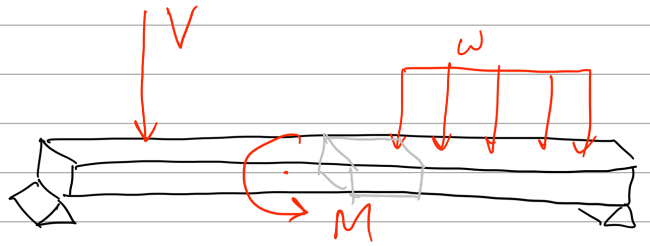
Chapter 6. Bending

Beam : members that are slender and support loads that are applied perpendicular to their longitudinal axis

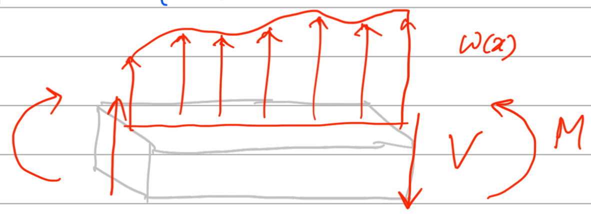
Loads (V, M, w)
shear force
bending moment



↓

stress

Sign convention



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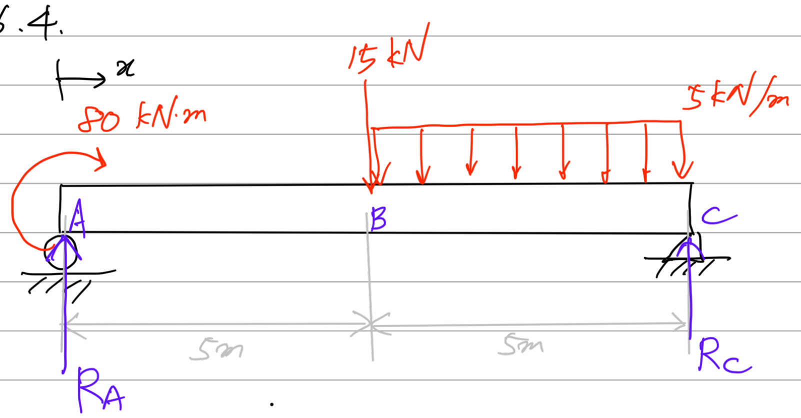
strain

↓

deformation

Shear and moment diagram
: expressing V(x) and M(x)

Example 6.4.



$\Sigma F = 0$: $R_A + R_B - 15 - 25 = 0$

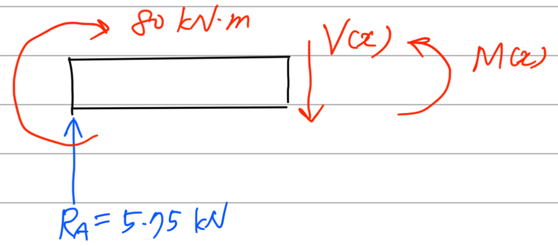
$\Sigma M = 0$: $5 \times 5 \times 2.5 + 15 \times 5 - 10 R_A - 80 = 0$
(w.r.t C)

