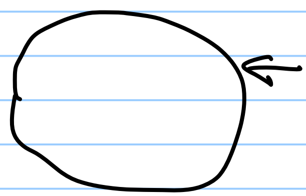
- Eulerian View



CV

$\bar{F}$: external force acting on the CV.

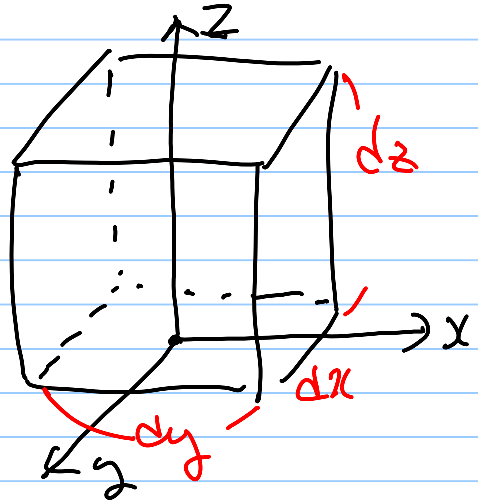
$$\frac{D}{Dt}\int_V \rho \bar{u}\, dV = \underbrace{\int_S \bar{P}\, dS}_{\text{surf.}} + \underbrace{\int_V \rho \bar{f}\, dV}_{\text{body}}$$

using RTT, Gauss Theorem

$$\Rightarrow \rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

(see Ch. 1.7, in Currie)

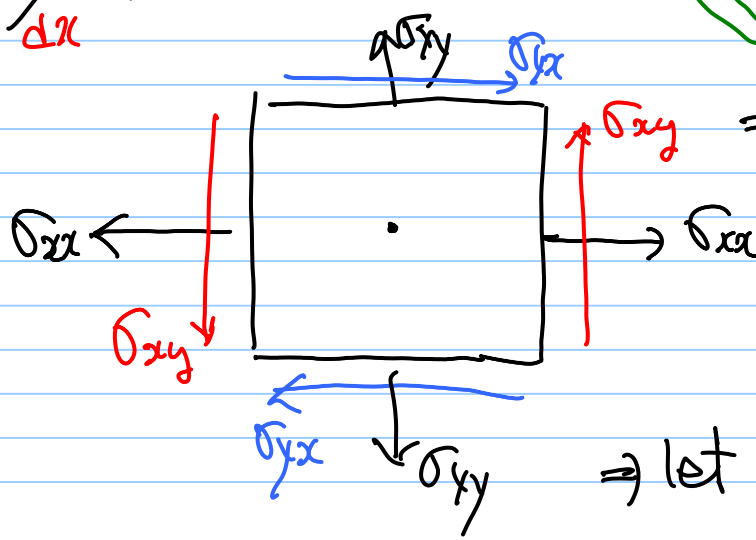
* Symmetry of $\bar{\bar{\sigma}}$ ($\sigma_{ij} = \sigma_{ji}$)



Torque, $T_z = I_z \cdot \alpha_z$

$$= \frac{\rho}{12}\left(dx^2 + dy^2\right)\left(dx\,dy\,dz\right)$$

or



$$\Rightarrow T_z = 2\cdot\frac{1}{2}dx\cdot\sigma_{xy}\,(dy\,dz)$$

$$-2\cdot\frac{1}{2}dy\cdot\sigma_{yx}\,(dx\cdot dz)$$

$$= (\sigma_{xy} - \sigma_{yx})\,dx\,dy\,dz$$

$\Rightarrow$ let $dx, dy, dz \to 0 \Rightarrow \boxed{\sigma_{xy} = \sigma_{yx}}$

$\bar{\bar{\sigma}}$: stress tensor

노트 제목 2019-09-10

· In a static ($\bar{u}=0$) fluid,

$$\bar{\bar{\sigma}} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}, \quad \sigma_{ij} = -p\,\delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

in a moving fluid, now we have a shear.

, we define mechanical pressure, $P_m = -\frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$

$$= -\frac{1}{3}\sigma_{ii}.$$

avg. of normal stress comp.

(measurable physically)

then, $\sigma_{ij} = -P_m\delta_{ij} + \sigma'_{ij}$

stress   mech. press.   shear stress (deviatoric)

(zero in static or inviscid fluid)

① $\sigma_{xx}' + \sigma_{yy}' + \sigma_{zz}' = 0$ from the definition of P_m.

② $\sigma_{ij}' = \sigma_{ji}'$ for symmetry

general conditions for all types of liq.

