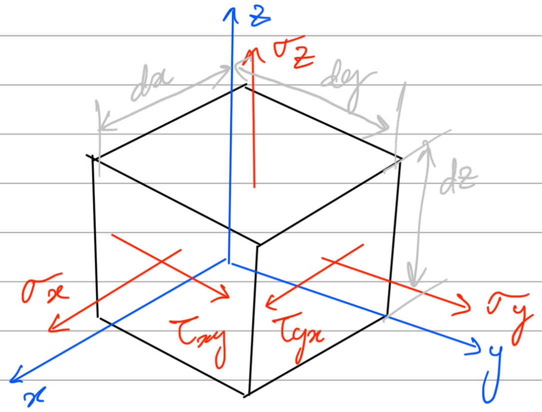
10.6. Material property relationships
(stress–strain relationship
or constitutive equation)



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}. \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}, \qquad \gamma_{xz} = \frac{\tau_{xz}}{E}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/G & & \\ & & & & 1/G & \\ & & & & & 1/G \end{bmatrix}}_{\text{compliance matrix}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

For homogeneous, isotropic material,

$$G = \frac{E}{2(1+\nu)} \rightarrow \text{prove in HW}$$

Example 10.9. $E = 200\ GPa$, $\nu = 0.3$

strain gage $\rightarrow$ strain state $\rightarrow$ principal strain $\rightarrow$ strain state?
($\varepsilon_a\ \varepsilon_b\ \varepsilon_c$) ($\varepsilon_x, \varepsilon_y, \gamma_{xy}$) ($\varepsilon_1, \varepsilon_2$)

$$\begin{aligned} \varepsilon_1 &= \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} \\ \varepsilon_2 &= \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} \end{aligned} \quad \rightarrow \quad \begin{aligned} \sigma_1 &= \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) \\ \sigma_2 &= \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) \end{aligned}$$

Using $\varepsilon_1 = 272 \times 10^{-6}$, $\varepsilon_2 = 33.9 \times 10^{-6}$
$\sigma_1 = 62.0\ MPa$
$\sigma_2 = 25.4\ MPa$

c.f. stiffness matrix

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & 0 & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & 1-2\nu & 0 & 0 \\ & 0 & & 0 & 1-2\nu & 0 \\ & & & 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

strain gage $\rightarrow$ strain state $\rightarrow$ stress state $\rightarrow$ principal stress
($\varepsilon_a\ \varepsilon_b\ \varepsilon_c$) ($\varepsilon_x, \varepsilon_y, \gamma_{xy}$) ($\sigma_x, \sigma_y, \tau_{xy}$)

$\varepsilon_x = 60 \times 10^{-6}$, $\varepsilon_y = 246 \times 10^{-6}$, $\gamma_{xy} = -149 \times 10^{-6}$

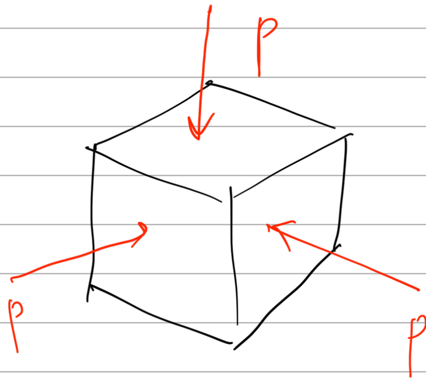
$$\left.\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}\sigma_y \\ \varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}\sigma_x \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \end{aligned}\right\} \rightarrow \quad \begin{aligned} \sigma_x &= 29.4\ MPa \\ \sigma_y &= 58.0\ MPa \\ \tau_{xy} &= -11.46\ MPa \end{aligned}$$

$\rightarrow$ Mohr's circle $\left(\begin{aligned} \sigma_1 &= 62\ MPa \\ \sigma_2 &= 25.4\ MPa \end{aligned}\right.$

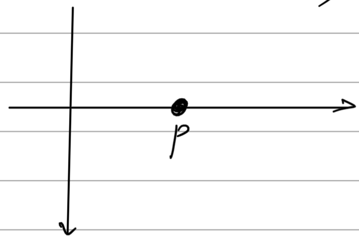
Dilatation

$$\begin{aligned} \delta V &= (1+\varepsilon_x)dx \cdot (1+\varepsilon_y)dy\ (1+\varepsilon_z)dz - dx\,dy\,dz \\ &\simeq (\varepsilon_x + \varepsilon_y + \varepsilon_z)\ \underbrace{dx\,dy\,dz}_{dV} \end{aligned}$$

dilatation
$$e = \frac{\delta V}{dV} = \left(\frac{1-2\nu}{E}\right) \cdot (\sigma_x + \sigma_y + \sigma_z)$$