$$R_A = \frac{25 \times 2.5 + 75 - 80}{10} = 5.75 \text{ kN}$$

$$R_B = 40 - 5.75 = 34.25 \text{ kN}$$

Method of section

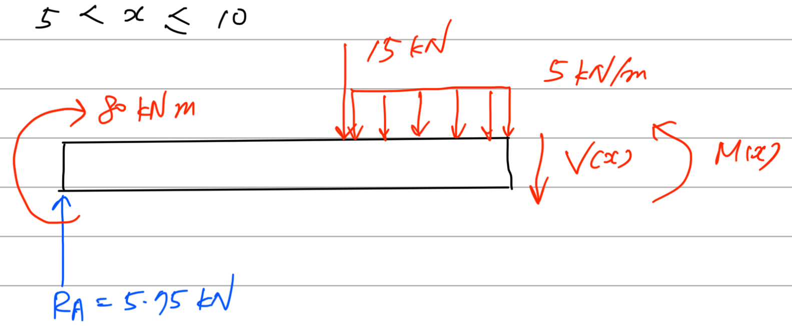
① $0 \le x < 5$



$$V(x) = 5.75 \text{ kN}$$

$$M(x) - 5.75x - 80 = 0 \rightarrow M(x) = 5.75x + 80$$

② $5 < x \le 10$



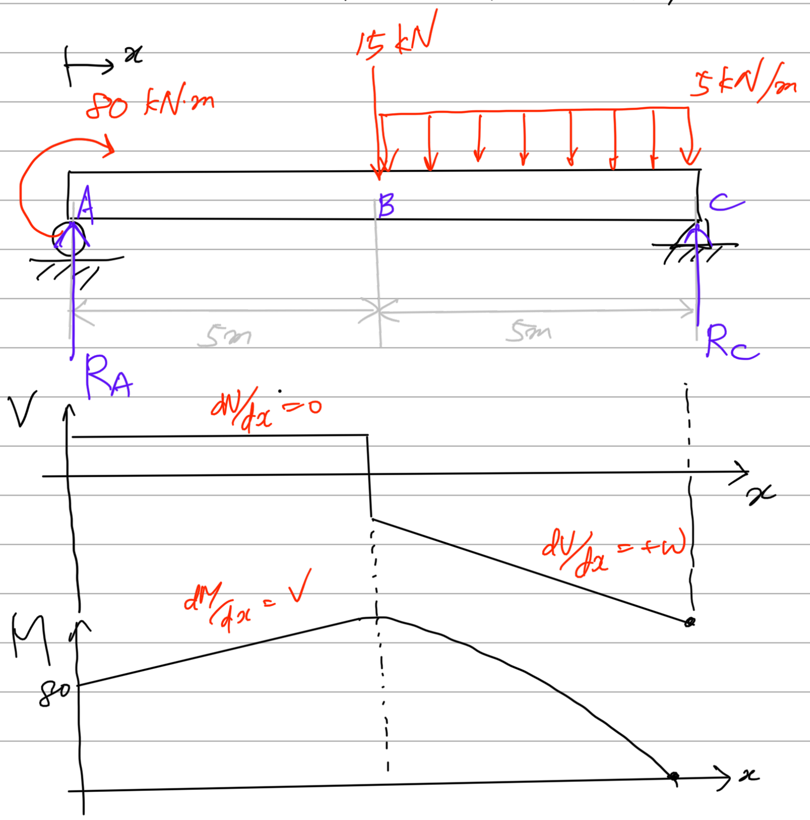
$$5.75 - 15 - 5(x-5) - V(x) = 0$$

$$V(x) = 15.75 - 5x$$

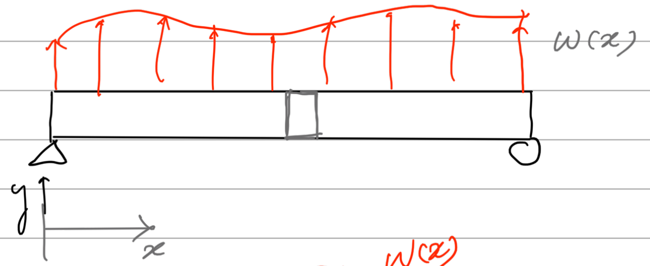
$$M(x) + 5(x-5)\cdot\left(\frac{x-5}{2}\right) + 15(x-5) - 5.75x - 80 = 0$$

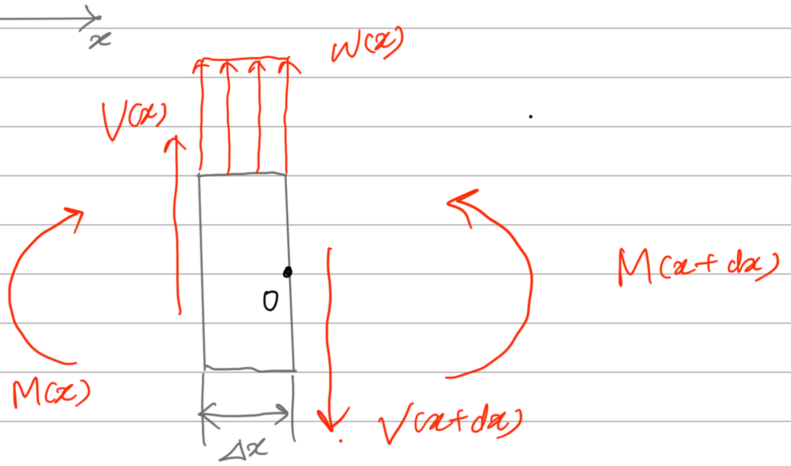
$$M(x) = -2.5(x^2 - 10x + 25) - 15x + 75 + 5.75x + 80$$

$$= -2.5x^2 + 15.75x + 92.5$$



6.2. Graphical method for constructing shear and moment diagrams





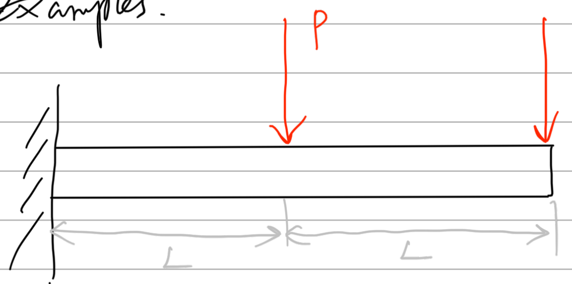
$$\Sigma F_y = 0 \quad : \quad V(x) + w(x)\,dx - V(x+dx) = 0$$

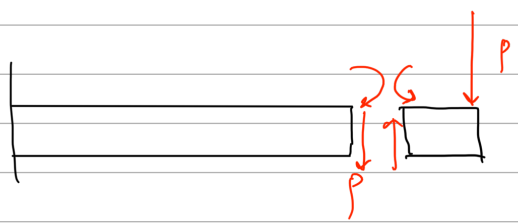
$$\rightarrow \quad \frac{V(x+dx) - V(x)}{dx} = \frac{dV(x)}{dx} = w(x)$$

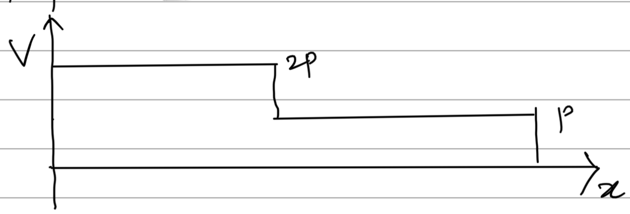
$$\Sigma M_0 = 0 \quad (\circlearrowleft +) \qquad M(x+dx) - M(x) - V(x)\cdot dx - w(x)\cdot dx\cdot\frac{dx}{2} = 0$$

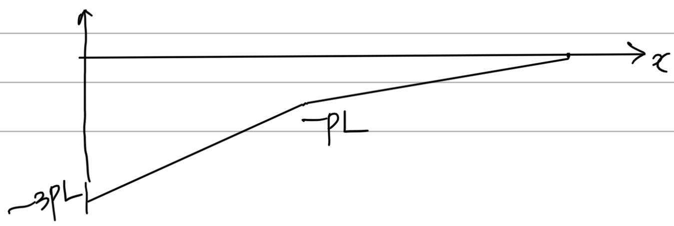
$$\rightarrow \quad \frac{M(x+dx) - M(x)}{dx} = \frac{dM(x)}{dx} = V(x)$$

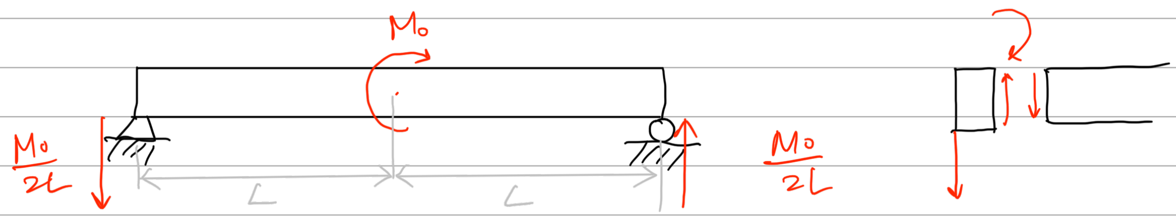
Examples.