③ In a Newtonian fluid, stresses are proportional to strain rate

④ relation of ③ is isotropic. (no preferential direction)

e.g.) $\sigma_{xy}' = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$, ...

for Newtonian liquids.

· For details, Currie, Batchelor, White,

$$\rightarrow \sigma_{ij} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\bar{u}\right)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

· If we substitute this into the mtm conserv. eq.

, we get N–S eq.

· for incompressible flows : $\nabla\cdot\bar{u} = 0$.

$$\rho\frac{D\bar{u}}{Dt} = -\nabla P_m + \mu\nabla^2\bar{u} + \rho\bar{f}.$$

· $\mu = 0$. $\rho\frac{D\bar{u}}{Dt} = -\nabla P_m + \rho\bar{f}$ Euler eq.

⊙ Helmholtz vorticity eqn. ($\nabla \times$ (N–S eq.)).

$$\nabla\times\left[\rho\frac{D\bar{u}}{Dt} = -\nabla P_m + \mu\nabla^2\bar{u} + \rho\bar{f}\,\right]$$

$$\frac{D\bar{u}}{Dt} = \frac{\partial\bar{u}}{\partial t} + \underline{\bar{u}\cdot\nabla\bar{u}} = \frac{\partial\bar{u}}{\partial t} + \underline{\frac{1}{2}\nabla(\bar{u}\cdot\bar{u}) - \bar{u}\times\bar{\omega}}$$

vorticity $(=\nabla\times\bar{u})$

vector identity

$\therefore$ N-S eq: $\rho\left(\dfrac{\partial\bar{u}}{\partial t} - \bar{u}\times\bar{\omega}\right) = -\nabla P_m + \mu\nabla^2\bar{u} + \rho\bar{f} - \nabla\left(\frac{1}{2}\rho\bar{u}^2\right)$

$$\frac{\partial\bar{u}}{\partial t} - \bar{u}\times\bar{\omega} = -\nabla\left(\frac{P_t}{\rho}\right) + \nu\nabla^2\bar{u} + \bar{f}$$

we use, $\nabla\times(\bar{u}\times\bar{\omega}) = \bar{\omega}\cdot\nabla\bar{u} - \bar{u}\cdot\nabla\bar{\omega} + \bar{u}\underset{0}{(\cancel{\nabla\cdot\bar{\omega}})} - \bar{\omega}\underset{0}{(\cancel{\nabla\cdot\bar{u}})}$

$$\frac{D\bar{\omega}}{Dt} = \underline{\bar{\omega}\cdot\nabla\bar{u}} + \nabla\times\bar{f} + \underline{\nu\nabla^2\bar{\omega}}$$

: Vorticity Eq.

vorticity generation by tilting / stretching.

viscous diffusion.

non-conservative force (e.g. coriolis force)

if "f" is conservative, $(f = \nabla\phi)$

"work that is indep. of path"

③ Conservation of energy (1st law of thermodyna. $dE = \delta W + \delta Q$).

. Eulerian View w/ fixed CV.

$$\frac{\partial}{\partial t}\int_V \underbrace{\rho e_t}_{\text{total energy}} dV + \int_S \rho e_t (\bar{u}\cdot\bar{n})\, dS = \dot{Q}_{cs}(t) + \int_S \bar{\bar{\sigma}}\cdot(\bar{u}\cdot\bar{n})\, dS.$$

/

· total energy, $e_t = e + \frac{1}{2}u^2 + \Psi$ ← P.E.

internal e, K.E.

Eq. for e_t.

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}u^2 + \Psi\right) = \bar{u}\cdot(-\nabla p) + u_i \frac{\partial \sigma_{ji}}{\partial x_j} - p\,\nabla\cdot\bar{u} + \Phi - \nabla\cdot\bar{q}$$

work done by net pressure in translation.

work done by net viscous stress

$p\,dV$: work.

Φ : viscous dissipation, $\equiv \sigma_{ji}\frac{\partial u_i}{\partial x_j}$

$\bar{q} = -k\nabla T$, $\nabla\cdot\bar{q} = -\nabla\cdot(k\nabla T)$: heat exchange.

· Total 'e' eqn.

$\hookrightarrow e_t = e + \frac{1}{2}u^2 + \Psi.$

· KE eqn. ($u_i \times$ N-S eqn)

$$\underbrace{\rho u_i \frac{Du_i}{Dt}}_{=\rho\frac{D}{Dt}\left(\frac{1}{2}u_i^2\right)} = -u_i\frac{\partial p}{\partial x_i} + u_i\frac{\partial \sigma'_{ij}}{\partial x_j} + \rho u_i f_i.$$

· PE eqn.

