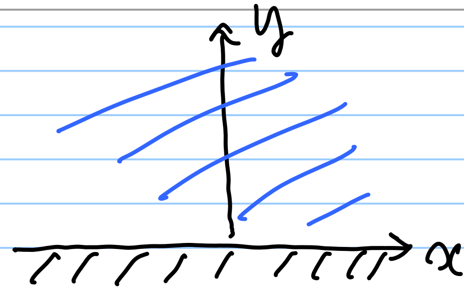
③ Stoke's First Problem (Rayleigh Prob.)



$u = 0$ @ $t = 0$

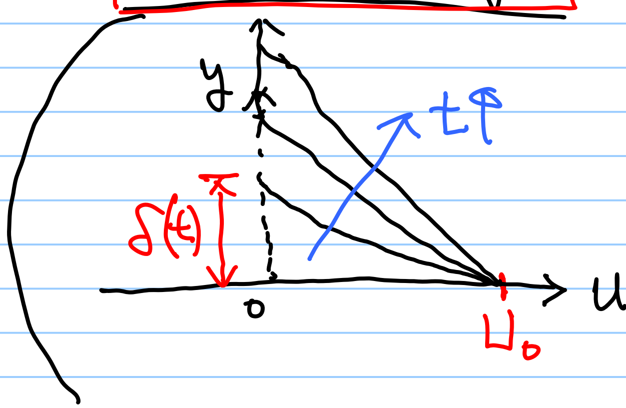
impulsely moving plate : $u = U_0$ (for $t > 0$)

$\hookrightarrow u = u(y)$, $\dfrac{\partial p}{\partial x} = 0$.

$$\text{※} \Rightarrow \boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$

w/ BC's $u(y,t)$: finite

$$u(0,t) \begin{cases} = 0 & \text{for } t \leq 0 \\ = U_0 & \text{for } t > 0 \end{cases}.$$



$\delta(t)$: penetration depth.

↘ simple scaling analysis : $\frac{U_0}{t} \sim \nu \frac{U_0}{\delta^2} \Rightarrow \underline{\delta \sim \sqrt{\nu t}}$.

: this problem describes a viscous diffusion process

(transfer of x-mtm into y-direction)

boundary-layer like behavior.

→ as ν increases, the penetration depth becomes thicker.

· Wall-shear stress, $\tau_w = \mu \frac{\partial u}{\partial y} \sim \frac{\mu U_0}{\underline{\delta}} \sim \frac{\mu U_0}{\sqrt{\nu t}}$.

$$\therefore \frac{\tau_w \delta}{\mu U_0} \sim 1.$$

* Similarity solution.

· shape of vel. profile is similar at all times.

(PDE ⟶ ODE)

indep. var. : y, t. $\longrightarrow$ dep. var. $u(y,t)$.

$\longrightarrow \dfrac{u(y,t)}{U_0} = f(\eta)$:

$\hookrightarrow$ dimensionless similarity variable.

$$\eta = \alpha \cdot \frac{y}{t^n}.$$

put η into the gov. eq.

$$\circledast \rightarrow -U_0 \cdot n \cdot f' \cdot \eta \cdot \cancel{t^{-1}} = \nu \cdot U_0 \cdot f'' \cdot \alpha^2 \cdot \cancel{t^{-2n}}$$

$\longrightarrow$ we want this to be ODE.

need $n = 1/2$.

to make 'η' dimensionless & for convenience,

we choose $\alpha^2 = \dfrac{1}{4\nu}$. $\Rightarrow \eta = \dfrac{y}{2\sqrt{\nu t}}$.

resulting ODE : $f'' + 2\eta \cdot f' = 0$.

$$\ln f' = -\eta^2 + \ln A, \quad f' = A e^{-\eta^2}.$$

$$\therefore f(\eta) = A \int_0^{\eta} e^{-\xi^2}\, d\xi + B. \qquad f = u/U_0.$$

· $\underline{BC}$: $u(0,t) = U_0$ $(t>0)$. $\rightarrow f(0) = 1 \rightarrow B = 1$.

$$\left(\eta = \frac{y}{2\sqrt{\nu t}}\right).$$

· IC : $u(y,0) = 0$. $\rightarrow \lim_{\eta \to \infty} f = 0 \rightarrow A = -\frac{2}{\sqrt{\pi}}$.

$$\therefore \frac{u(y,t)}{U_0} = f(\eta) = \underbrace{1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} \cdot d\xi}. \quad = 1 - \operatorname{erf}(\eta) = 1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right).$$

$$= \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right).$$

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \cdot dt, \quad 1 - \mathrm{erf}(x) \equiv \mathrm{erfc}(x).$$

. Non-dimensionalized vel. profile collapsed to one curve ← similarity.

