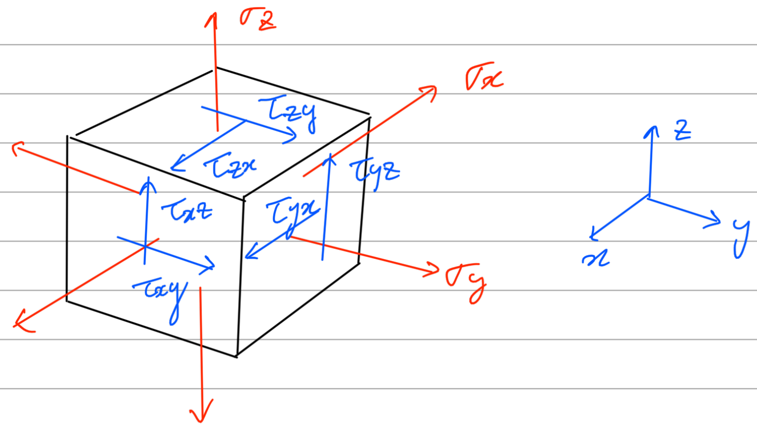
Chapter 9. Stress transformation

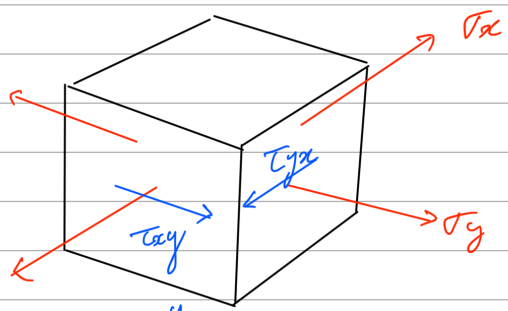
General state of stress

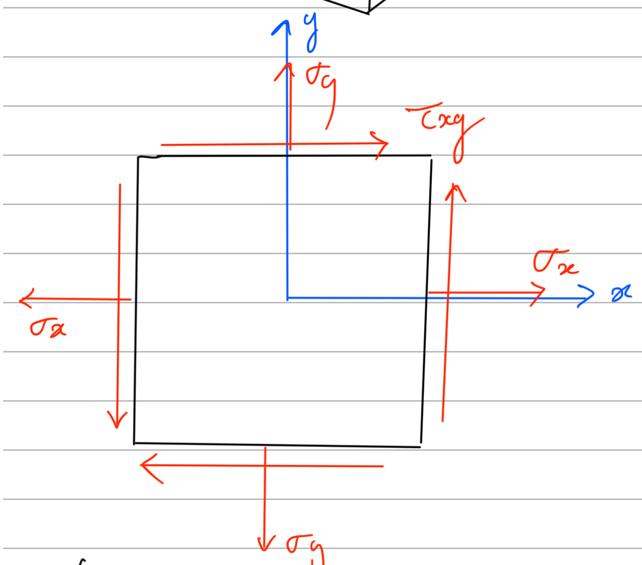


$\begin{cases} \sigma_x \ \sigma_y \ \sigma_z \\ \tau_{xy} \ \tau_{xz} \ \tau_{yz} \end{cases}$

Plane stress (x-y plane)

