몬테카를로 방사선해석 (Monte Carlo Radiation Analysis)

Random Number Generator (RNG)

Notice: This document is prepared and distributed for educational purposes only.

SNU/NUKE/EHK

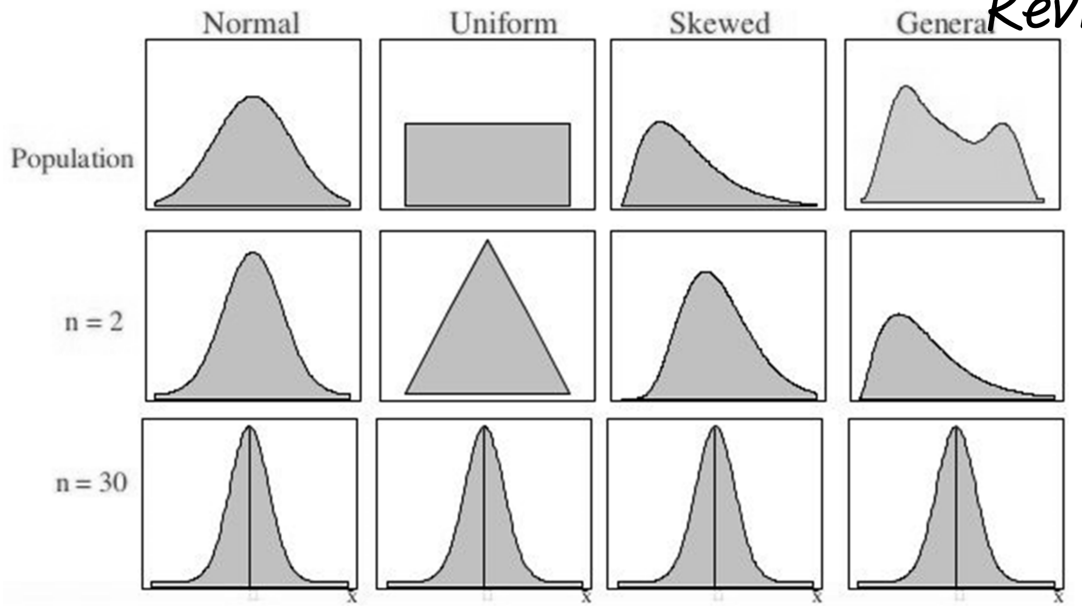
$$G = \frac{1}{N}\sum_{n=1}^{N} g(x_n) \qquad (4)$$

$$\bar{G} = E\left[\frac{1}{N}\sum_{n=1}^{N} g(x_n)\right] = \frac{1}{N}\sum_{n=1}^{N} E[g(x_n)] = \frac{1}{N}\sum_{n=1}^{N} \bar{g}(x_n) = \bar{g}(x) \quad (5)$$

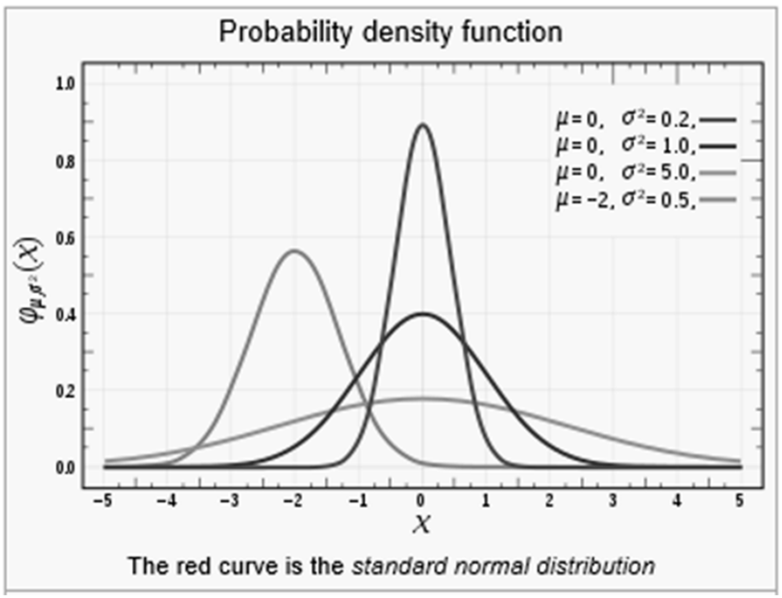
$$var[G] = var\left[\frac{1}{N}\sum_{n=1}^{N} g_n(x_n)\right] = \left(\frac{1}{N}\right)^2 \sum_{n=1}^{N} var[g(x_n)]$$