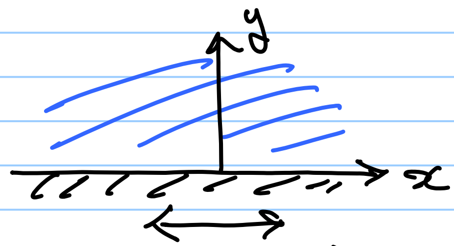
* penetration depth behaves like a boundary layer

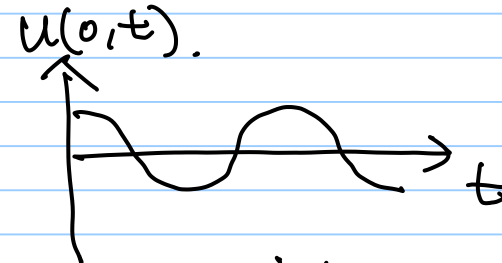
agree with the scaling analysis. ←

$$\frac{u}{U_0} = 0.01 \longrightarrow \delta = 3.6\sqrt{\nu t}.$$

wall shear, $\tau_w = \frac{\mu U_0}{\sqrt{\pi \nu t}}$.

7 Stokes' Second Prob : oscillating wall.





$$u(0,t) = U_0 \cdot \cos(nt)$$

$$= \mathrm{Re}\left[U_0 e^{int}\right]$$

↳ real part.

$$e^{int} = \cos(nt) + i\sin(nt)$$

$$\underline{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$

($u(y,t)$ has a finite value)

Assume, $u(y,t) = \mathrm{Re}\left[w(y)\cdot e^{int}\right]$ → into mtm eq.

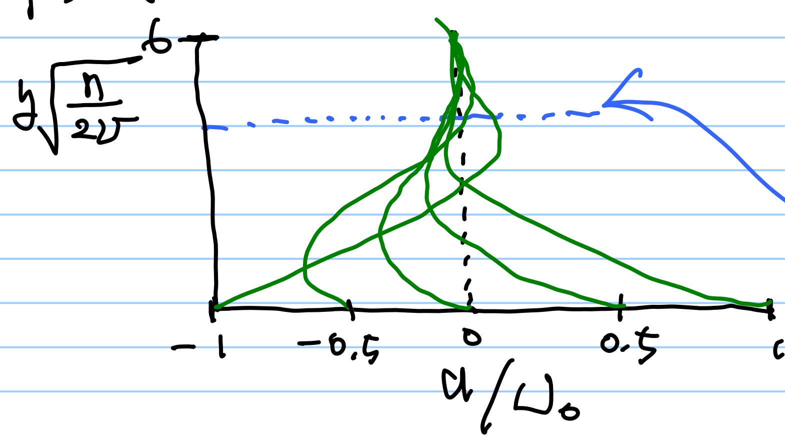
$$\Rightarrow w(y) = A\cdot \exp\left\{-(1+i)\sqrt{\frac{n}{2\nu}}\,y\right\} + B\cdot \exp\left\{(1+i)\sqrt{\frac{n}{2\nu}}\,y\right\}$$

then, $\dfrac{u(y,t)}{U_0} = \underbrace{\exp\left\{-\sqrt{\dfrac{n}{2\nu}}\,y\right\}}\cdot\cos\left(nt-\sqrt{\dfrac{n}{2\nu}}\,y\right)$.

i) exponentially decay in y-direction

ii) phase shift between wall motion and fluid velocity by $\sqrt{\dfrac{n}{2\nu}}\,y$.

- e.g.) for instantaneous vel. profiles.



White "Viscous Fluid Flow".

