Chapter 5. Torsion

Torsional member : e.g., shaft

Load : Torque

↓

stress $\tau = G \cdot \gamma$

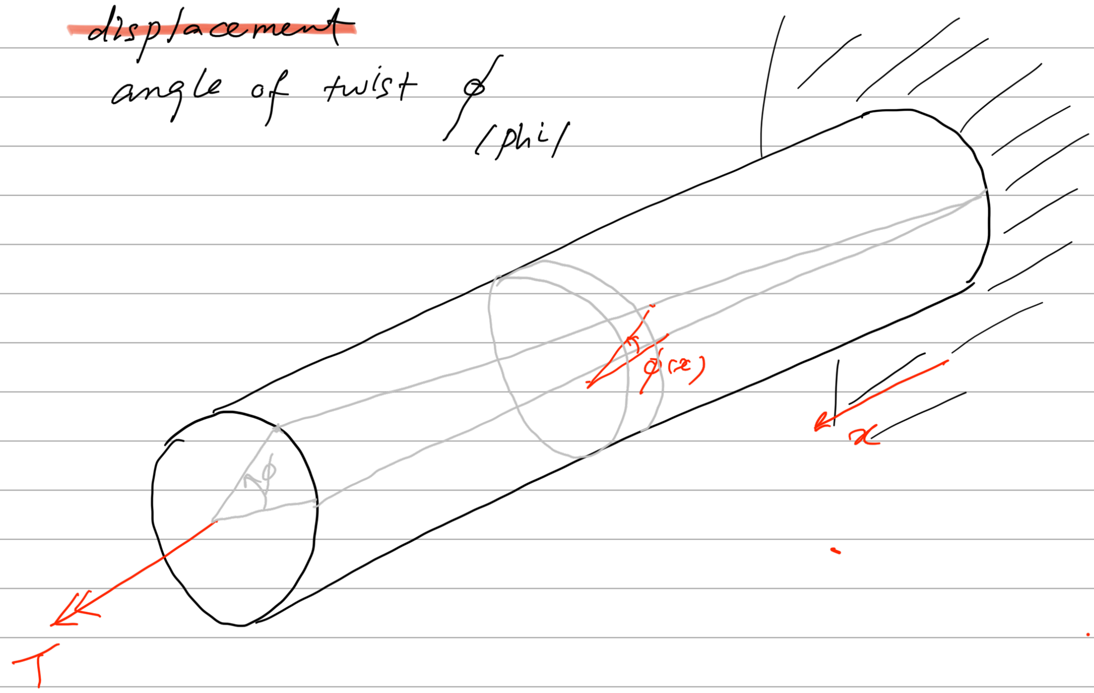
↓

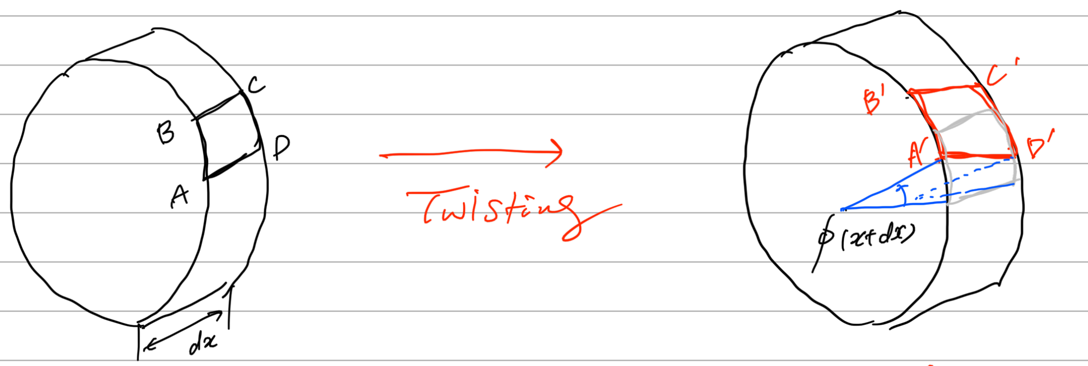
strain $\gamma = \rho \frac{d\phi}{dx}$

↓

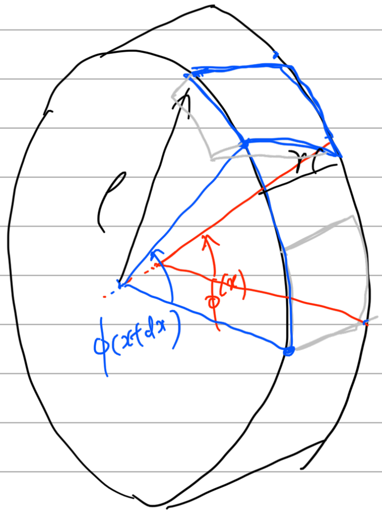
~~displacement~~

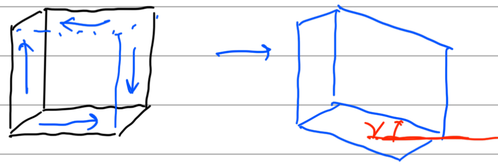
angle of twist ϕ (phi)





Before deformation After deformation



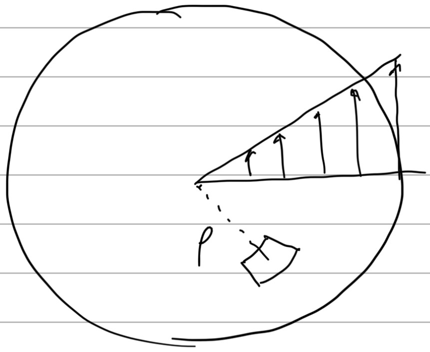


$$\rho \cdot d\phi = dx \cdot \tan\gamma$$

For small twist, $\tan\gamma \simeq \gamma$

$$\gamma = \rho \cdot \frac{d\phi}{dx} \quad : \text{shear strain}$$

$$= \rho \cdot \theta \qquad \theta = d\phi/dx : \text{rate of twist}$$



$$dT = \rho\,\tau \cdot dA$$

$$T = \int \rho \cdot \tau \, dA$$

$$= \int \rho \cdot G \cdot \rho \frac{d\phi}{dx} \cdot dA$$

$$= G\theta \cdot \underbrace{\int \rho^2 dA}_{\text{polar moment of inertia } J}$$