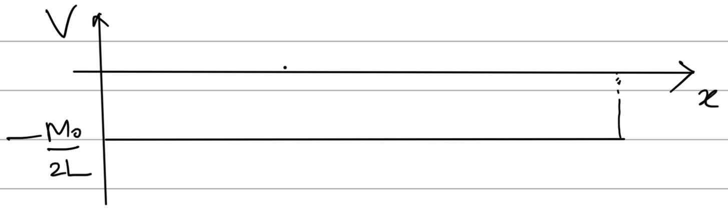


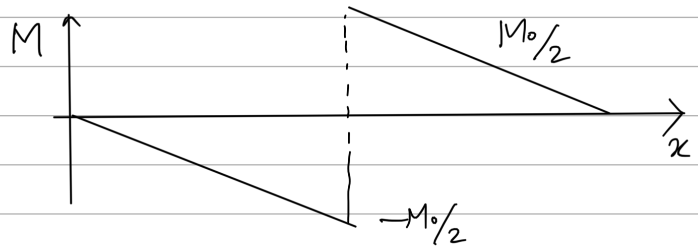












6.3. Bending deformation of a straight member

Loads (V, M, w)
shear force
bending moment

↓

stress $\quad \sigma = -kEy = -\frac{M}{I} \cdot y$

↓

$\kappa = \frac{M}{EI}$

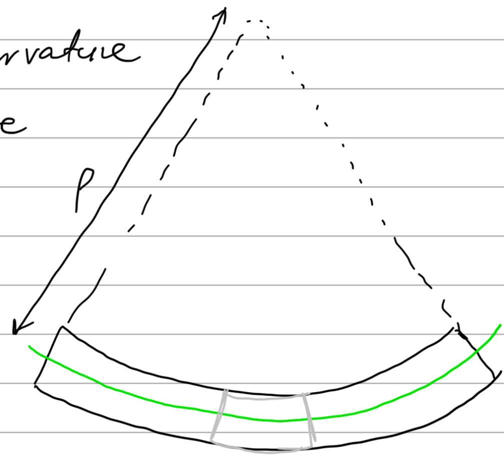
strain $\quad \varepsilon = -ky = -\frac{M}{EI} y$

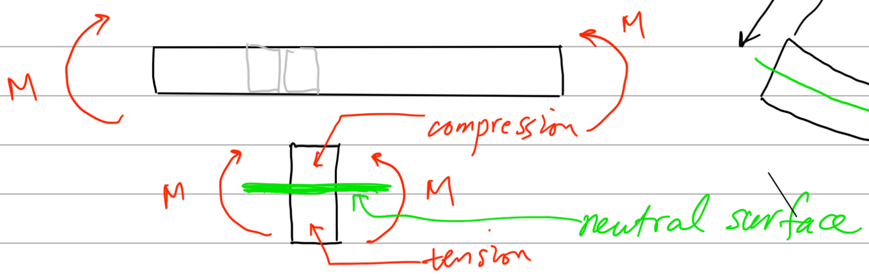
↓

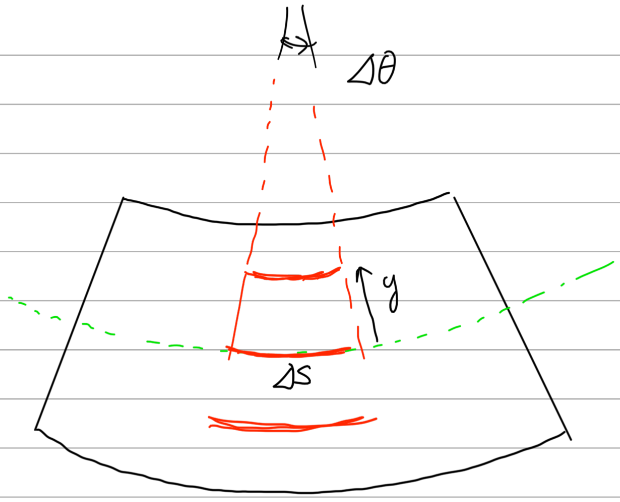
deformation

ρ : radius of curvature

$k = 1/\rho$: curvature







$$\rho\,\Delta\theta = \Delta s$$

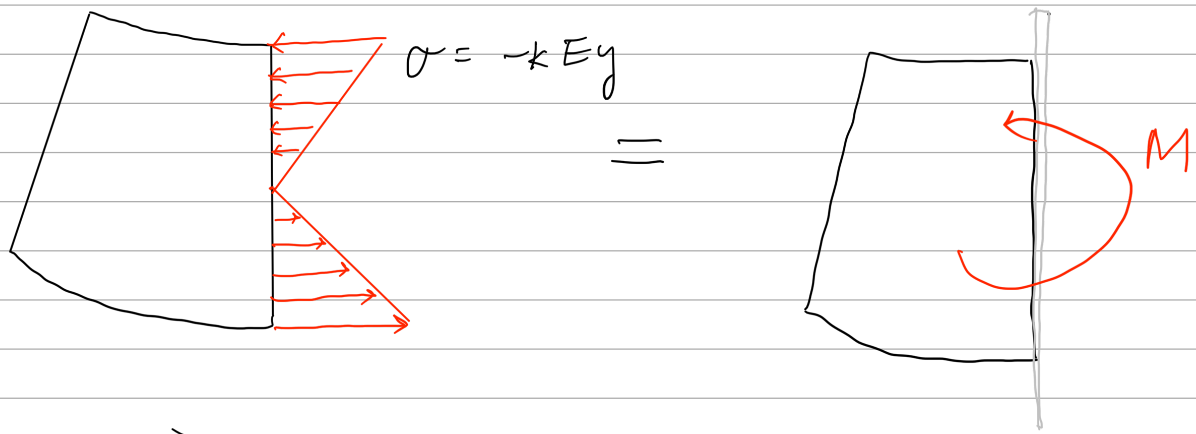
$$(\rho - y)\,\Delta\theta = \Delta s'$$