$\sigma_z = \tau_{xz} = \tau_{yz} = 0$

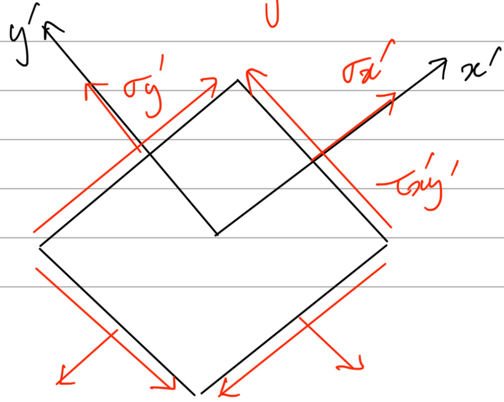




State of plane stress in x-y

$\sigma_x, \sigma_y, \tau_{xy}$

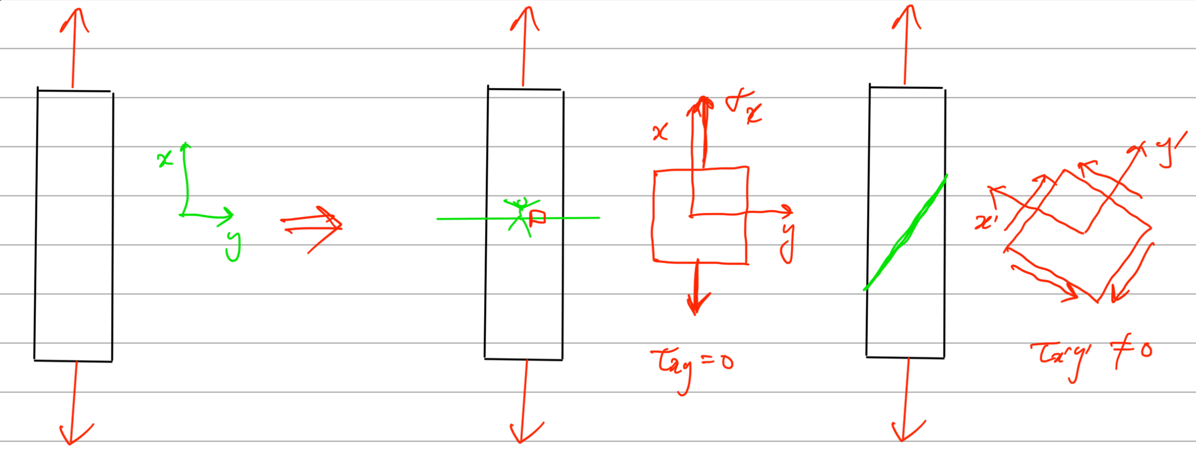
↓ Stress transformation



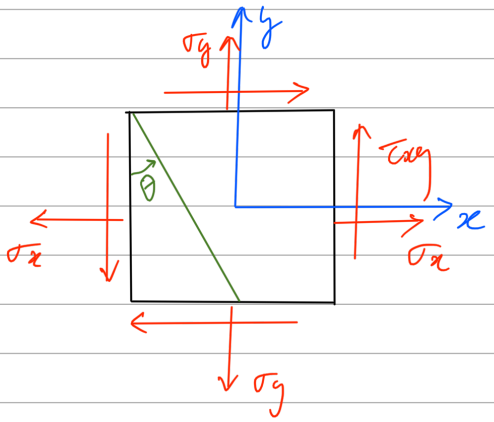
State of plane stress in x'-y'

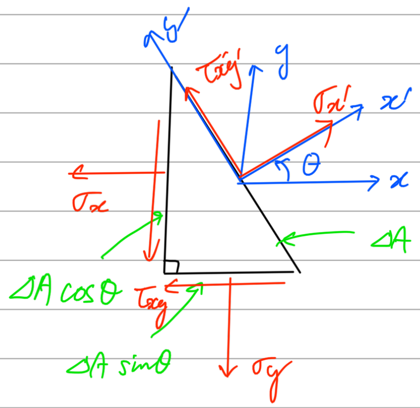
$\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}$

example.



9.2. General equations of plane stress transformation





Force equilibrium in x'-direction

$$\sigma_{x'} \cdot \Delta A - (\sigma_x \Delta A \cos\theta)\cdot\cos\theta - (\tau_{xy} \Delta A \cos\theta)\sin\theta - (\sigma_y \Delta A \sin\theta)\sin\theta - (\tau_{xy} \Delta A \sin\theta)\cos\theta = 0$$

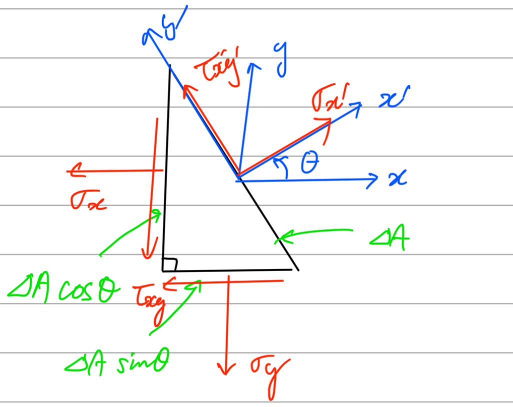
$$\sigma_{x'} = \sigma_x \cdot \cos^2\theta + \sigma_y \cdot \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\left(\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \right)$$

$$= \frac{\cos 2\theta + 1}{2}\sigma_x + \frac{1 - \cos 2\theta}{2}\sigma_y + \tau_{xy}\sin 2\theta$$

