⊙ Scales & dimensions (units).

Dimensions	Units (SI)
L	m
M	kg
T	sec, s
Θ	kelvin, K

· Natural units (characteristic units).

e.g.)

$L \to R$.

$V \to U_\infty$,

$M \to \rho R^3$.

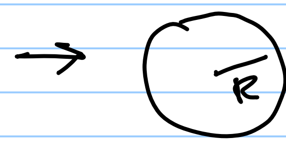
· Dimensional analysis.

① Buckingham Π theorem.

"In a physical situation with "s" scales (parameters) in "u" units, there are "$p = s - u$" dimensionless groups"

· inviscid, incompressible flow

U_∞, ρ

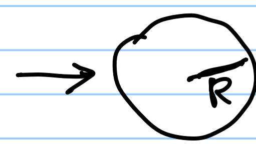


$s = 3$ (U_∞, ρ, R) $\quad p = 0.$

$u = 3$ (L, M, T).

· viscous, incompressible flow

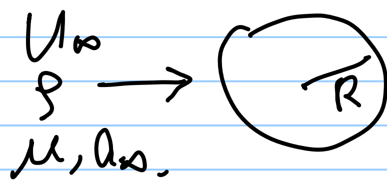
U_∞, ρ, μ.



$s = 4.$ (μ). $\quad p = 1.$ $\underline{\underline{Re}}$.

$u = 3$ (L, M, T).

· 비정상, compressible flow



$r = 5$, $p = 2$, Re, Ma.

$u = 3$.

② Non-dimensionalization of basic equations

· Length : $x \rightarrow x^* \equiv x/R$. ($R$: natural unit.)

· Time, : $t \rightarrow t^* \equiv t/(R/U_\infty)$. ($R/U_\infty$: residence time)

↑ flow characteristic time scale.

· pressure : $P \rightarrow P^* = (P - P_\infty) / (\frac{1}{2}\rho U_\infty^2)$.

· temperature : $T \rightarrow T^* = (T - T_\infty) / (T_w - T_\infty)$.

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad \left(\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \bar{u}^*) = 0 \right)$$

$$\rho^* \frac{D\bar{u}^*}{Dt^*} = -\nabla^* p^* + \frac{1}{Re} \nabla^* \left[\mu^* \left(\frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right) \right], \quad Re \equiv \frac{\rho U_\infty R}{\mu_0}.$$

$$\rho^* c_p^* \frac{DT^*}{Dt^*} = Ec \cdot \frac{Dp^*}{Dt^*} + \frac{1}{Re \cdot Pr} \nabla^* \cdot (k^* \nabla^* T^*) + \frac{Ec}{Re} \Phi^*.$$

$$Pr \equiv \frac{\mu_0 c_{p_0}}{k_0} \sim \frac{\text{viscous diffusion rate}}{\text{thermal} \quad \text{"}} . \qquad \left(\begin{array}{ll} 0.7 & \text{air} \\ 7 & \text{water} \end{array} \right)$$

$$Ec \equiv \frac{U_\infty^2}{c_p (T_w - T_0)} \sim \frac{\text{kinetic energy}}{\text{enthalpy difference}} \leftarrow \text{dissipation by heat tr.}$$

* Concept of "order of magnitude".

if $f(p) = \underline{O}(g(p))$ as $p \to 0$ or $p \to \infty$, we say function f is the same order of magnitude as function g as $p \to 0$ or $p \to \infty$

- $\lim\limits_{\substack{p \to 0 \\ p \to \infty}} \dfrac{f(p)}{g(p)} = \text{constant}$. (don't need to be 1.)

find "dominant balance".

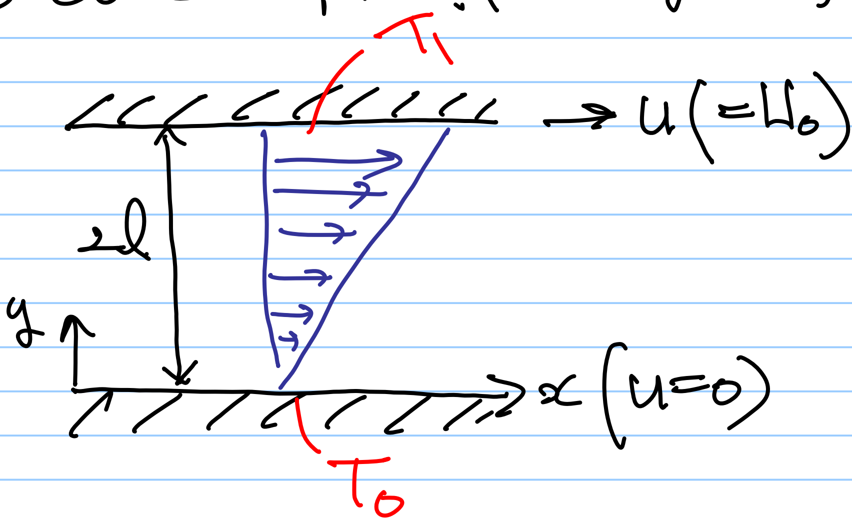
$$\underbrace{\frac{\partial u^*}{\partial t^*}}_{①} + \underbrace{u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*}}_{②} = \underbrace{-\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*}}_{③} + \underbrace{\frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right]}_{④};$$