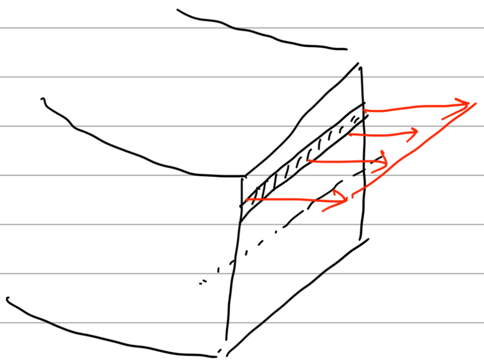
$$\varepsilon = \lim_{\Delta\theta\to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

$$= \lim_{\Delta\theta\to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

$$= -\frac{y}{\rho} = -\kappa y$$

(c.f. Torsional member
$\gamma = \rho \cdot \theta$)





$$dM = -dF \cdot y$$

$$= -\sigma\, dA\, y$$

$$= +\kappa E\, y^2\, dA$$

$$M = \int dM$$

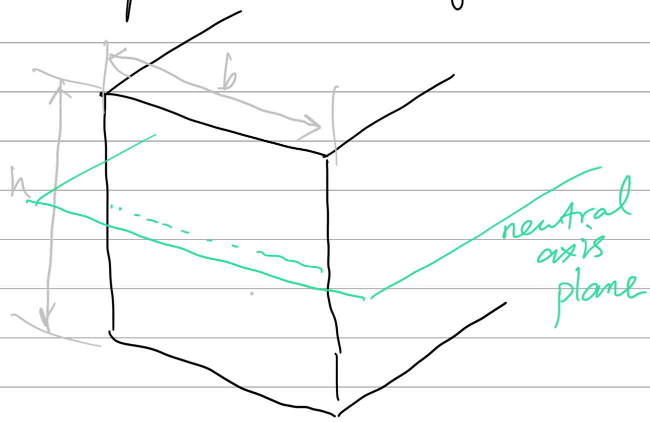
$$= +\kappa E \int y^2\, dA$$

$$= \kappa E I$$

$I = \int y^2\, dA$: moment of inertia

$$\kappa = \frac{M}{EI} \quad \left(\text{c.f. } \theta = \frac{T}{GJ}\right)$$

Example. Rectangular beam



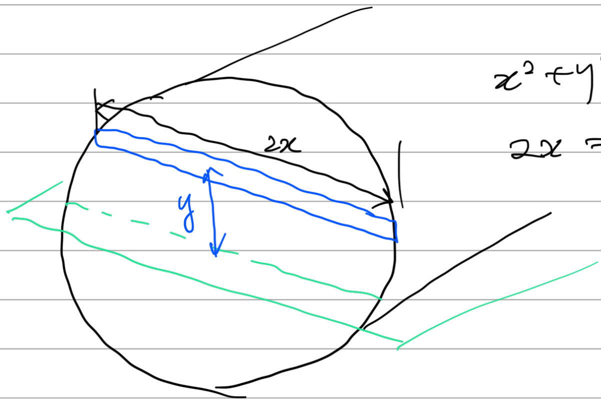
$$I = \int y^2 dA$$

$$= \int_{-h/2}^{h/2} y^2 b \, dy \quad (dA = b\,dy)$$

$$= b \cdot \left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = \frac{b}{3}\left[\frac{h^3}{8} - \left(-\frac{h^3}{8}\right)\right]$$

$$= \frac{bh^3}{12}$$

Circular beam



$$x^2 + y^2 = r^2$$

$$2x = 2\sqrt{r^2 - y^2}$$

$$I = \int y^2 dA$$

$$= \int y^2 \cdot 2x \, dy$$

$$= \int 2y^2 \cdot \sqrt{r^2 - y^2} \, dy$$

$$(\text{put } y = r\sin\theta)$$

$$= 2\int_0^{\pi/2} 2r^2 \sin^2\theta \cdot r\cos\theta \cdot r\cos\theta \, d\theta$$

$$(2\sin\theta\cos\theta = \sin 2\theta)$$

$$= r^4 \int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$(\cos^2\theta - \sin^2\theta = \cos 2\theta = 1 - 2\sin^2\theta)$$

$$= r^4 \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= r^4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

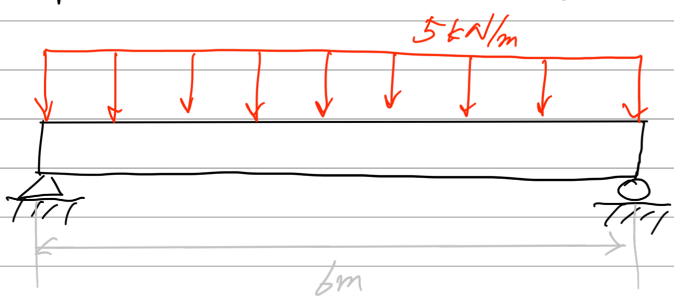
$$= \frac{\pi r^4}{4}$$

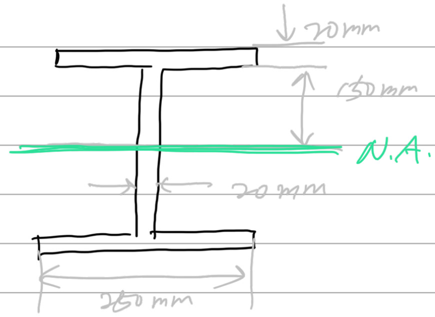
$$\int \cos 2\theta \, d\theta = \sin 2\theta / 2$$