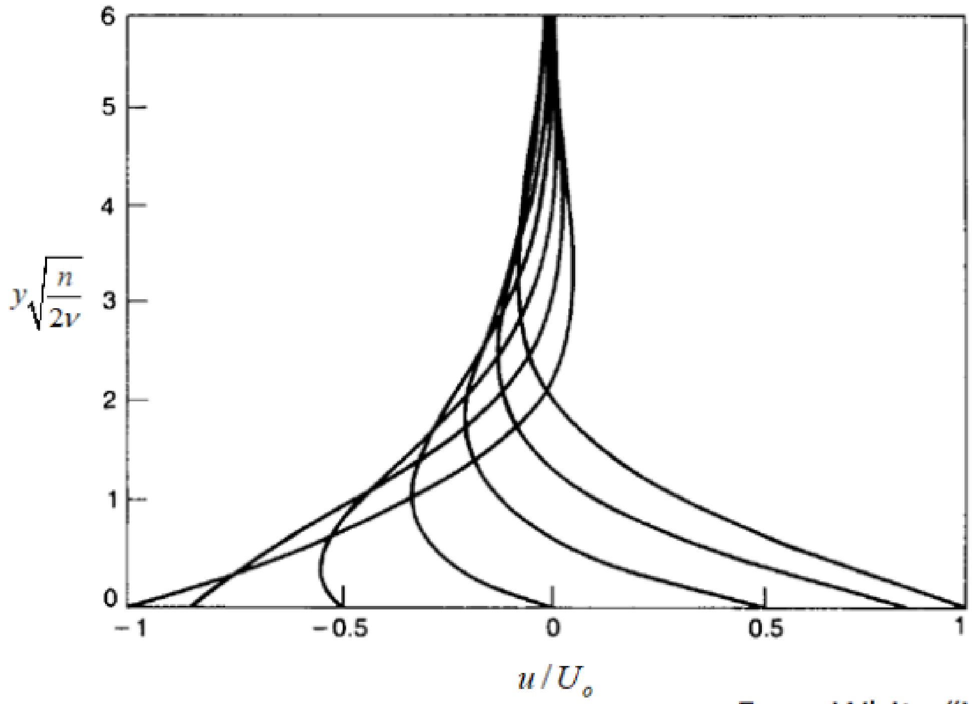
- if $\exp\left\{-\sqrt{\frac{n}{2\nu}}\,y\right\} = 0.01$, or $\sqrt{\frac{n}{2\nu}}\,\underline{\underline{y}} = 4.6$

~~then~~, $\delta \doteq 6.5\sqrt{\frac{\nu}{n}} \sim \sqrt{\frac{\nu}{n}}$.

so, we can say

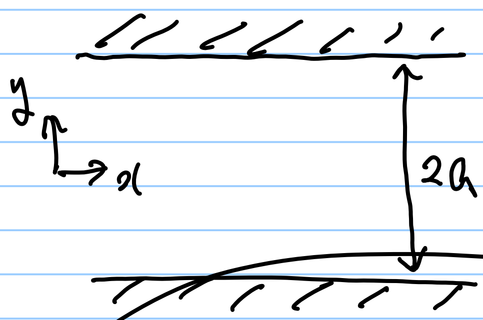
i) low-frequency motion will be dominant in diffusing x-mtm into y-direction.

ii) small phase shift for low-freq. motion.



From White "Viscous Fluid Flow"

⑤ Pulsating flow between parallel surfaces.



1D, incompressible, viscous.

$$\frac{\partial p}{\partial x} = P_x \cdot \cos(nt).$$

amp. of pressure-grad. oscillation

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2},$$

BC's : $u(-a,t) = u(a,t) = 0$.

$$\frac{\partial p}{\partial x} = Re\left[P_x \cdot e^{int}\right] \longrightarrow u(y,t) = Re\left[w(y)\, e^{int}\right].$$

$$\therefore\ u(y,t) = Re\left[i\frac{P_x}{\rho n}\left\{1 - \frac{\cosh\{(1+i)\sqrt{n/2\nu}\, y\}}{\cosh\{(1+i)\sqrt{n/2\nu}\, a\}}\right\} e^{int}\right]$$

i) $u(y,t)$ oscillates @ frequency of n.

ii) " has a phase shift, function of y.

$\sqrt{\frac{n}{2\nu}}\,a \ll 1$: quasi-steady.

$\sqrt{\frac{n}{2\nu}}\,a \gg 1$: unsteady flow

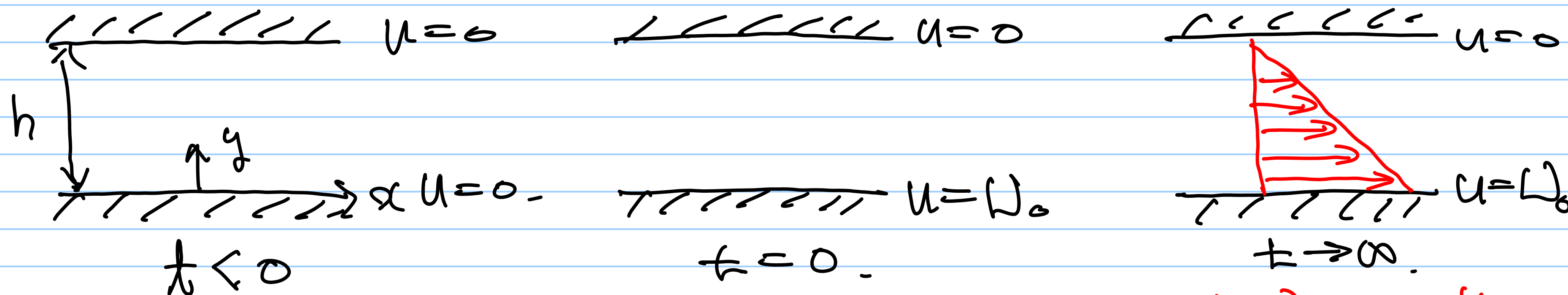
① thin viscous region will occur only near the wall

② phase difference between pressure and velocity, pressure and shear.