· In case of conservative force, $\bar{f} = -\nabla\Psi(r)$.

$$\rho\frac{D\Psi}{Dt} = \rho\left(\cancel{\frac{\partial\Psi}{\partial t}}^{\,0} + \bar{u}\nabla\Psi\right) \Rightarrow \rho\frac{D\Psi}{Dt} = -\rho\bar{u}\cdot\bar{f}$$

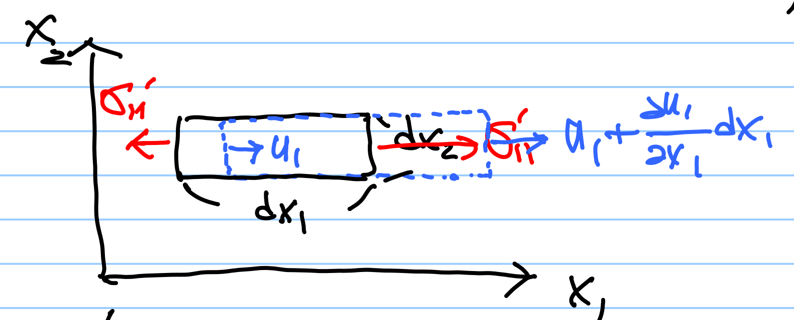
· Internal energy eqn : Total E − KE − PE.

$$: \rho\frac{De}{Dt} = -p\cdot\nabla\cdot\bar{u} + \Phi - \nabla\cdot\bar{q}.$$

⊙ viscous dissipation, Φ

$\Phi \equiv \sigma'_{ji}\frac{\partial u_i}{\partial x_j}$. ; work done by viscous forces to deform / distort the fluid (not translate).

$$= \sigma'_{11}\frac{\partial u_1}{\partial x_1} + \sigma'_{12}\frac{\partial u_2}{\partial x_1} + \sigma'_{13}\frac{\partial u_3}{\partial x_1}$$
$$+ \sigma'_{21}\frac{\partial u_1}{\partial x_2} + \sigma'_{22}\frac{\partial u_2}{\partial x_2} + \sigma'_{23}\frac{\partial u_3}{\partial x_2}$$
$$+ \sigma'_{31}\frac{\partial u_1}{\partial x_3} + \sigma'_{32}\frac{\partial u_2}{\partial x_3} + \sigma'_{33}\frac{\partial u_3}{\partial x_3}.$$



← incremental work due to distortion in x_1-direction, (elongation)
not due to the translation,

$$= \underbrace{\sigma_{11}' \cdot dx_2 \cdot dx_3}_{\text{force}} \cdot \underbrace{\left(\frac{\partial u_1}{\partial x_1} dx_1 \cdot dt\right)}_{\text{elongation}} = \sigma_{11}' \frac{\partial u_1}{\partial x_1} dV \cdot dt.$$

⊙ 2^{ND} Law of Thermo-dynamics. : $T \cdot dS \geq \delta Q$

change in sys. entropy.

temp. of heat source.

heat added to sys.

· for a fluid particle.

s : entropy/mass. $\rho T \cdot \frac{Ds}{Dt} \geq -\nabla \cdot \bar{q}$.

· Gibbs energy eqn for entropy production

$$dE = \delta Q_{rev} - P \cdot dV.$$

↳ reversible, for reversible system, $\delta Q_{rev} = T \cdot dS$

$$\rightarrow T \cdot dS = dE + P \cdot dV.$$

↳ for a fluid particle.

$$T \cdot d(\rho \cdot \delta V \cdot s) = d(\rho \cdot dV \cdot e) + P \cdot d\left(\frac{\rho\, dV}{\rho}\right).$$

$$\rightarrow T \cdot \frac{Ds}{Dt} = \frac{De}{Dt} + P \cdot \frac{D}{Dt}\left(\frac{1}{\rho}\right).$$

using mass conservation eq. $\left(\frac{D\rho}{Dt} + \rho(\nabla \cdot \bar{u}) = 0\right)$

$$= \frac{De}{Dt} + \frac{P}{\rho} \nabla \cdot \bar{u}$$

$$= \frac{1}{\rho}\left(\Phi - \nabla \cdot \bar{q}\right).$$

using eqn for e.