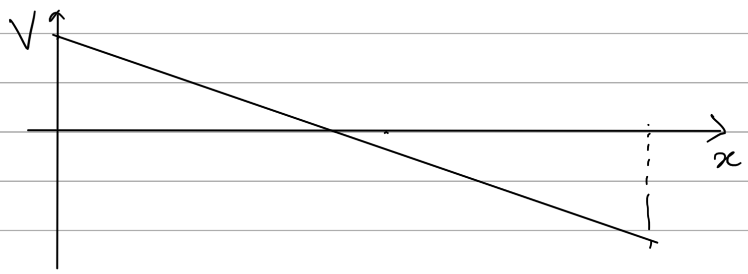
$$= \int \frac{\cos 2\theta \, d(2\theta)}{2}$$

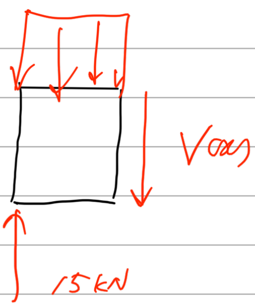
$$J = \int r^2\,dA = \int (x^2+y^2)\,dA$$
$$= \int x^2\,dA + \int y^2\,dA = 2I = \frac{\pi r^4}{2}$$
$$\therefore I = \frac{\pi r^4}{4}$$

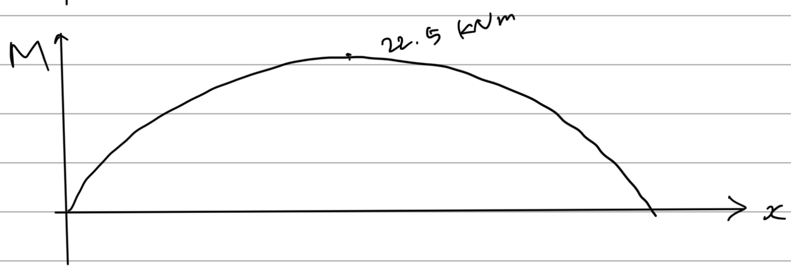
Example 6.12. Maximum bending stress in I-beam

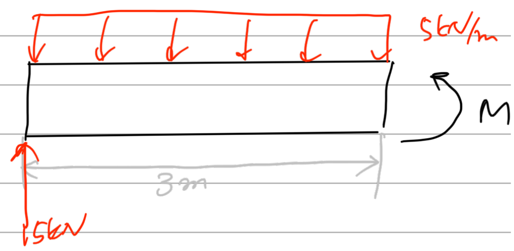






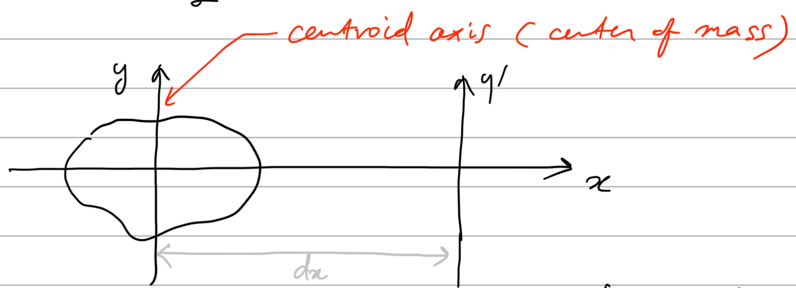






$$M + 5\times3\times1.5 - 15\times3 = 0$$
$$\therefore M = 45 - 22.5 = 22.5$$

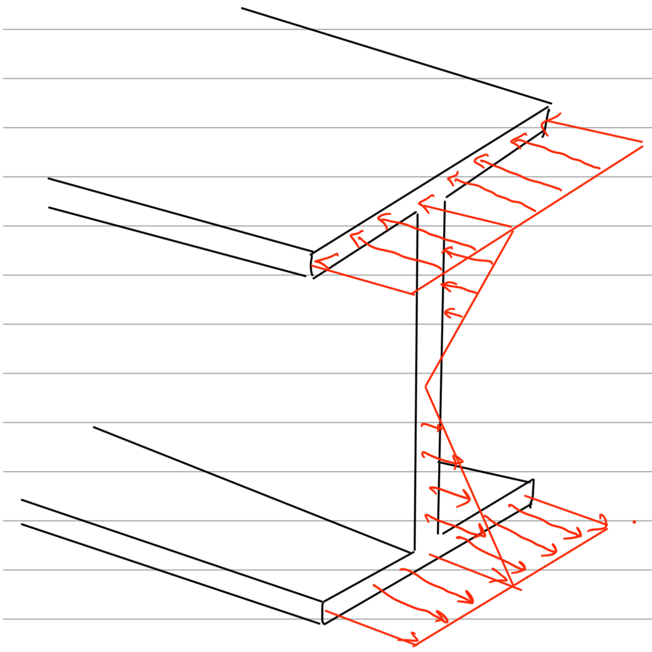
$$\sigma = \frac{M}{I}y$$



$$I_y = \int x^2\,dA \qquad I_{y'} = \int (x-d)^2\,dA$$
$$= \int x^2\,dA - \cancel{2d\int x\,dA} + \int d^2\,dA$$
$$= I_y + A d_x^2$$

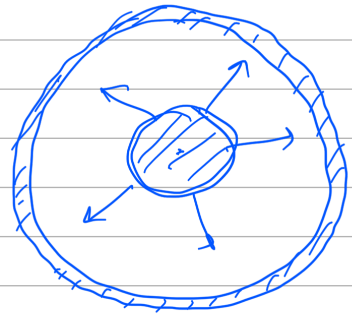
$$I = 2\left[\frac{1}{12}\cdot 0.25\cdot 0.02^3 + 0.02\times0.25\times0.160^2\right] + \frac{1}{12}\cdot 0.02\cdot 0.30^3 = 301.3\times10^{-6}\ m^4$$

$$\sigma_{max} = \frac{22.5 \times 10^3 \ Nm}{301.3 \times 10^{-6} \ m^4} \cdot \times 0.170 \ m = 12.7 \ MPa$$