$$\theta = \frac{T}{GJ}$$

T: loading, J: geometry, G: material property

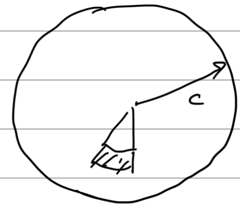
$$\tau = G\gamma$$

$$= G \cdot \rho \cdot \theta$$

$$= G \cdot \rho \cdot \frac{T}{GJ} = \frac{T}{J} \cdot \rho$$

J : polar moment of inertia

- solid shaft

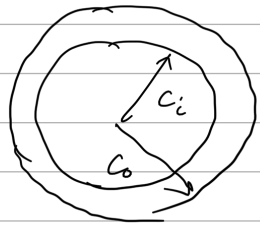


$$J = \int \rho^2 dA = \int_0^c \int_0^{2\pi} \rho^2 \rho \, d\theta \, d\rho$$

$dA = \rho \, d\theta \cdot d\rho$

$$= \frac{c^4}{4} 2\pi = \frac{\pi}{2} \cdot c^4$$

- Tube

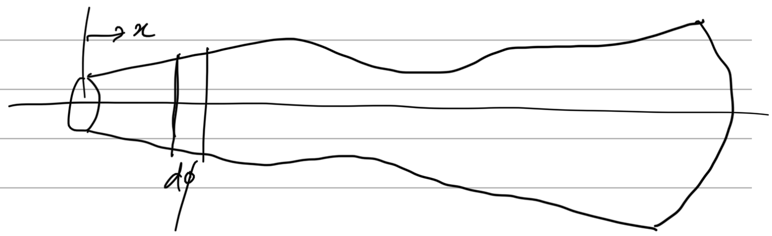


$$J = \int_{c_i}^{c_o} \int_0^{2\pi} \rho^3 \, d\theta \, d\rho$$

$$= \frac{\pi}{2} (c_o^4 - c_i^4)$$

Angle of twist

$$\gamma = \rho \cdot \frac{d\phi}{dx}$$



$$\rightarrow \quad \phi(x) = \int d\phi$$

$$= \int \frac{T(x)}{G(x) J(x)} \cdot dx$$

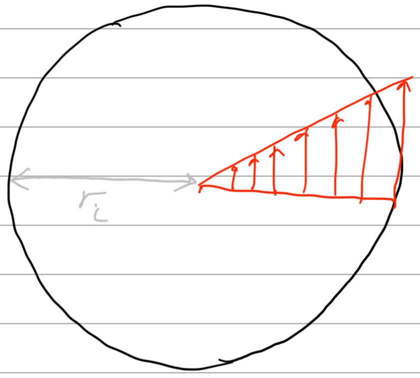
$$\left(\theta = \frac{T}{G \cdot J} = \frac{d\phi}{dx} \text{ : rate of twist} \right)$$

Constant torque and cross sectional area

$$\phi = \frac{TL}{GJ}$$

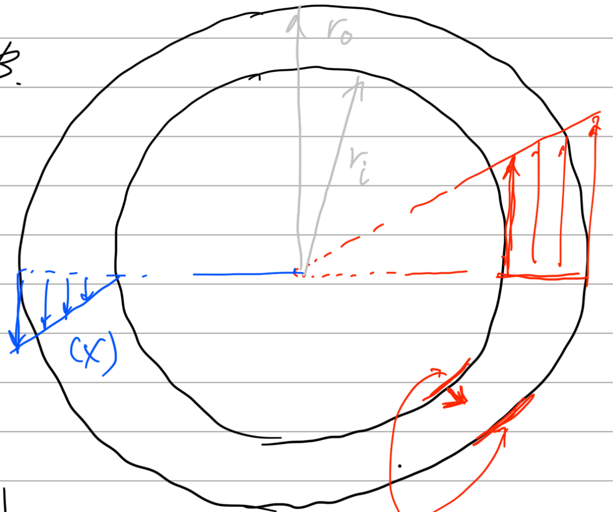
Example - Solid shaft vs. Hollow tube

A.



vs.

B.



free wall