Hydrostatic pressure
(no shear stress)



$$\sigma_x = \sigma_y = \sigma_z = -p$$

$$\frac{p}{e} = -\frac{E}{3(1-2\nu)} = -K \quad : \text{bulk modulus}$$

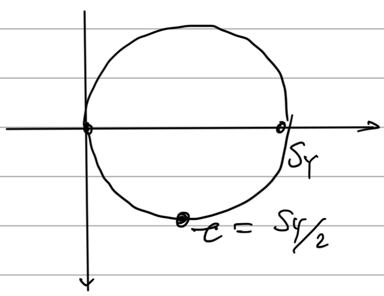
if $\nu \simeq 1/3$, $\quad K \simeq E$

$\quad \nu \simeq 0.5$, $\quad K \to \infty$ (upper ceiling of Poisson's ratio)

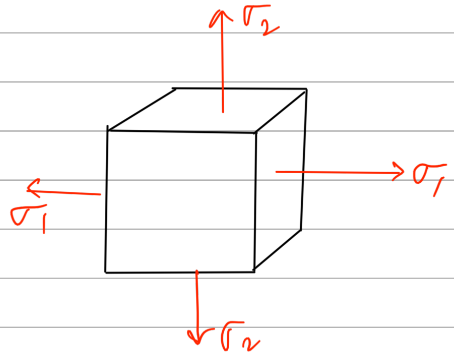
10.7. Theories of Failure

Maximum shear stress theory
(Tresca yield criterion)
: the maximum shear stresses in a multi-axial stress state reaches the value of the maximum shear stress in uni-axial tension
(For ductile materials)

- Uni-axial tension



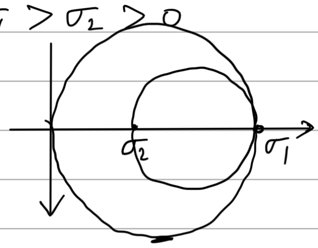
- Multi-axial stress state



(Note: principal stresses and axes)

(1) $\sigma_1 > 0$, $\sigma_2 > 0$ (both tension)

if $\sigma_1 > \sigma_2 > 0$



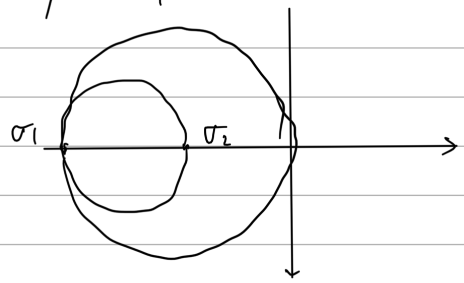
$$\frac{\sigma_1}{2} = \frac{\sigma_Y}{2}$$

if $\sigma_2 > \sigma_1 > 0$

$$\frac{\sigma_2}{2} = \frac{\sigma_Y}{2}$$

(2) $\sigma_1 < 0$, $\sigma_2 < 0$ (both compression)

if $\sigma_1 < \sigma_2 < 0$



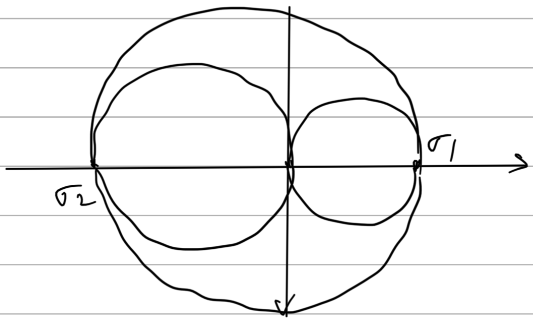
$$-\frac{\sigma_1}{2} = \frac{\sigma_Y}{2}$$

if $\sigma_2 < \sigma_1 < 0$

$$-\frac{\sigma_2}{2} = \frac{\sigma_Y}{2}$$

(3) σ_1 and σ_2 have different sign

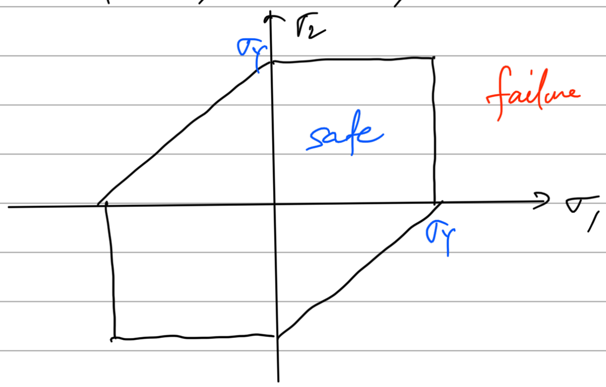
$\sigma_1 > 0$, $\sigma_2 < 0$



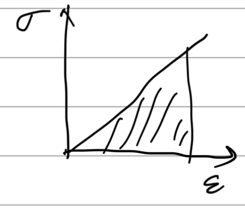
$$\frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_Y}{2}$$