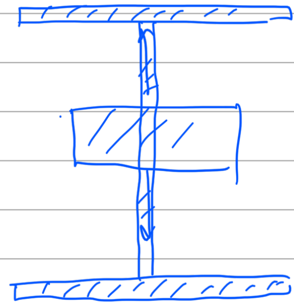
$J \uparrow$

$(\int r^2 dA)$

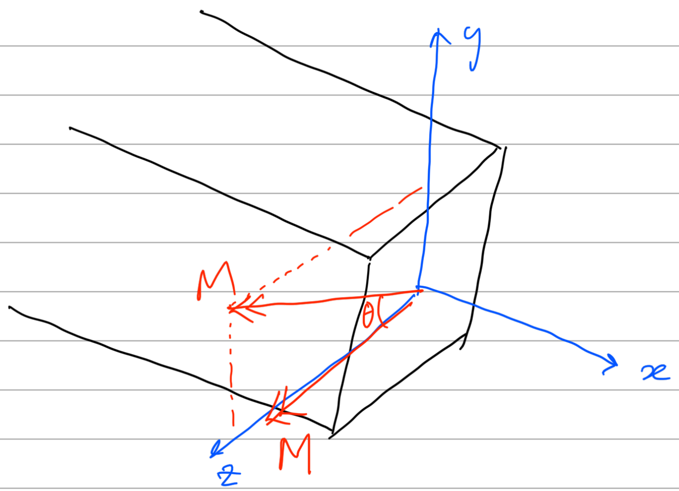


$I \uparrow$

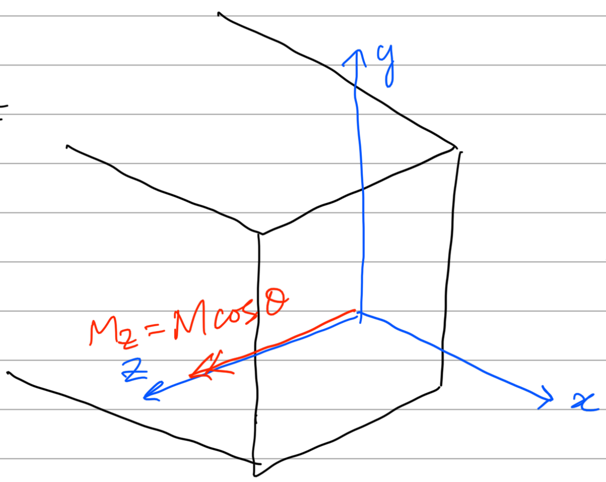
$(\int y^2 dA)$



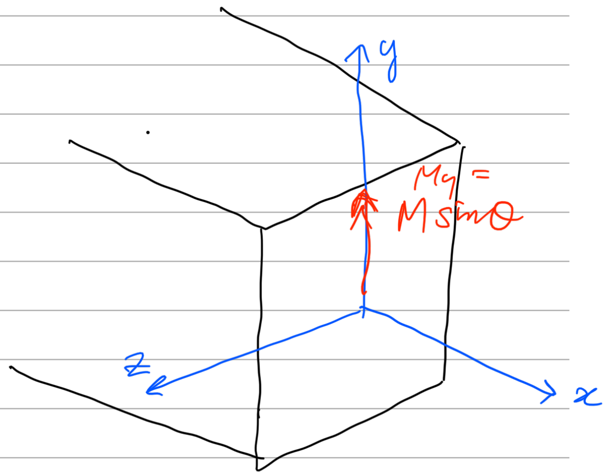
6.5. Unsymmetric bending



$=$



$+$



$$\sigma = -\frac{M_z}{I_z} y \qquad\qquad \sigma = \frac{M_y}{I_y} z$$

$$\therefore \sigma = -\frac{M_z}{I_z} y + \frac{M_y}{I_y} z$$

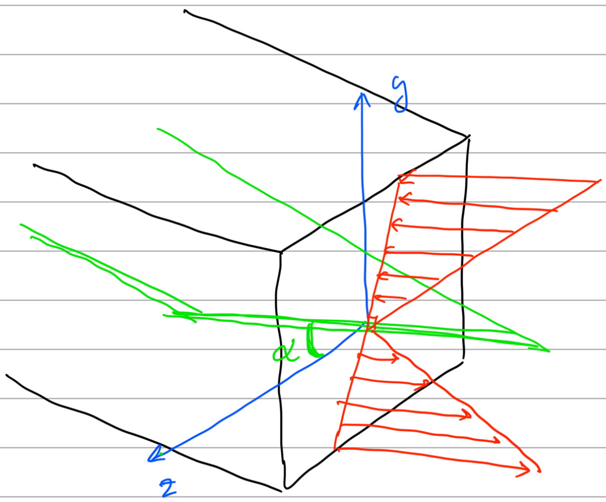
Neutral axis plane

$$\sigma = 0 \rightarrow y = \frac{M_y}{M_z} \frac{I_z}{I_y} z$$

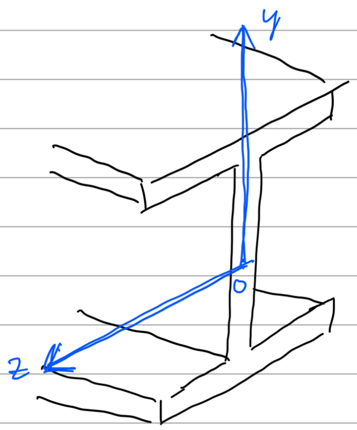
$$= \frac{M \sin\theta}{M \cos\theta} \cdot \frac{I_z}{I_y} z$$

$$= \frac{I_z}{I_y} \tan\theta \cdot z$$

$$\frac{y}{z} = \frac{I_z}{I_y} \tan\theta = \tan\alpha$$



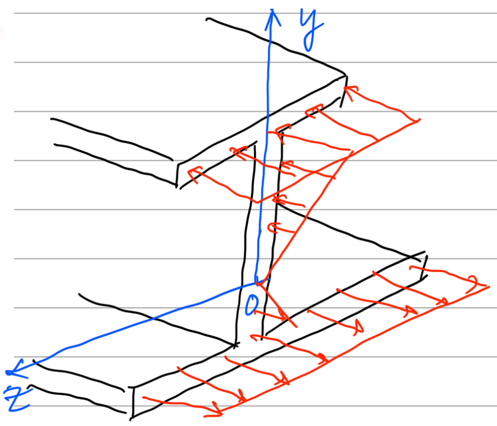
Q: How do we find y- and z-axis for arbitrary shaped cross-section?