: shear in the radial direction cannot be non-zero, but tangential ones can be non-zero.

① Under the same torsional load (T), what should be r_o to make the same τ_{max}?

$$\tau_A\Big|_{max} = \frac{T}{J_A} r_i \qquad \tau_B\Big|_{max} = \frac{T}{J_B} \cdot r_o$$

$$J_A = \frac{\pi r_i^4}{2} \qquad J_B = \frac{\pi (r_o^4 - r_i^4)}{2}$$

$$\tau_A|_{max} = \tau_B|_{max} \rightarrow \frac{T r_i}{\frac{\pi r_i^4}{2}} = \frac{T r_o}{\frac{\pi (r_o^4 - r_i^4)}{2}}$$

$$\rightarrow \frac{r_o^4 - r_i^4}{r_i^4} = \frac{r_o}{r_i}$$

$$(r_o/r_i)^4 - (r_o/r_i) - 1 = 0$$

$$r_o/r_i = 1.2207$$

Material saving

$$A_B/A_A = \frac{\pi (r_o^2 - r_i^2)}{\pi r_i^2} = \left(\frac{r_o}{r_i}\right)^2 - 1 = 1.2207^2 - 1 = 0.4902$$

② To make identical ϕ

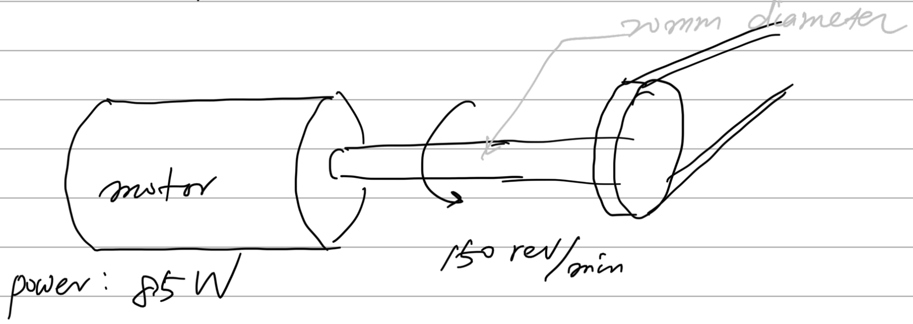
$$\phi_A = \frac{T \cdot L}{G J_A} \qquad \phi_B = \frac{T L}{G J_B}$$

$$r_i^4 = r_o^4 - r_i^4$$

$$\rightarrow \quad \left(\frac{r_o}{r_i}\right)^4 = 2 \quad , \quad \beta = 1.1892$$

$$\frac{A_B}{A_o} = \left(\frac{r_o}{r_i}\right)^2 - 1 = \sqrt{2} - 1 = 0.414$$

5.3. Power transmission



Maximum shear stress ?