- Stokes Flow : ③, ④.
- Potential Flow : ②, ③ + $\vec{\omega} = 0$. (irrotationality)
- Acoustic Prob. : ①, ③.
- Rayleigh Prob. (unsteady viscous) : ①, ④.

Exact Solutions in Viscous Flows (Ch.? in Currie)

① Couette Flow. (steady flow, between a fixed and a moving plate)

1D, $\frac{\partial p}{\partial x} = 0$.



. continuity : $\frac{\partial u}{\partial x} = 0$ (fully developed)

. momentum

$$\rho \frac{D\bar{u}}{Dt} = -\cancel{\nabla p} + \mu \nabla^2 \bar{u} + \cancel{\rho \bar{f}}$$

$$\cancel{\frac{\partial \bar{u}}{\partial t}} + \cancel{u\frac{\partial \bar{u}}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}$$

$$\rightarrow \mu \frac{d^2 u}{dy^2} = 0 \quad \text{(linear)}$$

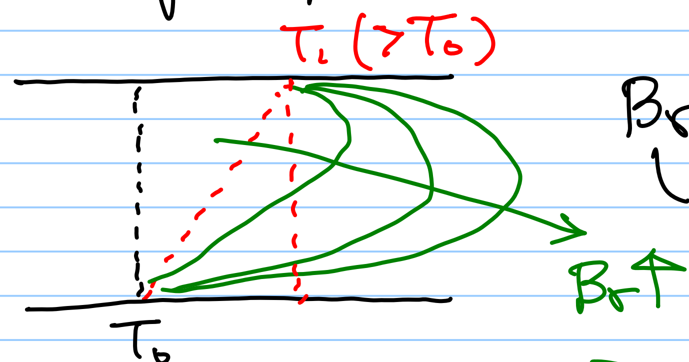
$$\rightarrow u(y) = c_1 y + c_2 .$$

· energy : $0 = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 \rightarrow T = T(y)$.

from $u=u(y)$, → shear stress, $\tau = \frac{\mu U_0}{2l}$ (const.)

friction coeff. $C_f \equiv \frac{\tau}{\frac{1}{2}\rho U_0^2} = \frac{1}{Re_l}$

$u(y) \rightarrow T(y)$: parabolic profile. $\left(Re_l \equiv \frac{\rho U_0 l}{\mu}\right)$



$Br \equiv \frac{\mu U_0^2}{k(T_1-T_0)} \sim \frac{\text{viscous dissipation}}{\text{conduction}}$.

Brinkman number

as more viscous dissipation → more convex temperature profile.

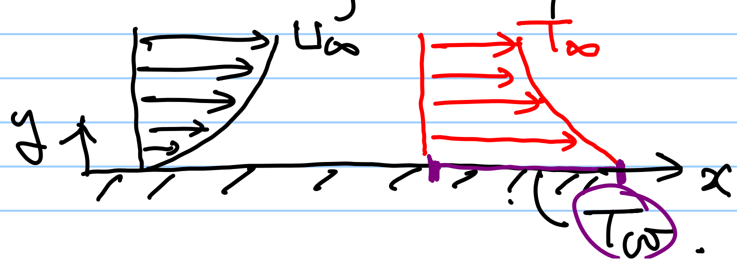
$u(y) \xrightarrow{\text{linear}} T(y) \rightarrow$ parabolic, Br.

$\rightarrow \bar{q} = -k\frac{\partial T}{\partial y}$ (analogous to τ, $\tau = \mu\frac{\partial u}{\partial y}$)

Nusselt number, $Nu \equiv \frac{h \cdot l}{k}$ (h : convective heat tr. coeff.)

* Reynolds analogy

: like the velocity, a thermal boundary layer develops when a fluid at the specific temperature flows over a surface w/ different temp.



