

N-S eq : elliptic type.

BL eq : parabolic type, can be marched along "x".

• Recall from Stoke's 1st Prob.

$$\delta \sim \sqrt{\nu x}. \quad \text{if } x \sim L/U_0 \Rightarrow \delta \sim \sqrt{\frac{\nu L}{U_0}}$$
$$\Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}.$$

① Similarity Solution to BDLF eqns.

(Blasius Solution).

• Flat plate, 2D, no-pressure grad.

• BC's : (a) $\hat{y} = 0 \rightarrow \hat{u} = 0, \hat{v} = 0.$

(b) $\hat{y} \rightarrow \infty \rightarrow \hat{u} \equiv u/U_0 = 1.$

(Let's drop "n" for convenience)

• To make a single equation, $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$.

• To have a similarity solution, introduce similarity variable.

$\Rightarrow \psi(x, y) = f(\eta)$, $\eta \sim \frac{y}{x^n}$ for a flat plate, $n = 1/2$

$$\eta \equiv \frac{y}{\sqrt{2\nu x/U_0}}, \quad \psi(x, y) = \sqrt{2\nu x U_0} f(\eta).$$

$$\eta = y / \sqrt{\nu x / U_0} \rightarrow \psi(x, y) = \sqrt{\nu x U_0} \cdot f(\eta), \quad \leftarrow \eta \sim y / x^{1/2}$$

u/U_0 should remain unchanged when plotted with y/δ

local δ body thickness

w/o dp/dx , $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \right)$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$u \sim U_0$
 $y \sim \delta$

$$\frac{U_0}{x} + \frac{v}{\delta} \sim 0 \rightarrow v \sim \frac{U_0 \delta}{x}$$

$$U_0 \frac{U_0}{x} + \frac{U_0 \delta}{x} \cdot \frac{U_0}{\delta} \sim \nu \frac{U_0}{\delta^2} \rightarrow \delta^2 \sim \frac{\nu x}{U_0} \quad \text{or} \quad \delta \sim \sqrt{\frac{\nu x}{U_0}}$$

we want $\frac{u}{U_0} = f'(\frac{y}{\delta})$ \rightarrow $\frac{u}{U_0} = f'(\frac{y}{\sqrt{2\nu x/U_0}}) = f'(\eta)$.
 \rightarrow substitute into BDLF eqn.

$\rightarrow \underline{f'''' + \frac{1}{2}ff'' = 0}$ BC: $f(0) = 0, f'(0) = 0, f'(\infty) = 1$.
 \rightarrow to be solved numerically.

* $\delta: u/U_0 \approx 0.99 \rightarrow \eta \approx 5.0$ $\frac{\delta}{\sqrt{2\nu x/U_0}} = 5.0$.

$\therefore \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$

* $v = -\frac{\partial \psi}{\partial x} = \frac{1}{2} \sqrt{\frac{\nu U_0}{x}} (\eta f' - f)$ \leftarrow wall normal velocity
 (if $\eta \rightarrow \infty$ (i.e., far away from the wall))

$$\Downarrow U_{\infty} = 0.86 U_0 \sqrt{\frac{25}{x L_0}} \neq 0 \text{ why?}$$

→ there is a flow outward due to the displaced fluid with increasing δ .

$$* \tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

$$= \mu U_0 \sqrt{\frac{U_0}{2\nu x}} \cdot f''(0)$$

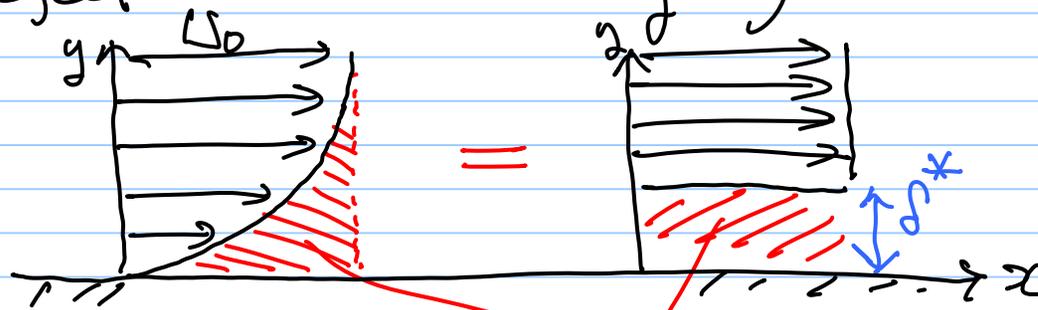
$$\rightarrow c_f \equiv \frac{\tau_w}{\frac{1}{2} \rho U_0^2} = \frac{0.664}{\sqrt{Re_x}}$$

$$\cdot \text{ Drag force, } D = b \int_0^L \tau_w dx = 0.664 U_0 \cdot b \sqrt{\mu \rho U_0 L}$$

$$C_p \equiv \frac{D}{\frac{1}{2} \rho U_0^2 (bL)} = \frac{1.328}{\sqrt{Re_L}}$$

* Displacement thickness (mass flow defect)

: distance which the outer flow is displaced from the wall by a stagnant (zero velocity) layer that has the mass defect as the boundary layer.