· for irreversible sys. (e.g. viscosity effects)

$$T \cdot \frac{Ds}{Dt} \geq -\nabla \cdot \bar{q} \quad (\because \Phi \geq 0)$$

→ irreversibility, sign of loss source of entropy.

· HW#1. Solve Probs. 1.1–1.9, 3.3 in Currie.
(Due 10/1) (301–210)

① Eqn for temperature, T.

(we need thermodynamic constitutive relations for)

$\rho(P,T)$
$e(P,T)$
$h(P,T)$
} in terms of β, K_T & C_P.

$\rho(P,T)$;

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p .$$: coeff. of volumetric thermal expansion.

$$\kappa_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T .$$: isothermal conductivity

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p$$: specific heat at const pressure.

$$d\rho = \rho \kappa_T \, dp - \rho \cdot \beta \, dT$$

$$de = \left(\frac{\kappa_T p - \beta T}{\rho}\right) dp + \left(c_p - \frac{\beta p}{\rho}\right) dT .$$

$$dh = \left(\frac{1-\beta T}{\rho}\right) dp + c_p \cdot dT$$

$$ds = -\frac{\beta}{\rho}\, dp + \frac{c_p}{T}\, dT .$$

· Back to 'e' eqn.

$$\rho \frac{De}{Dt} = -p(\nabla\cdot\bar{u}) + \Phi - \nabla\cdot\bar{q}.$$

· from continuity (mass conserv.)

$$\rho\cdot\nabla\cdot\bar{u} = -\frac{D\rho}{Dt}, \quad \bar{q} = -k\nabla T.$$

$$\rho C_p \frac{DT}{Dt} = \underbrace{\beta T \frac{DP}{Dt}}_{\text{isentropic}} + \underbrace{\Phi + \nabla\cdot(k\nabla T)}_{\text{non-isentropic}} \quad — \ *$$

* Isentropic & ideal gas case.

$$\rho = \frac{P}{RT}, \quad C_p = \frac{\gamma R}{\gamma-1}, \rightarrow \frac{\gamma}{\gamma-1}\cdot\frac{dT}{T} = \frac{dP}{P}. \quad \text{or} \quad \frac{T^{\gamma/(\gamma-1)}}{P} = \text{const}.$$

* Isentropic & liquid case.

$\Delta\rho/\rho \ll 1$, $C_p \simeq$ const, $\beta \simeq$ const.

$\therefore \rho C_p dT \simeq \beta T \cdot dp$.

$$\rightarrow T = T_0 \cdot \exp\left(\frac{\beta(p-p_0)}{\rho C_p}\right).$$

· for water, $\frac{\beta}{\rho C_p} \simeq 4\times 10^{-11}$. $\rightarrow p-p_0 \simeq O(10^{11})$ for appreciable change in T.

in (*),

· heat transfer effect.

T changes $\rho C_p \frac{DT}{Dt} = \nabla\cdot(k\nabla T)$

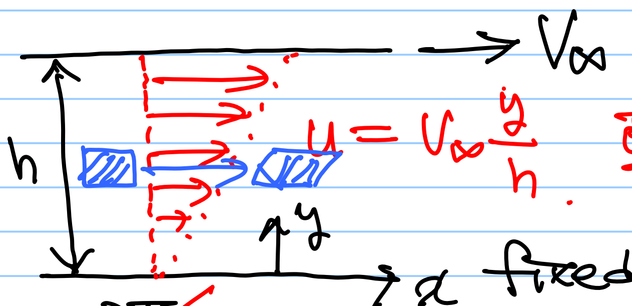
in time!

$\hookrightarrow T = T(\bar{x})$

· viscous dissipation effect.

$$\rho C_p \frac{DT}{Dt} = \Phi$$

ex) Couette flow.

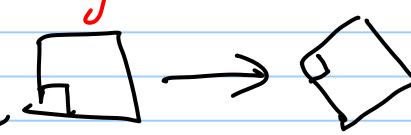


$$u = V_\infty \frac{y}{h}. \qquad \Phi = \mu \left(\frac{\partial u}{\partial y}\right)^2$$

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \cancel{\frac{\partial T}{\partial x}} + v \cancel{\frac{\partial T}{\partial y}}\right) = \Phi = \mu \left(\frac{\partial u}{\partial y}\right)^2$$

$$\rho C_p \frac{\partial T}{\partial t} = \mu \left(\frac{\partial u}{\partial y}\right)^2 \rightarrow T \uparrow \text{ as time goes on.}$$

if just rotational motion? (no deformation),



$$\Rightarrow \Phi = 0.$$