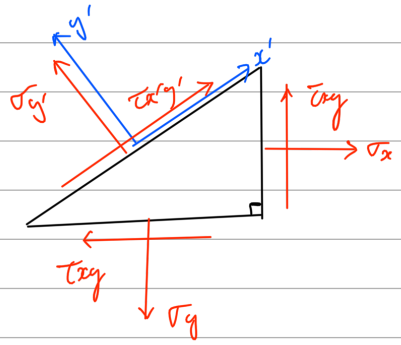
$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{—①}$$

Force equilibrium in y'-direction



$$\tau_{x'y'}\cdot\Delta A + (\sigma_x \Delta A\cos\theta)\sin\theta - (\tau_{xy}\Delta A\cos\theta)\cos\theta$$
$$- (\sigma_y \Delta A\sin\theta)\cos\theta + (\tau_{xy}\Delta A\sin\theta)\sin\theta = 0$$

$$\tau_{x'y'} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta$$
$$+ \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \text{—②}$$



$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

(or by plugging in $\theta + 90°$ into eq. ①)

9.4. Stress transformation trajectory

From eqns. ① and ②

given coefficients

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{—③}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \text{—④}$$

parameters that we want to know (output)

coordinate of our choice (input)

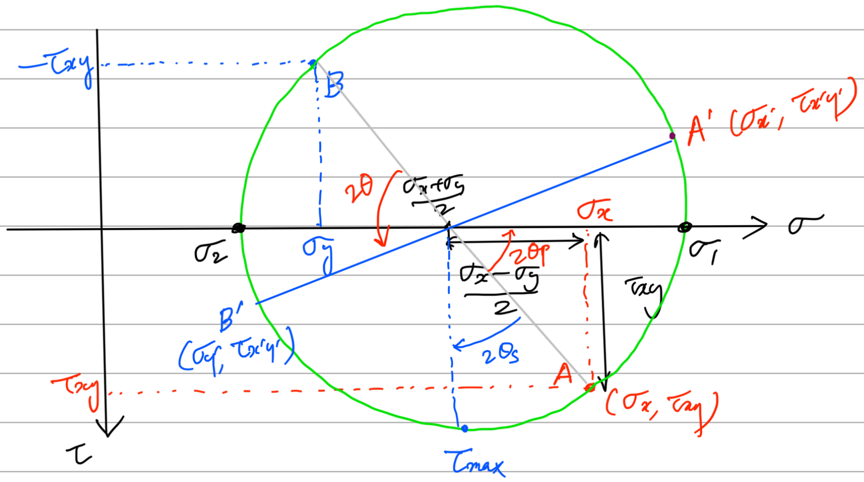
$③^2 + ④^2$

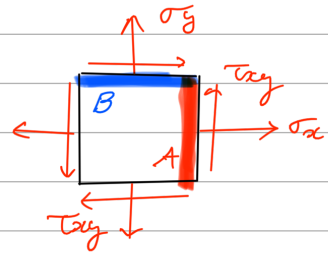
$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2$$

$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\cos^2 2\theta + \underline{(\sigma_x - \sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta} + \tau_{xy}^2\sin^2 2\theta$$

$$+ \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\sin^2 2\theta - \underline{(\sigma_x - \sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta} + \tau_{xy}^2\cos^2 2\theta$$

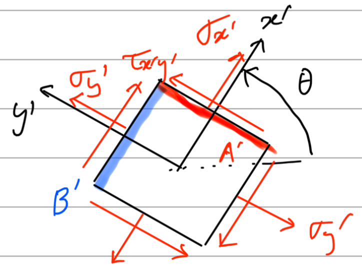
$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$





plane A : (σ_x, τ_{xy})

plane B : $(\sigma_y, -\tau_{xy})$



(insert a practice problem)