$\Rightarrow$ mntm transfer, $C_f \left(\equiv \tau_w / \frac{1}{2}\rho u^2 \right)$

$\Leftrightarrow$ heat transfer, Nu.

$$\left\{ \begin{array}{l} u^* \dfrac{\partial u^*}{\partial x^*} + v^* \dfrac{\partial u^*}{\partial y^*} \sim \dfrac{\partial^2 u^*}{\partial y^{*2}} \\ u^* \dfrac{\partial T^*}{\partial x^*} + v^* \dfrac{\partial T^*}{\partial y^*} \sim \dfrac{\partial^2 T^*}{\partial y^{*2}} \end{array} \right.$$

$\therefore\ C_f \cdot \dfrac{Re_x}{2} = Nu_x$.

① No pressure gradient.

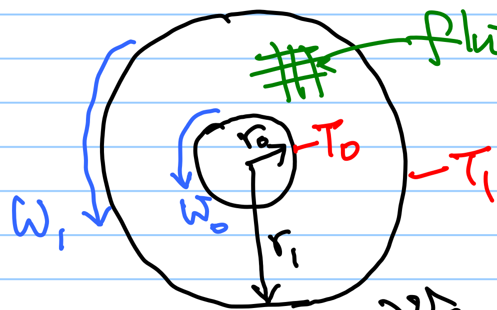
② $Pr \left(\equiv \nu / \alpha \right) \simeq 1.0$.

let $St \equiv Nu / (Re \cdot Pr)$ $\rightarrow$ $\frac{1}{2} C_f = St$.

· modified Reynolds Analogy

$$\frac{1}{2} C_f = St \cdot Pr^{2/3} \quad (0.6 < Pr < 60)$$

⓪ Flow between two rotating concentric cylinders.



· 1D, steady.

· BC's @ $r = r_0$, $v_\theta = r_0 \omega_0$, $T = T_0$,

@ $r = r_1$, $v_\theta = r_1 \omega_1$, $T = T_1$

· continuity : $\dfrac{\partial v_\theta}{\partial \theta} = 0$

· r-mtm : $\dfrac{\partial p}{\partial r} = \dfrac{\rho}{r} v_\theta^2$ → centrifugal effect is balanced by pressure gradient.

· θ-mtm : $\dfrac{d^2 v_\theta}{dr^2} + \dfrac{d}{dr}\left(\dfrac{v_\theta}{r}\right) = 0$

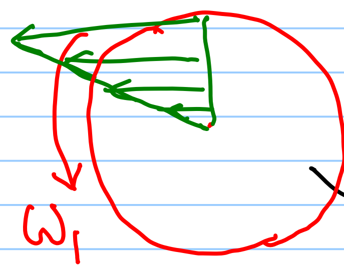
· energy : $0 = \dfrac{k}{r}\dfrac{d}{dr}\left(r \dfrac{dT}{dr}\right) + \mu\left(\dfrac{dv_\theta}{dr} - \dfrac{v_\theta}{r}\right)^2$.

$$\rightarrow v_\theta(r) = r_0\omega_0\left[\frac{r_1/r - r/r_1}{r_1/r_0 - r_0/r_1}\right] + r_1\omega_1\left[\frac{r/r_0 - r_0/r}{r_1/r_0 - r_0/r_1}\right] \rightarrow \text{"}c_1 r + \frac{c_2}{r}\text{" form.}$$

$\hookrightarrow$ energy eq. to get $T(r)$.

* special case

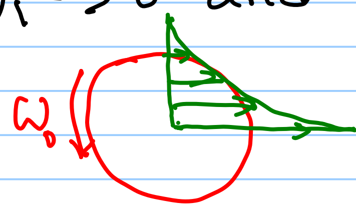
1. $r_0 \to 0$ and $\omega_0 \to 0$. (inner cylinder disappearing)



$v_\theta(r) = r\omega_1$. ; Solid-body rotation.

$\rightarrow$ "forced vortex".

2. $\omega_1 \to 0$ and $r_1 \to \infty$. (outer cylinder disappearing).

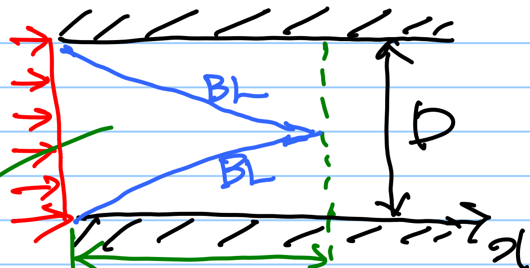


$v_\theta(r) \simeq \frac{r_0^2\omega_0}{r}$ "free vortex".

② Poiseuille flow.

· Couette flow : relative motion

→ pressure-driven flow, fully developed, $\left(\frac{\partial u}{\partial x}=0\right)$.



inviscid core

Le → fully developed $(x > L_e)$ → viscous everywhere.

entrance length, $L_e \sim O(10 \sim 10^2)D$.