$$P = \frac{dW}{dt} = \frac{T\,d\theta}{dt} = T \cdot \omega$$

$$85\ W = T \cdot \frac{150 \times 2\pi}{60}\ \text{rad/s}$$

$$\therefore T = \frac{85 \times 60}{150 \times 2\pi}\ N\cdot m$$

$$= 17/\pi\ N\cdot m$$

$$\theta = \frac{T}{G \cdot J}$$

$$\tau|_{max} = G \cdot \gamma|_{max}$$

$$= G \cdot R\,\theta$$

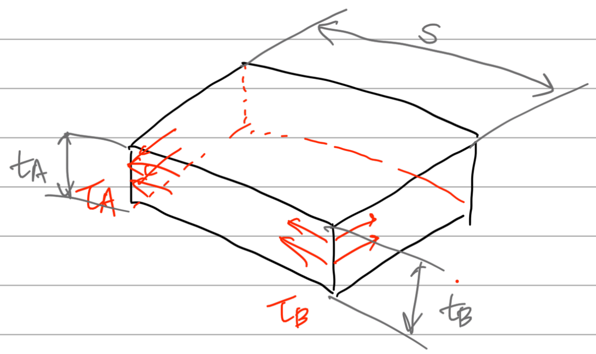
$$= G \cdot R \cdot \frac{T}{G \cdot J}$$