Maximum normal stress

σ_1, σ_2 : principal stresses

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y/2)}$$

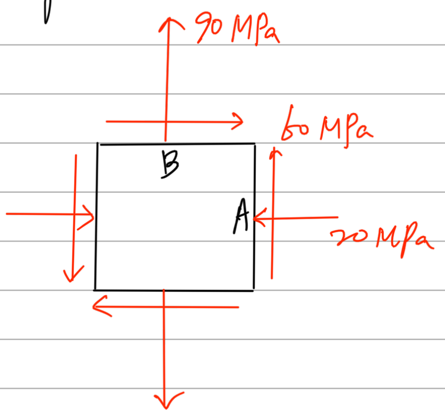
* No shear stress acts on the principal planes

Maximum shear stress

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

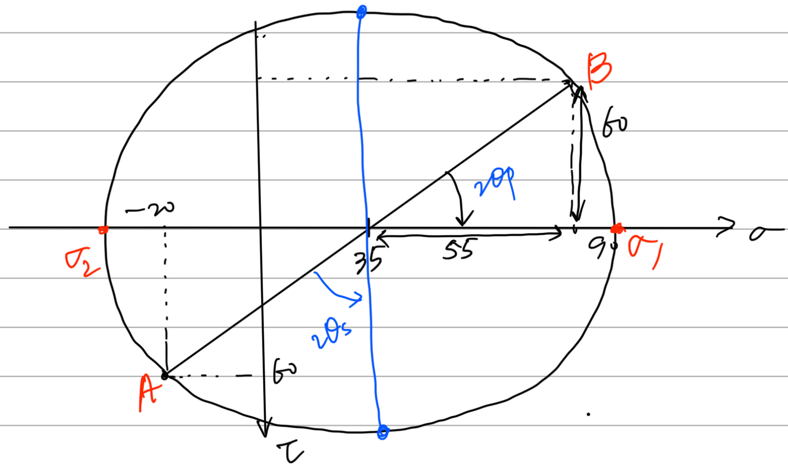
$$\tan 2\theta_s = -\frac{(\sigma_x-\sigma_y)/2}{\tau_{xy}}$$

Example 9.3.



principal stresses?

Maximum shear stress?



Plane A : $(-20, 60)$

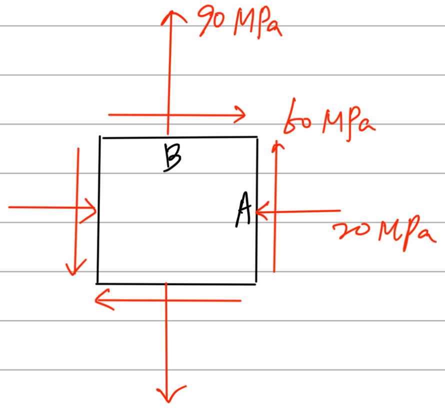
Plane B : $(90, -60)$

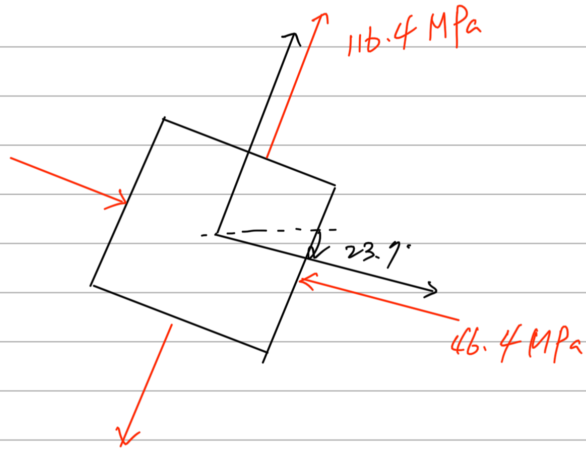
$$R = \sqrt{55^2 + 60^2} = 81.4$$

$$\sigma_1 = 35 + 81.4 = 116.4 \text{ MPa}$$

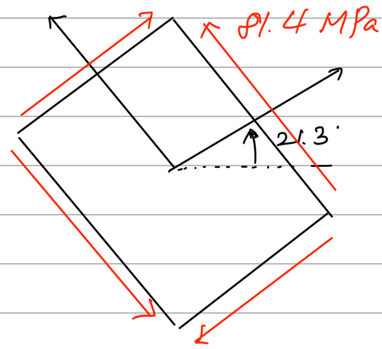
$$\sigma_2 = 35 - 81.4 = -46.4 \text{ MPa}$$