$$= \left(\frac{1}{N}\right)^2 N \cdot var[g(x_n)] = \left(\frac{1}{N}\right) var[g(x_n)] \qquad (6)$$

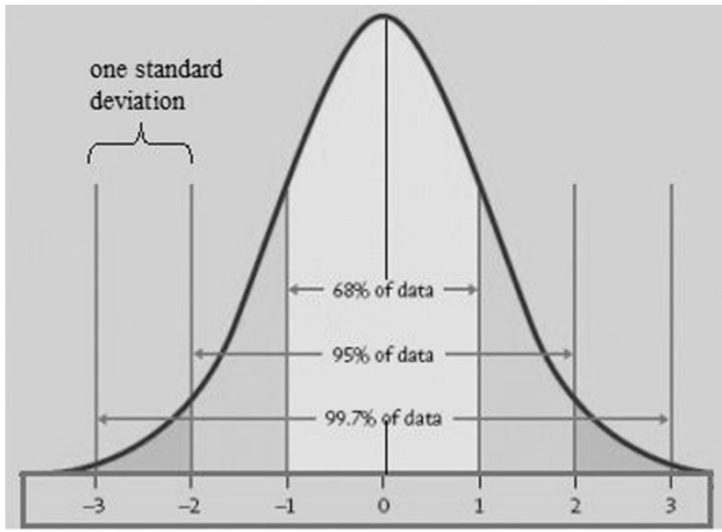
Central Limit Theorem



Normal distribution



parametric properties in normal distribution



Transformation of pdf's to cdf's

- Given a pdf $f(x)$, one define a new variable $y(x)$ with the goal of finding the pdf $g(y)$.

  – Restrict the transformation $y(x)$ to be a unique transformation, that is, a given value of x corresponds unambiguously to a value of y.

  – $f(x)dx = g(y)dy$ for strictly (monotone) increase: $dy/dx > 0$

  where $f(x)dx = prob(x \le x' \le x+dx)$ and
  $g(y)dy = prob(y \le y' \le y+dy)$

- Given a cdf $F(x)$: $y(x) = F(x) \equiv \int_{-\infty}^{x} f(x')\,dx'$,

  one finds that the pdf $g(y)$: $g(y) = 1$, for $0 \le y \le 1$

  vs. random sampling of a number in $0 \le \xi \le 1$

Random Number Generation

- Desirable Attributes:
  - Uniformity: RNs distributed uniformly on (0, 1)
  - Independence: no correlation b/w RNs
  - Efficiency: fast and minimal need for storage
  - Replicability*
    - debugging
    - compare various scenarios or different systems
  - Long Cycle Length
- Independent and identically distributed (i.i.d.) RV

➢ *Independent and identically distributed (i.i.d.) RV*

- *Identically Distributed means that there are no overall trends: the distribution doesn't fluctuate and all items in the sample are taken from the same probability distribution.*
- *Independent means that the sample items are all independent events. In other words, they aren't connected to each other in any way.*