$$= \frac{T}{J} \cdot R \quad , \quad J = \frac{\pi}{2} R^4$$

$$= \frac{17/\pi}{\pi/2 \cdot (10 \times 10^{-3})^3}$$

$$= \frac{34}{\pi^2} \times 10^6\ Pa$$

5.7. Thin-walled Tubes having closed cross sections

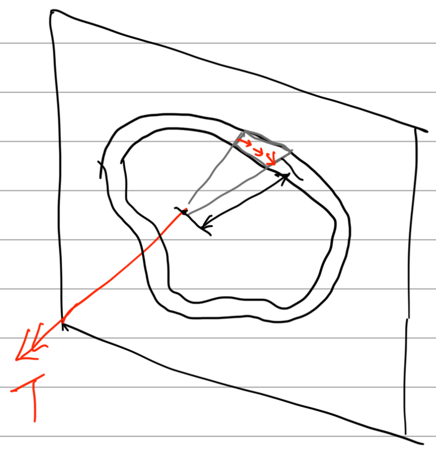


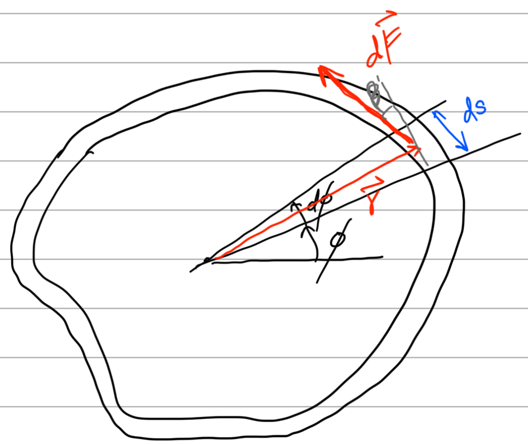
Force equilibrium

$$\tau_A \cdot dx \cdot t_A = \tau_B \cdot dx \cdot t_B$$

$$\rightarrow \quad \tau_A t_A = \tau_B \cdot t_B$$

$$= q \text{ : shear flow}$$





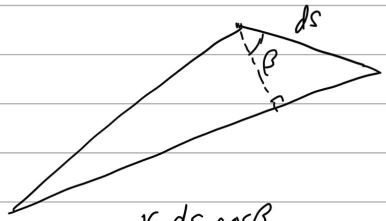
$$d\vec{T} = \vec{r} \times d\vec{F}$$

$$= r \, dF \cos\beta$$

thickness

$$= r \, \tau(\phi) \cdot ds \; t(\phi) \cdot \cos\beta$$

$$\tau(\phi)\, t(\phi) = q \quad (\text{const.})$$



$$\frac{r\, ds \cos\beta}{2} = dA$$

$$T = \int dT = q \cdot \int r\cos\beta \cdot ds$$

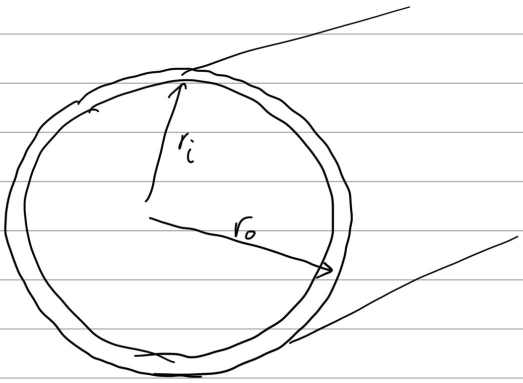
$$= q \int 2\, dA$$

$$= 2q\, A_m$$

mean area enclosed within the boundary

$$q = \frac{T}{2A_m}$$

example. circular cross section



Method - 1

$$\tau = \frac{T}{J} \cdot r$$

$$J = \frac{\pi}{2}(r_o^4 - r_i^4)$$

$$\left(r_o = r + t/2, \quad r_i = r - t/2\right)$$

$$= \frac{\pi}{2}\left[r^4\left(1+\frac{t}{2r}\right)^4 - r^4\left(1-\frac{t}{2r}\right)^4\right]$$

$$= \frac{\pi}{2} r^4 \left[1 + \frac{2t}{r} - \left(1 - \frac{2t}{r}\right)\right]$$

$$= \frac{\pi r^4}{2} \cdot \frac{4t}{r} = 2\pi r^3 t$$

$$\left(\begin{array}{l}\text{Taylor series} \\ f = (1+x)^n \simeq 1 + xn \quad \text{if } x \simeq 0\end{array}\right)$$

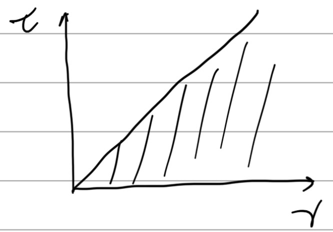
$$\therefore \tau = \frac{T}{2\pi r^3 t} \cdot r = \frac{T}{2\pi r^2 t}.$$

Method - 2

$$q = \frac{T}{2A_m} = \frac{T}{2\pi r^2} = \tau \cdot t$$

$$\therefore \tau = \frac{T}{2\pi r^2 t}$$

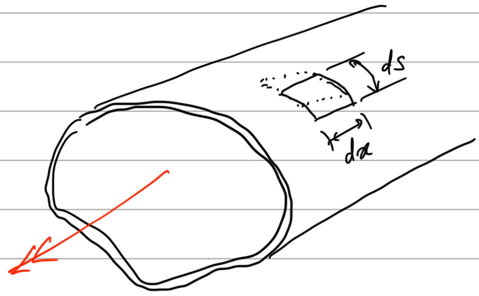
Angle of twist for a thin-walled tube

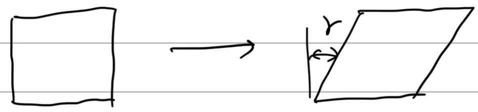


strain energy density under shear

$$u = \frac{1}{2}\tau\gamma = \frac{\tau^2}{2G}$$

$$\left(\text{c.f. } u = \frac{1}{2}\sigma\varepsilon = \frac{\sigma^2}{2E} \text{ under axial}\right)$$