$$\tan 2\theta_p = -\frac{60}{55} \rightarrow \theta_p = -23.7^\circ$$

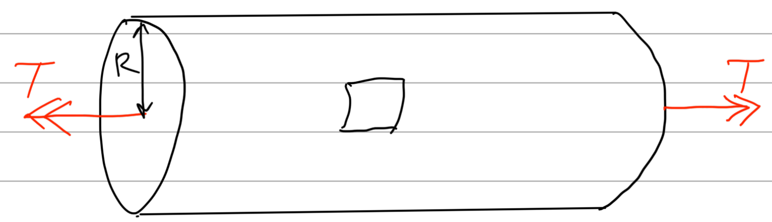




$$\tan 2\theta_s = \frac{55}{60} \rightarrow \theta_s = 21.3^\circ$$

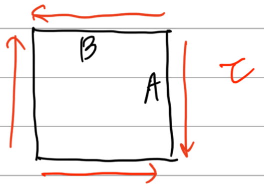


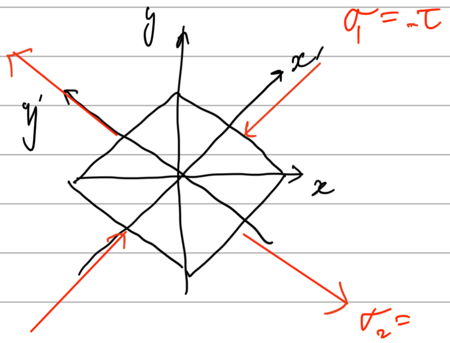
Example 9.5.

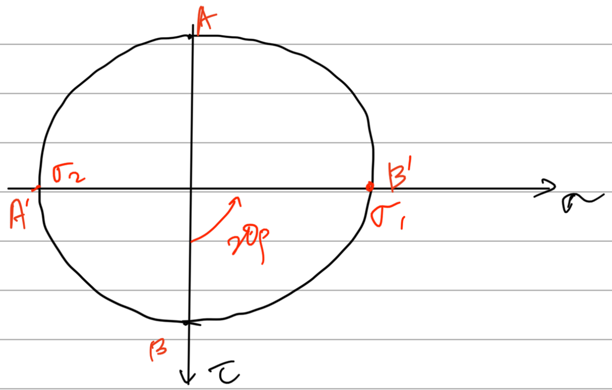


Failure mode for brittle material (e.g. concrete, chalk, cast iron) and ductile material (e.g. mild steel, aluminum)