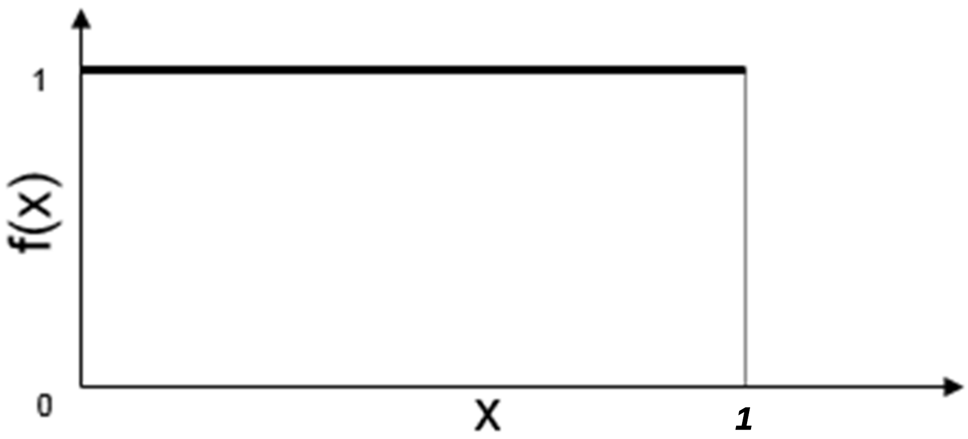
Random Number Generation (cont.)

- Each random number R_t is an independent sample drawn from a continuous uniform distribution between 0 and 1

$$\text{pdf:} \quad f(x) = \begin{cases} 1 \ , & 0 \le x \le 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

Random Number Generation (cont.)

PDF:



$$E(R) = \int_0^1 x\,dx = [x^2/2]_0^1 = 1/2$$

$$V(R) = \int_0^1 x^2 dx - [E(R)]^2$$

$$= [x^3/3]_0^1 - (1/2)^2 = 1/3 - 1/4 = 1/12$$

Random Number Generation (cont.)

- Most random-number generators are of the form:

  1. Start with a seed number X_0.

  2. Generate $X_n = f(X_{n-1})$ for $n = 1, 2, 3, \cdots$.

  3. Obtain $R_n = g(X_n)$

  where f is a pseudo-random generator and g is the output function

Random Number Generation Method #1

- Midsquare Method

$X_0 = 7182$ (seed), $X_0^2 = 51\underline{5811}24$

==> $R_1 = 0.5811$

$X_1^2 = (5811)^2 = 33\underline{7677}21$

==> $R_2 = 0.7677$

➢ Midsquare Method (cont.)

Note: Cannot choose a seed that guarantees that the sequence will not degenerate and will have a long period. Also, zeros, once they appear, are carried in subsequent numbers.

Ex 1. $X_0 = 5197$ (seed), $X_0^2 = 27\underline{0088}09$

==> $R_1 = 0.0088$, $X_1^2 = 00\underline{0077}44$

==> $R_2 = 0.0077$

Ex 2. $X_0 = 4500$ (seed), $X_0^2 = 20\underline{2500}00$

==> $R_1 = 0.2500$, $X_1^2 = 06\underline{2500}00$

==> $R_2 = 0.2500$

Note: Modular Mathematics

- ✓ When we divide a number into two integers, we will have an equation that looks like the following:

  A / B = Q with remainder R

  A is the dividend; B is the divisor; Q is the quotient; and R is the remainder.

- ✓ Sometimes, we are only interested in what the remainder will be when we divide A by B. For these cases there is an operator called the modulo operator (abbreviated as mod).

  Using the same A, B, Q,and R as above, we would have:

  A mod B = R

  We would say this as "A modulo B is congruent to R". Where B is referred to as the modulus. (2進法, 10進法, etc.)

  e.g. 13/5 = 2 with remainder 3, or 13 mod 5 = 3

Random Number Generation Method #2

➢ Linear Congruential Generator

- Basic Relationship

  $X_i = (a\, X_{i-1} + c) \bmod m$, where $a \geq 0$ and $m \geq 0$

  $R_i = X_i / m$

  a = multiplier, c = increment

- Most natural choice for m is one that equals to the capacity of a computer word.
- $m = 2^b$ (binary machine), where b is the number of bits in the computer word.
- $m = 10^d$ (decimal machine), where d is the number of digits in the computer word.

➢ Linear Congruential Generator (cont.)