principal axes

: axes of symmetry when exist

$\Rightarrow$ Principal axes will always be oriented along the axis of symmetry



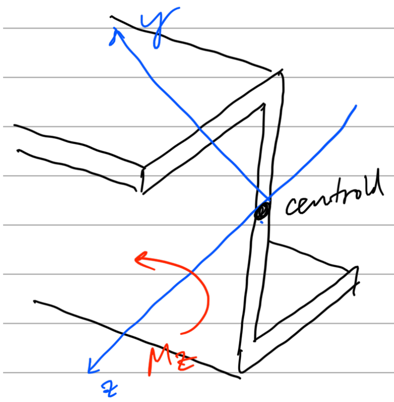
Net force at the cross section is zero

$$0 = \int \sigma \, dA$$

$$= \int \frac{M}{I} y \, dA$$

$$\Rightarrow \int y \, dA = 0$$

→ Passes through the centroid (center of mass for uniform density material)



$$0 = \int \sigma \, dA$$

$$M_y = -\int z \sigma \, dA = 0$$

$$M_z = \int y \sigma \, dA$$

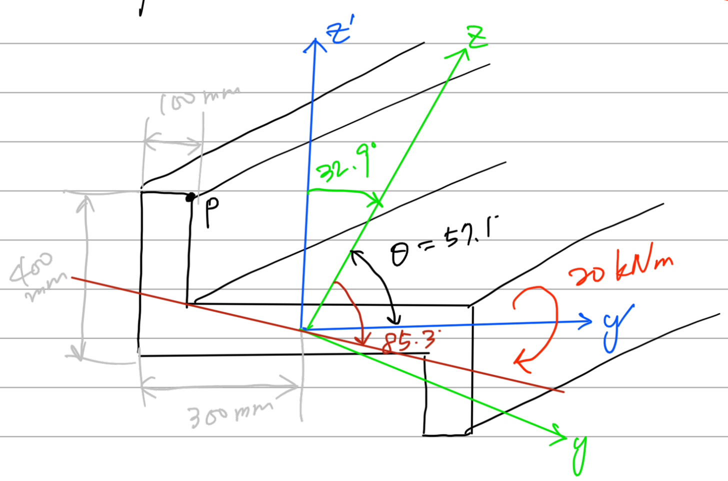
$$\sigma = \frac{M_z}{I_z} \cdot y$$

$$\therefore M_y = -\int \frac{M_z}{I_z} \cdot y z \, dA = 0$$

$$I_{zy} = \int y z \, dA = 0$$

Example 6.16

(Reference A. 4)



$$I_y = 0.960 \times 10^{-3} \ m^4$$

$$I_z = 7.54 \times 10^{-3} \ m^4$$

$$M_y = 20 \cos 32.9° = 16.79 \ kNm$$

$$M_z = 20 \sin 32.9° = 10.86 \ kNm$$

Stress at point P

$$y_p = -0.3580 \ m$$

$$z_p = 0.1852 \ m$$

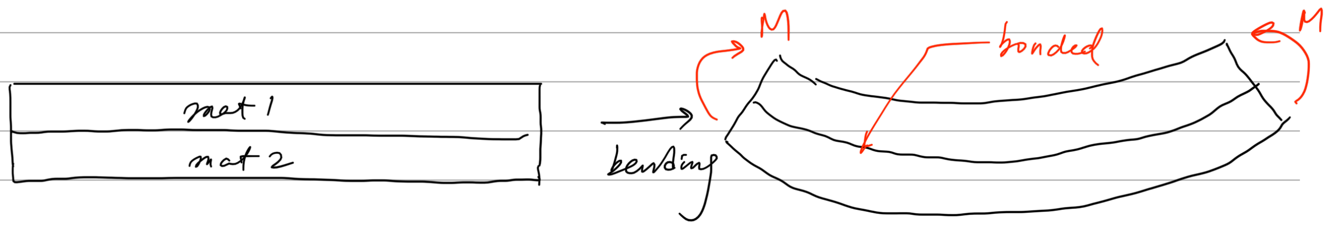
$$\sigma_p = -\frac{M_z}{I_z} y_p + \frac{M_y}{I_y} z_p$$

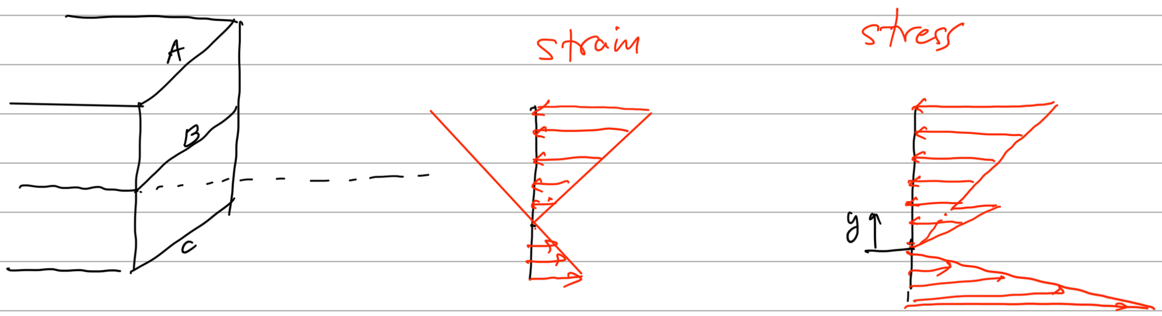
$$= 3.76 \text{ MPa}$$

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$

$$= \frac{7.54}{0.960}\tan 57.1^\circ \rightarrow \alpha = 85.3^\circ$$

6.6. Composite beams





$$\rightarrow \int \sigma \, dA = 0$$

$$\sigma_1 = -E_1 \kappa y$$

$$\sigma_2 = -E_2 \kappa y$$

$$M = -\int y\,\sigma(y)\,dA$$

$$= -\int_{mat1} y\,\sigma_1(y)\,dA - \int_{mat2} y\,\sigma_2(y)\,dA$$

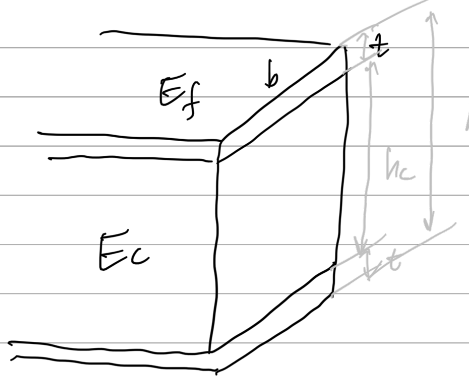
$$= \int E_1 \kappa y^2 dA + \int E_2 \kappa y^2 dA$$