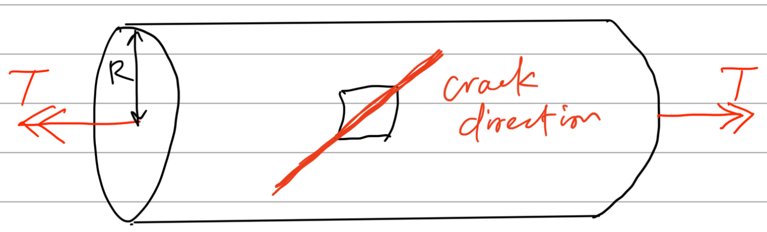
$$\tau = \frac{T}{J} \cdot R$$



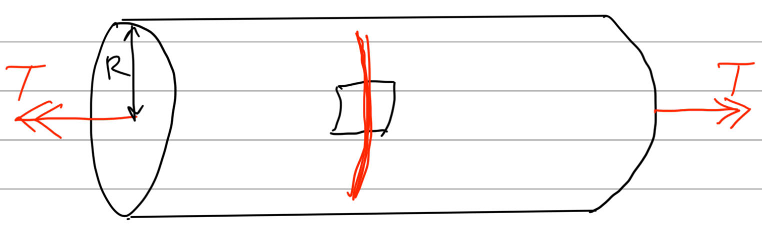




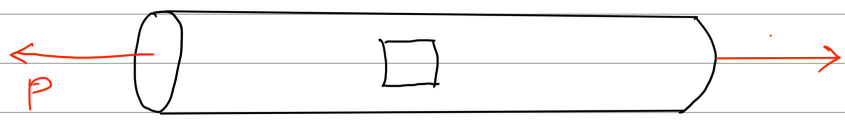
Brittle material fails under tension

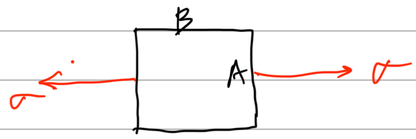


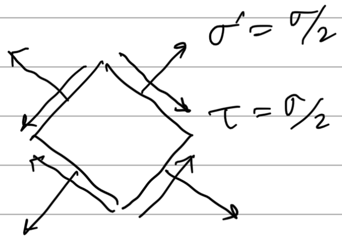
Ductile material fails under shear

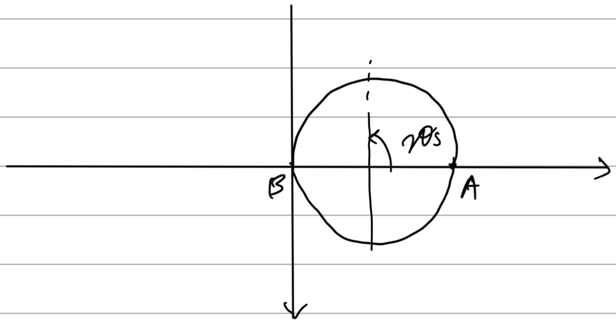


Example 9.6.

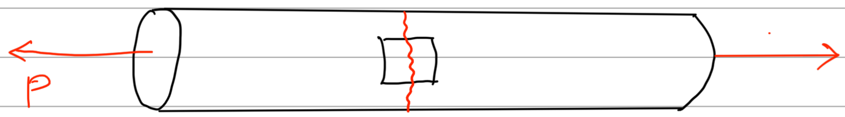


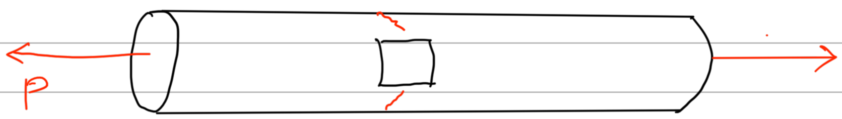




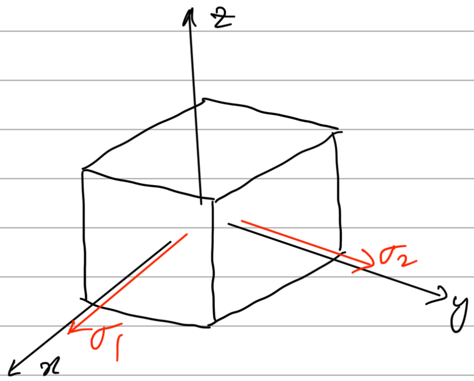


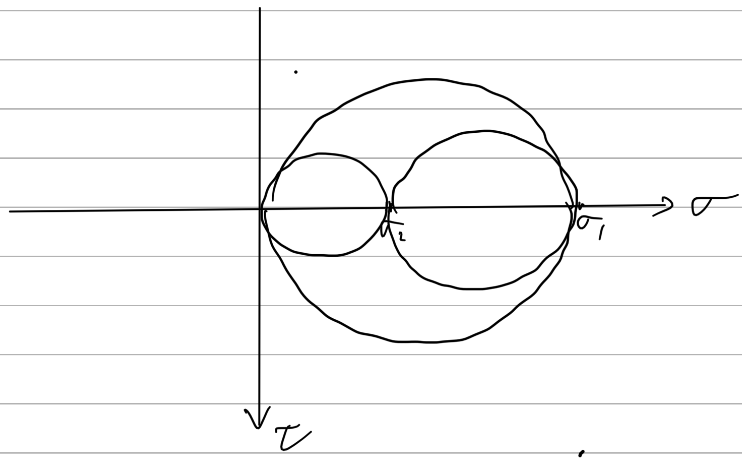
Cast iron



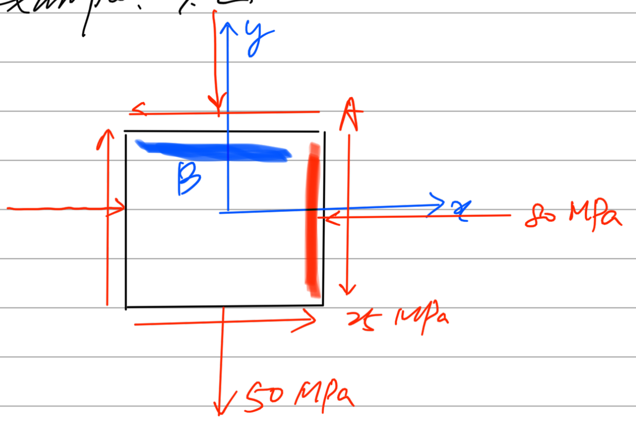


Mohr's circle in 3D





Example. 9.2.