$\sigma_1 < 0$, $\sigma_2 > 0$,

$$\frac{\sigma_2 - \sigma_1}{2} = \frac{\sigma_Y}{2}$$



Maximum distortion energy theory
(von Mises criterion)



- Strain energy density

$$u = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$
$$= \frac{1}{2}\sigma_1\left[\frac{\sigma_1}{E} - \frac{\nu}{E}(\sigma_2+\sigma_3)\right] + \frac{1}{2}\sigma_2\left[\frac{\sigma_2}{E} - \frac{\nu}{E}(\sigma_1+\sigma_3)\right]$$
$$+ \frac{1}{2}\sigma_3\left[\frac{\sigma_3}{E} - \frac{\nu}{E}(\sigma_1+\sigma_2)\right]$$
$$= \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)\right]$$

- Hydrostatic pressure ($\sigma_{avg} = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$) does not contribute to the failure, but stress deviator (i.e., $\sigma_1 - \sigma_{avg}$, $\sigma_2 - \sigma_{avg}$, $\sigma_3 - \sigma_{avg}$) does.

- Distortional energy u_d

$$u_d = \frac{1}{2E}\left[(\sigma_1-\sigma_{avg})^2 + (\sigma_2-\sigma_{avg})^2 + (\sigma_3-\sigma_{avg})^2\right.$$
$$\left. -2\nu\left\{(\sigma_1-\sigma_{avg})(\sigma_2-\sigma_{avg}) + \cdots\right\}\right]$$
$$= \frac{1+\nu}{6E}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]$$

From uni-axial tension test

$$(u_d)_Y = \frac{1+\nu}{6E}\cdot 2\sigma_Y^2$$

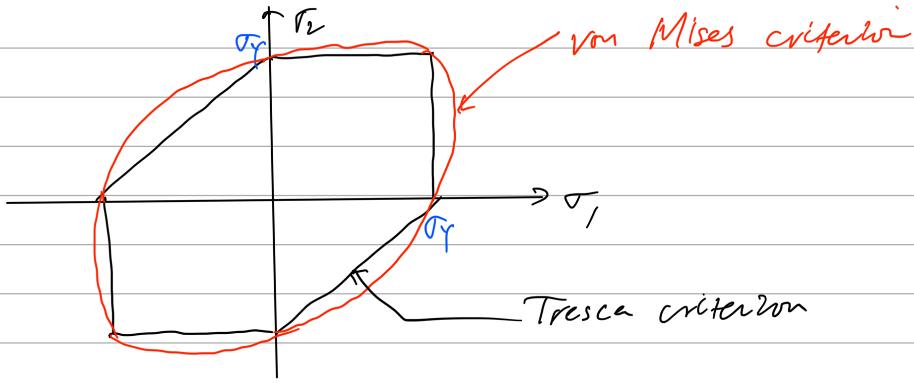
$$\therefore \quad \frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right] = \sigma_Y^2$$

$$\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]} = \sigma_{vM}$$

: von Mises stress

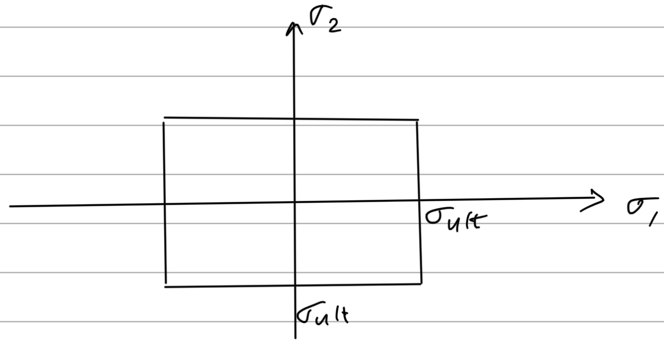
Under plane stress condition

$$\frac{1}{2}\left(\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_2^2 + \sigma_1^2\right) = \sigma_Y^2$$
$$\rightarrow \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$



Maximum normal stress theory
(for brittle materials)

$$|\sigma_1| = \sigma_{ult}$$
$$|\sigma_2| = \sigma_{ult}$$



Mohr - Coulomb failure criterion