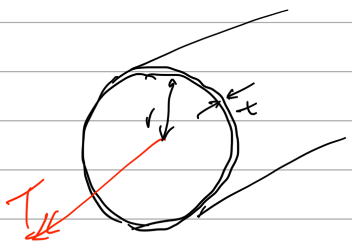


$$\begin{aligned} dU &= u\,dV \\ &= \frac{\tau^2}{2G}\cdot t\cdot ds\cdot dx \\ &= \frac{\tau^2 t^2}{2G}\cdot\frac{ds\,dx}{t} \\ &= \frac{q^2}{2G}\cdot\frac{ds\,dx}{t} \end{aligned}$$

$$\begin{aligned} U = \int dU &= \int_0^L \oint \frac{q^2}{2G}\cdot\frac{ds}{t}\cdot dx \\ &= \frac{q^2 L}{2G}\cdot\oint\frac{ds}{t} \qquad \left(q = \tau\cdot t = \frac{T}{2A_m}\right) \\ &= \frac{T^2 L}{8GA_m^2}\oint\frac{ds}{t} \\ &= \frac{1}{2}T\phi \end{aligned}$$

$$\therefore\ \phi = \frac{TL}{4GA_m^2}\oint\frac{ds}{t}$$

Example. Circular tube



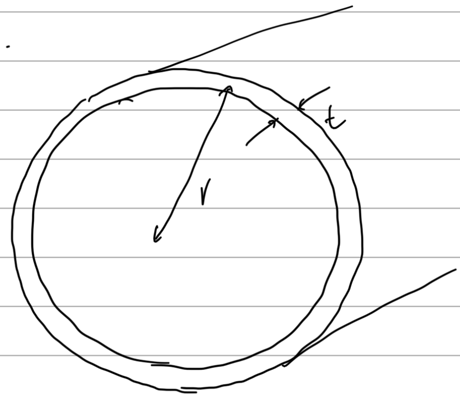
$$\begin{aligned} \phi &= \frac{T\cdot L}{4G\cdot(\pi r^2)^2}\cdot\frac{1}{t}2\pi r \\ &= \frac{TL}{G\,2\pi t r^3} \end{aligned}$$

$$\left(\text{c.f. } \phi = \frac{TL}{GJ} = \frac{TL}{G\cdot 2\pi r^3 t}\right)$$

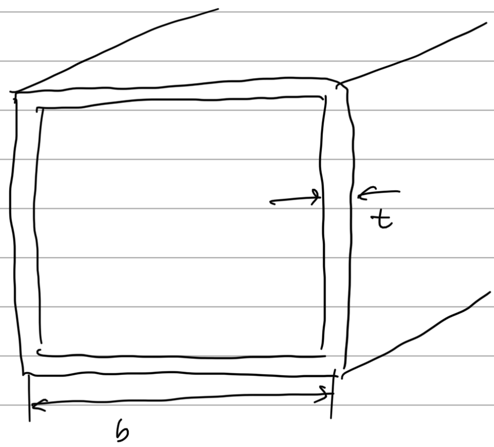
	Shaft	Thin-walled tube
Load	T	T $q = \tau \cdot t = \dfrac{T}{2A_m}$
stress	$\tau = \dfrac{T}{J}\rho$	$\tau = \dfrac{T}{2tA_m}$
strain	$\gamma = \rho\theta = \dfrac{T}{GJ}\cdot\rho$	$\gamma = \dfrac{\tau}{G} = \dfrac{T}{2tGA_m}$
displacement (deformation)	$\phi = \dfrac{TL}{GJ}$ $\left(\theta = \dfrac{T}{GJ}\right)$	$\phi = \dfrac{TL}{4GA_m^2}\oint\dfrac{ds}{t} = \dfrac{TL}{GJ'}$ $J' = \dfrac{4A_m^2}{\oint ds/t}$: torsional constant

Example. Circular vs. Square tubes

C.



S.



When the same amount of material and thickness is used, which structure is torsionally stiffer?

$$2\pi r\cdot t = 4b\cdot t \qquad \therefore \frac{b}{r} = \frac{\pi}{2}$$

$$\frac{\phi_c}{\phi_s} = \frac{J'_s}{J'_c} = \left[\frac{4A_m^2}{\oint \frac{ds}{t}}\right]_s \Big/ \left[\frac{4A_m^2}{\oint ds/t}\right]_c$$

$$= \frac{4(b^2)^2}{4b/t} \Big/ \frac{4(\pi r^2)^2}{2\pi r/t}$$

$$= \frac{b^3 \cdot t}{2\pi r^3 \cdot t} = \frac{1}{2\pi}\left(\frac{b}{r}\right)^3$$

$$= \frac{1}{2\pi} \cdot \left(\frac{\pi}{2}\right)^3$$

$$= \frac{\pi^2}{16}$$

$= 0.617 \rightarrow$ Circular tube is stiffer!