$$= E_1 \kappa I_1 + E_2 \kappa I_2$$

$$\therefore \kappa = \frac{M}{E_1 I_1 + E_2 I_2}$$

with respect to common neutral axis plane

Sandwich structure



$$I_f = 2\int y^2 dA = 2\cdot\frac{h^2}{4}\cdot bt = \frac{bh^2t}{2}$$

$$I_c = \frac{1}{12}bh^3$$

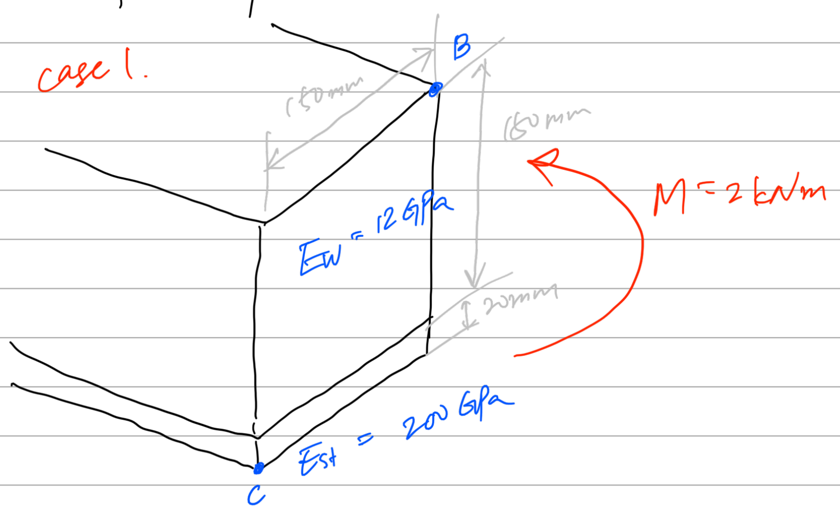
$E_c \ll E_f$, thus $E_cI_c \ll E_fI_f$

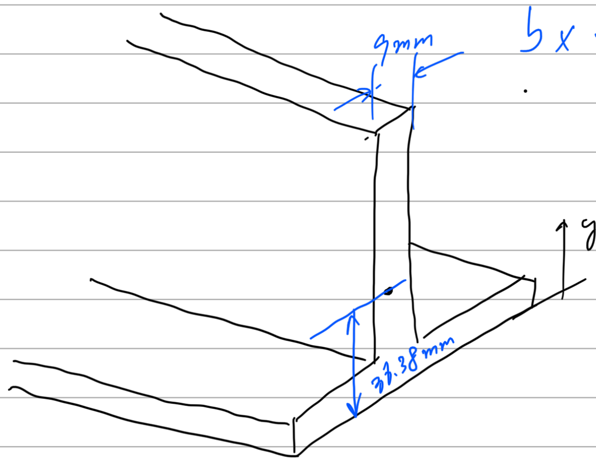
$$\therefore \kappa \doteq \frac{M}{\frac{1}{12}bh^3E_f}$$

Example. 6.17

case 1.

Normal stress at B & C





$b \times E_W = b' \times E_s$

$$\therefore b' = b\cdot\frac{E_W}{E_s} = \cancel{150}^{3} \times \frac{12}{\cancel{200}_{4}} = 9mm$$

$$\bar{y} = \frac{\int y\,dA}{A} = 36.38\text{ mm}$$

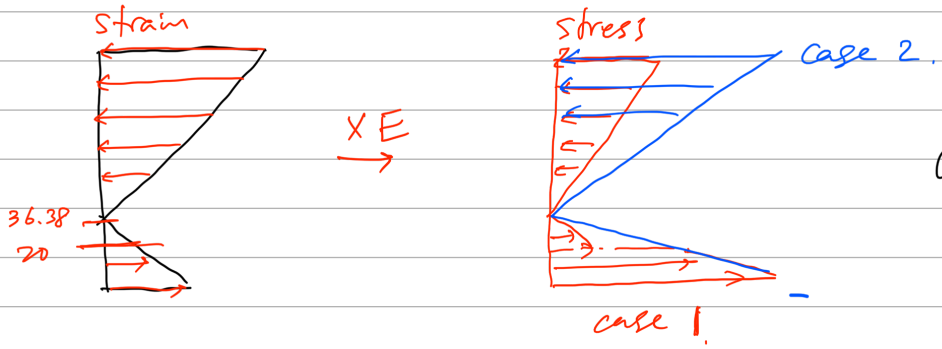
$$I = \int y^2 dA = 9.358\times10^{-6}\text{ m}^4$$

$$\sigma_c = -\frac{M}{I}y_c = -\frac{2\text{ kNm}}{9.358\times10^{-6}\text{ m}^4}\times(-36.38\text{ mm})$$

$$= 7.78\text{ MPa}$$

$$\sigma_A = -\frac{M}{I}y_A = -\frac{2\text{ kNm}}{9.358\times10^{-6}\text{ m}^4}\times(170-37.38)\text{ mm}$$

$$= 28.6\text{ MPa}$$



$$\sigma_A = \frac{28.6\ \text{MPa}}{(E_{st}/E_w)}$$

$$= 1.71\ \text{MPa}$$