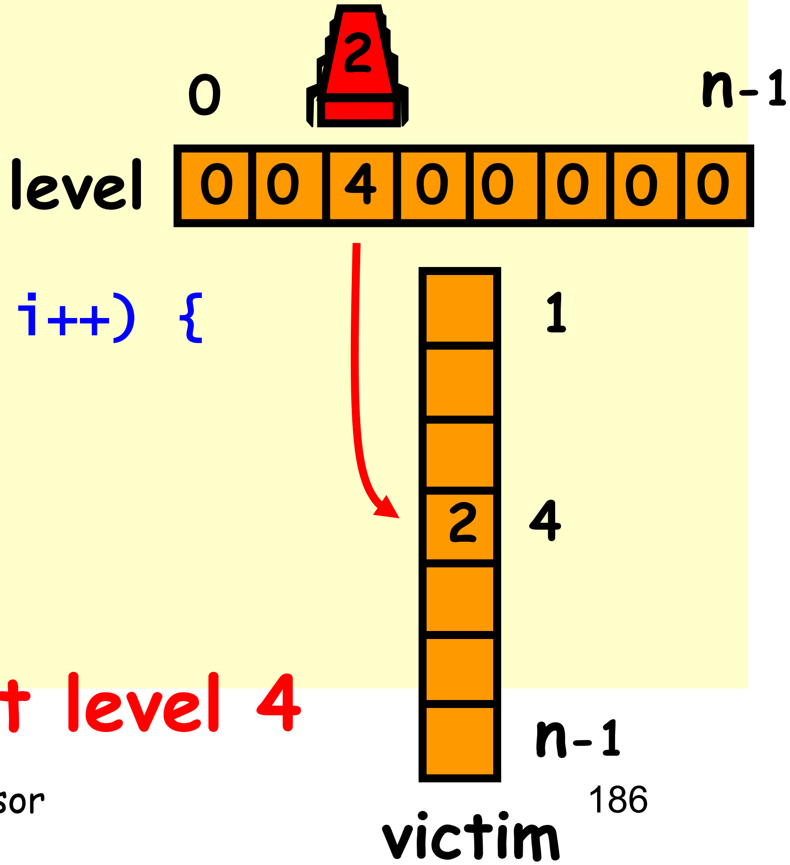


No Precedence



Filter

```
class Filter implements Lock {
  volatile int[] level;  // level[i] for thread i
  volatile int[] victim; // victim[L] for level L

  public Filter(int n) {
      level  = new int[n];
      victim = new int[n];
      for (int i = 1; i < n; i++) {
          level[i] = 0;
      }}
   …
}
```



Thread 2 at level 4

Filter

```
class Filter implements Lock {
  …

  public void lock(){
    for (int L = 1; L < n; L++) {
      level[i]  = L;
      victim[L] = i;
      while ((∃ k != i level[k] >= L) &&
                victim[L] == i );
    }}
  public void unlock() {
    level[i] = 0;
  }}
```

Filter Lock (n=3)

```
level[1]=level[2]=level[3]=0; victim[1]=victim[2]=0;

public void lock(){
 j = (i mod 3)+1;  k=(j mod 3)+1;
 level[i]  = 1;
 victim[1] = i;
 while (level[j]>= 1||level[k]>=1)&&victim[1]==i );
 level[i]  = 2;
 victim[2] = i;
 while (level[j]>= 2||level[k]>=2)&&victim[2]==i );
 }

public void unlock() {
    level[i] = 0;
  }
```

Filter Lock

	p1	p2	p3	p2	p2	p2	p3	p3	p3	p2	p2	
level[1]	0	1	1	1	1	1	1	1	1	1	1	1
level[2]	0	0	1	1	2	0	1	1	1	1	2	0
level[3]	0	0	0	1	1	1	1	2	0	1	1	1
victim[1]	0	1	2	3	3	3	2	2	2	3	3	3
victim[2]	0	0	0	0	2	2	2	3	3	3	2	2
p1			Sleeps…									
p2		blocked		CS	unlock	blocked				CS	unlock	
p3			blocked				CS	unlock	blocked			