$$U_0 \delta^* = \int_0^{\infty} (U_0 - u) dy \rightarrow \delta^* = \int_0^{\infty} \left(1 - \frac{u}{U_0}\right) dy$$

(mass flow defect)

$$\textcircled{a} \eta = 5.0 : \frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}} \sim \frac{1}{3} \delta = \sqrt{\frac{5x}{U_0}} (\eta - f) \text{ (Blasius)}$$

* Momentum thickness (θ)

: thickness of stagnant layer which has the same momentum defect as the boundary layer.

$$\rho U_0^2 \theta = \int_0^{\infty} \rho u (U_0 - u) dy \rightarrow \theta = \int_0^{\infty} \frac{u}{U_0} \left(1 - \frac{u}{U_0}\right) dy$$

$$= \sqrt{\frac{5x}{U_0}} \int_0^{\infty} f'(1 - f') d\eta$$

$$\textcircled{a} \eta = 5.0, \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

* shape factor, $H = \delta^*/\theta = \begin{cases} 2.59 & \text{(laminar)} \\ 1.5 & \text{(turbulent)} \end{cases}$

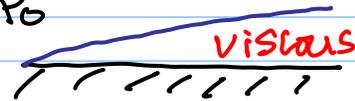
(also, H indicates flow separation

: $H > 2.8 - 4 = 0$, separation might occur.

① Boundary layer w/ dp/dx . $\frac{U_0}{\rho_0}$ inviscid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$



→ apply at the outer edge of Boundary layer.

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = U_0 \frac{dU_0}{dx} \quad \leftarrow \text{(inviscid analysis)}$$

↓
 $u \rightarrow U_0$
 $\partial u / \partial y \rightarrow 0$

① Falkner-Skan solutions, $U_0(x) \sim x^m$.

② other simple flows, $U_0(x) = (x-d)_+$.

③ arbitrary, $U_0(x)$.

* Falkner-Skan flows.

- a family of similarity solutions.

$$u(x,y) = \underline{U_0(x)} \cdot f'(\eta), \quad \eta = y/g(x).$$

$\rightarrow \psi(x,y) = U_0(x) \cdot g(x) \cdot f(\eta)$. \rightarrow substitute into the
continuity eq in terms of η .

$$\Rightarrow f''' + \underbrace{\left[\frac{g}{\nu} \cdot \frac{d}{dx}(U_0 g) \right]}_{=\alpha \text{ ①}} f f'' + \underbrace{\left[\frac{g^2}{\nu} \cdot \frac{dU_0}{dx} \right]}_{=\beta \text{ ②}} \{1 - (f')^2\} = 0$$

→ For similarity solution to exist, α, β should be constant!

So, eqns ①, ②

i) determine
 α, β

$$\text{or } \frac{d}{dx}(U_0 g^2) = U(2\alpha - \beta) \quad \text{③} \quad \text{① \& ③ or ② \& ③}$$

$$\underline{f^{(4)} + \alpha f f'' + \beta [1 - (f')^2] = 0.}$$

• procedure for Falkner-Skan flows.

i) choose α, β

ii) find $U_0(x)$ & $g(x)$ by ① & ②, ② & ③, ① & ③

iii) determine $f(\eta)$.

iv) $\psi(x, y) = U_0(x) \cdot g(x) \cdot f(\eta) \rightarrow u, v$.

* Flat-plate BL, w/ dp/dx , ($\alpha = 1/2$, $\beta = 0$) ← i)

$$\frac{d}{dx} (U_0 g^2) = \nu, \quad \frac{g^2}{\nu} \cdot \frac{dU_0}{dx} = 0. \quad \leftarrow \text{ii)}$$

$$\rightarrow U_0 = c \text{ (constant)}, \quad g = \sqrt{\frac{\nu x}{U_0}}$$

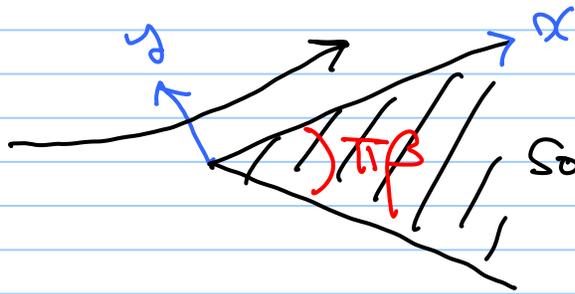
* 2D Stagnation point flow, $\alpha = \beta = 1$.

Falkner-Skan flows.

* flow over a wedge

$$f''' + \alpha f f'' + \beta [1 - (f')^2] = 0.$$

$$\alpha = \frac{2}{\nu} \frac{d}{dx} (U_0 g), \quad \beta = \frac{g^2}{\nu} \frac{dU_0}{dx}.$$



Solid wedge $\leftarrow \alpha = 1,$
 $\beta = \text{arbitrary} \left(\sim \frac{2P}{2\alpha} \right)$

$$\left(\frac{d}{dx} (g^2 U_0) = \nu(2-\beta) \right)$$

integrate $g^2 U_0 = \nu(2-\beta)x$