③ phase difference between velocities at different y's.

⑤ Transient (unsteady) flow for Couette flow formation.

(lower wall moves, for convenience)



$$\frac{u(y,\infty)}{U_0} = 1 - \frac{y}{h}.$$

Let's define, $w(y,t)$ as.

$$\frac{w(y,t)}{U_0} = \frac{u(y,t)}{U_0} - \left(1 - \frac{y}{h}\right).$$

; difference between $\boxed{\text{'u'}}$ and final steady solution.

$$t^* = \frac{\nu t}{h^2}. \quad y^* = \frac{y}{h}. \quad \frac{u}{U_0} = u^*, \quad \frac{w}{U_0} = w^*$$

$$w^* = u^* - (1 - y^*).$$

then, mtm. conservation eq. : $\underline{\dfrac{\partial w^*}{\partial t^*} = \dfrac{\partial^2 w^*}{\partial y^{*2}}}$.

$$\begin{cases} w^*(0, t^*) = 0 \\ w^*(1, t^*) = 0 \\ w^*(y^*, 0) = -1 + y^*. \end{cases}$$

sep. of variables

$$w^*(y^*, t^*) = f(y^*) \cdot g(t^*). \therefore \frac{\dot{g}}{g} = \frac{f''}{f} = \text{const.} = -\lambda^2.$$

$$\rightarrow \begin{cases} f = A \cdot \sin(\lambda y^*) + \cancel{B \cdot \cos(\lambda y^*)} \\ g = C \cdot \exp(-\lambda^2 \cdot t^*). \end{cases}$$

$$w^*(0,t^*) = 0 \rightarrow B = 0.$$
$$w^*(1,t^*) = 0 \rightarrow \eta = n\pi.$$
$$\rightarrow f = A_n \cdot \sin(n\pi y^*)$$

$$w^*(y^*,0) = \sum A_n \cdot \sin(n\pi y^*) = -1 + y^*.$$

· Fourier orthogonality.

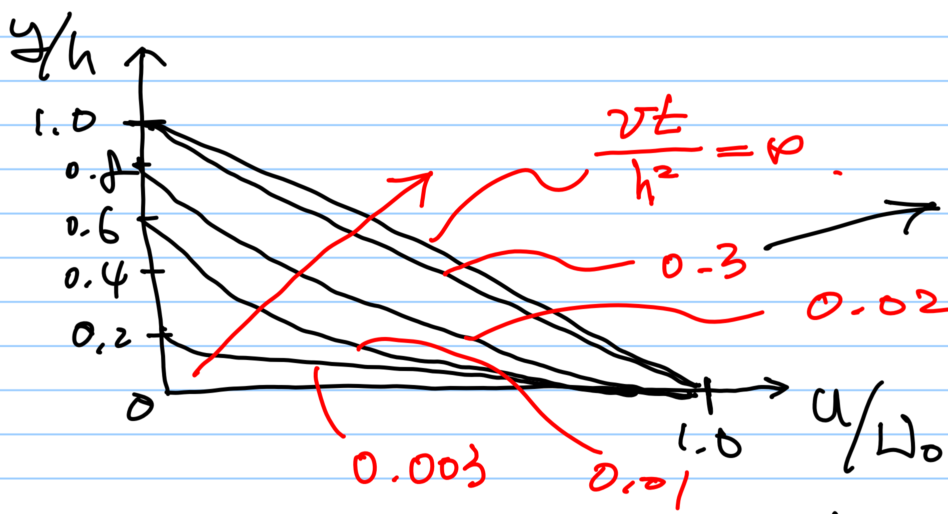
$$f(x) = a_0 + \sum_{n=1}^{\infty} \{ a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \}$$

$$\rightarrow \int_0^{2\pi} \cos(nx) \cdot \sin(mx)\, dx = 0 \quad (m \neq n \neq 0)$$

$$\underline{\underline{A_n}} = \int_{-1}^{1} (-1+y^*) \cdot \sin(n\pi y^*)\, dy^* = \underline{\underline{-\frac{2}{n\pi}}}.$$

$$\omega^* = u^* - (1-y^*), \quad u^* = \underline{\underline{\omega^*}} + \underline{\underline{(1-y^*)}}$$

$$\therefore \frac{u(y,t)}{U_0} = \left(1 - \frac{y}{h}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-n^2\pi^2 \cdot \frac{\nu t}{h^2}\right)\cdot \sin\frac{n\pi y}{h}.$$



if $h = 10$ mm, air.

$\rightarrow t \simeq 2.0$ sec.

- upper plate prevents the mtm diffusion
  $\rightarrow$ steady flow
- "n = 1" component survives the longest.