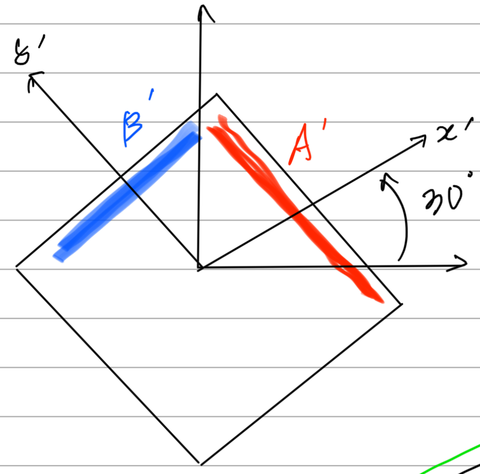


Stress state on another element oriented 30° clockwise

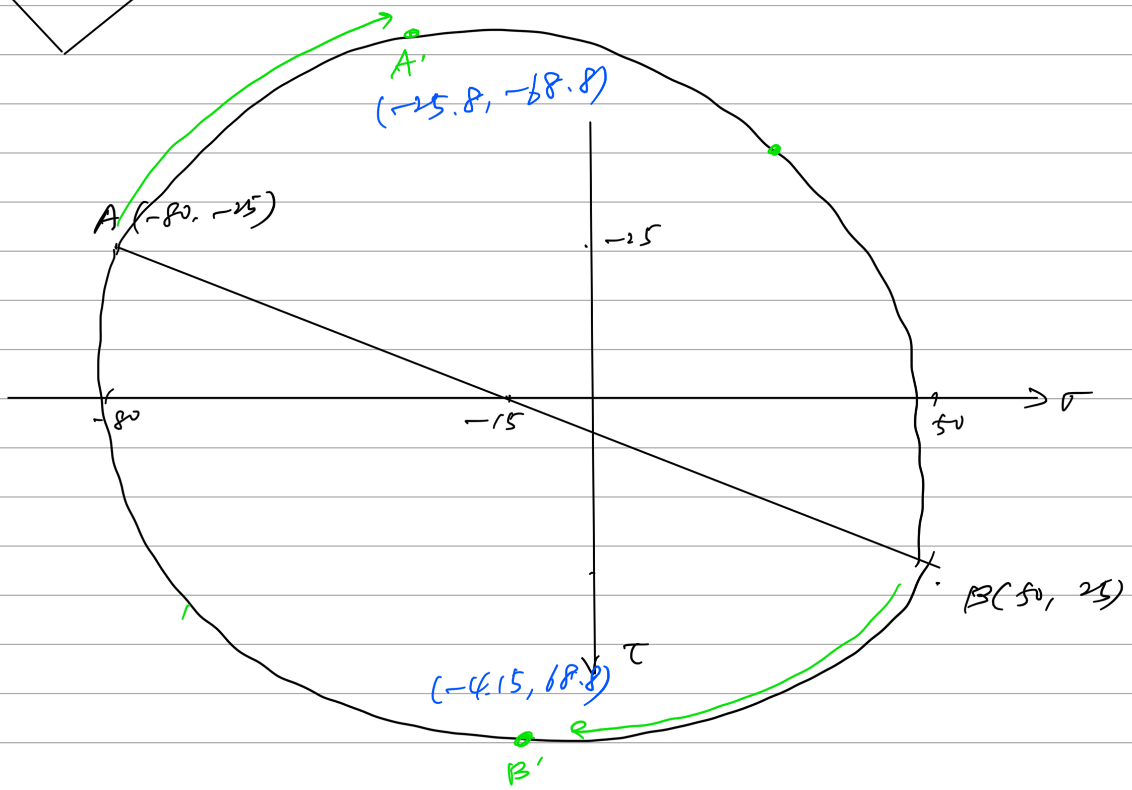
(σ_x, τ_{xy})

A : $= (-80, -25)$

B : $(\sigma_y, -\tau_{xy})$
$= (50, 25)$



← wrong. (−30°)



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$