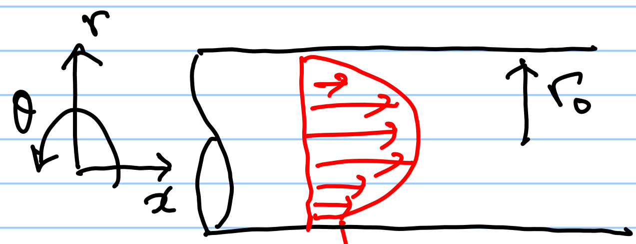
· 2D, steady, incompressible.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad 0=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

→ flow scales with $\boxed{\frac{1}{\mu}\frac{\partial p}{\partial x}}$

2-1) Hagen–Poiseuille flow.

· Circular pipe, fully-developed, axisymmetric $\left(\frac{\partial}{\partial\theta}=0\right)$.



parabolic

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{1}{\mu}\frac{dp}{dx}.$$

$$\hookrightarrow u(r)=\frac{1}{\mu}\frac{dp}{dx}\cdot\frac{1}{4}r^2+C_1\cdot\ln r+C_2.$$

BC ⓐ $r=r_0$, $u=0$.

ⓐ $r=0$, $\partial u/\partial r=0$.

$$\rightarrow u(r)=-\frac{1}{4\mu}\frac{dp}{dx}\left(r_0^2-r^2\right).$$

w/ $u=u(r)$.

· volumetric flow rate, $Q=\int u\cdot dA$.

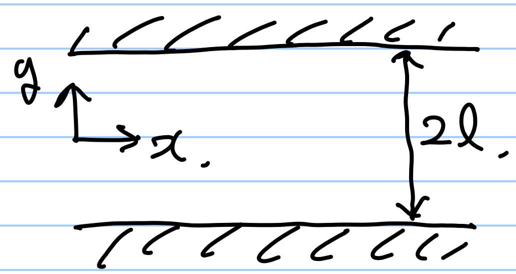
· bulk (average) velocity, $\bar{u}=Q/A$. $\left(=\frac{1}{2}U_{max}=\frac{1}{2}u(r=0)\right)$.

· shear stress @ $r=r_0$, $\tau_w = \mu \left.\frac{du}{dr}\right)_{r=r_0}$.

· Darcy's friction factor, $\lambda = \frac{8\tau_w}{\rho \bar{u}^2}$.

· skin-friction coefficient, $C_f = \frac{\tau_w}{\frac{1}{2}\rho \bar{u}^2}$.

2-2) Poiseuille flow between two flat plates.

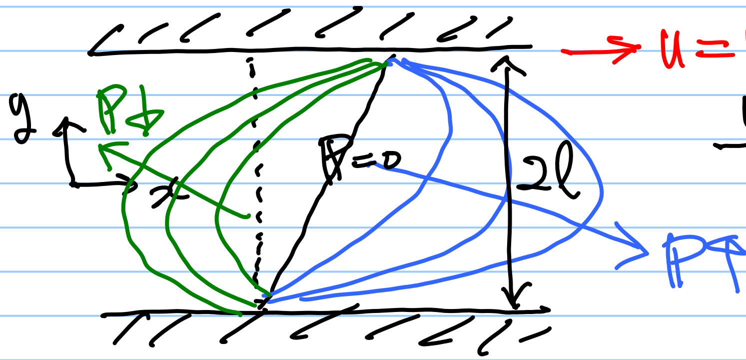


$\mu \frac{\partial^2 u}{dy^2} = \frac{dp}{dx}$. w/ @ $y=\pm l$, $u=0$.

$\rightarrow u(y) = -\frac{dp}{dx}\left(\frac{l^2}{2\mu}\right)\left[1-\left(\frac{y}{l}\right)^2\right]$.

$\rightarrow \tau_w = \mu \left.\frac{\partial u}{\partial y}\right)_{y=\pm l} = \mp \frac{2\mu \cdot U_{max}}{l}$.

2-3) two flat plates, Couette + Poiseuille flow.



$\rightarrow u = U_0$.

$$\frac{u(y)}{U_0} = \frac{1}{2}\left(1+\frac{y}{l}\right) + \mathbb{P}\left(1-\left(\frac{y}{l}\right)^2\right).$$

$$\mathbb{P} \equiv \left(-\frac{dp}{dx}\right)\cdot\frac{l^2}{2\mu U_0}.$$

$\mathbb{P} > 0$, $\frac{\partial p}{\partial x} < 0$: favorable.

$\mathbb{P} < 0$, $\frac{\partial p}{\partial x} > 0$: adverse

↳ backflow, separation.