- 16-bit machine

  $a = 1217,\ c = 0, X_0 = 23,\ m = 2^{15}-1 = 32767$

  $X_1 = (1217*23) \text{ mod } 32767 = 27991$

  $R_1 = 27991/32767 = 0.85424$

  $X_2 = (1217*27991) \text{ mod } 32767 = 20134$

  $R_2 = 20134/32767 = 0.61446$

➢ Linear Congruential Generator (cont.)

$X_{i+1} = (a X_i + c) \bmod m$, where $m>0$, $0<a<m$ and $0 \leq c<m$.

– The maximum period, P

- (case 1) For $c \neq 0$,

  $P = m$ provided that c is prime to m (greatest common divisor of c and m is 1) and the multiplier $a-1 = 4k$, where k is an integer.

- (case 2) For $m = 2^b$, and $c = 0$,

  $P = m/4 = 2^{b-2}$ provided that the seed X_0 is odd and the multiplier $a = 3 + 8k$ or $a = 5 + 8k$ for some $k = 0, 1, ...$

- (case 3) For m = a prime number and $c = 0$,

  $P = m-1$ provided that the multiplier, a, has the property that the smallest integer k, such that $a^k - 1$ is divisible by m, is $k = m - 1$,

.

Multiplicative Congruential Generator

$X_i = aX_{i-1} \mod m$, where $m>0$ and $a>0$

$R_i = X_i/m$

- Can not have full period ($P=m$), but can have $P = m-1 = 2^b-1$

Additive Congruential Generator

$X_i = (X_{i-1} + X_{i-k}) \mod m$, $i = 1, 2, \ldots$

$R_i = X_i/m$

- With consecutive numbers ($k=2$) R_{n-2}, R_{n-1}, and R_n, it will never happen that $R_{n-2} < R_n < R_{n-1}$ or $R_{n-1} < R_n < R_{n-2}$, which occurs by 1/6 for true uniform variables.

Choosing the initial seed

- e.g., time (wall-clock and since booting),

➢ Linear Congruential Generator (cont.)

$X_i = (a\,X_{i-1} + c) \bmod m$, where $a \geq 0$ and $m \geq 0$

$R_i = X_i/m$

- (case 1) For $c \neq 0$,

  $P = m$ provided that c is prime to m (greatest common divisor of c and m is 1) and the multiplier $a-1 = 4k$, where k is an integer.

– Examples

- For (a, c, m) = (1, 5, 13) and $z_0 = 1$, we get the sequence 1, 6, 11, 3, 8, 0, 5, 10, 2, 7, 12, 4, 9, 1, which has full period of 13. (case 1)

- For (a, c, m) = (2, 5, 13) and $z_0 = 1$, we get the sequence 1, 7, 6, 4, 0, 5, 2, 9, 10, 12, 3, 11, 1, which has a period of 12.

  With $z_0 = 8$, we get the sequence 8, 8, 8, (period of 1).

➢ *Linear Congruential Generator* (cont.)

$X_i = (a X_{i-1} + c) \bmod m$, where $a \geq 0$ and $m \geq 0$

$R_i = X_i / m$

– Examples

Using the multiplicative congruential method, find the period of the generator for $a = 13$, $m = 2^6$, $c=0$ (case 2) and $X_0 = 1, 2, 3,$ and 4. The solution is given in next slide. When the seed is 1 and 3, the sequence has period 16. However, a period of length eight is achieved when the seed is 2; and a period of length four occurs when the seed is 4.

- (case 2) For $m = 2^b$, and $c = 0$,

  $P = m/4 = 2^{b-2}$ provided that the seed X_0 is odd and the multiplier $a = 3 + 8k$ or $a = 5 + 8k$ for some $k = 0, 1,..$

- *Example results.*

i	X_i	X_i	X_i	X_i
0	1	2	3	4
1	13	26	39	52
2	41	18	59	36
3	21	42	63	20
4	17	34	51	4
5	29	58	23	
6	57	50	43	
7	37	10	47	
8	33	2	35	
9	45		7	
10	9		27	
11	53		31	
12	49		19	
13	61		55	
14	25		11	
15	5		15	
16	1		3	