$$g^2 \frac{dU_0}{dx} = \beta \nu$$

$$\frac{1}{U_0}\frac{dU_0}{dx} = \frac{\beta}{2-\beta}\cdot\frac{1}{x}$$

$$\therefore U_0(x) = C\cdot x^{\frac{\beta}{2-\beta}}, \quad g(x) = \sqrt{\frac{\nu(2-\beta)}{C}\cdot x^{\frac{1-\beta}{2-\beta}}}.$$

$\beta > 0 \rightarrow dU_0/dx > 0 \rightarrow dp/dx < 0$ (favorable)

⓪ Approximate Integral Methods

· Boundary-layer integral eqn.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_0\frac{dU_0}{dx}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \quad \text{— ①}$$

$$= \frac{\partial}{\partial x}(u^2)+\frac{\partial}{\partial y}(uv).$$

→ integrate ① across the boundary-layer. $(0 \le y \le \delta)$

w/ $u(x,0)=0$, $v(x,0)=0$. $u(x,\delta)=U_0$.

$\partial u/\partial y=0$ @ $y=\delta$, $\partial u/\partial y=\tau_w/\mu$ @ $y=0$.

$\left(\tau_w \equiv \mu \frac{\partial u}{\partial y}\Big|_{y=0}\right)$

from the continuity, $\int_0^\delta \frac{\partial u}{\partial x}dy+\int_0^\delta \frac{\partial v}{\partial y}dy=0$.

$$\rightarrow v(x,\delta)=-\int_0^\delta \frac{\partial u}{\partial x}dy$$

$$\therefore \int_0^\delta \text{①}\, dy \Rightarrow \frac{d}{dx}\int_0^\delta \underline{\underline{u(U_0-u)}}\, dy + \frac{dU_0}{dx}\int_0^\delta \underline{\underline{(U_0-u)}}\, dy = \frac{\tau_w}{\rho}.$$

integrands tend to be zero, at $y > \delta$

$$\therefore \frac{d}{dx}\left[U_0^2 \underbrace{\int_0^\infty \frac{u}{U_0}\left(1-\frac{u}{U_0}\right) dy}_{\equiv\,\theta.}\right] + \frac{dU_0}{dx}U_0 \underbrace{\int_0^\infty \left(1-\frac{u}{U_0}\right) dy}_{\equiv\,\delta^*.} = \frac{\tau_w}{\rho}.$$

$$\therefore \frac{d}{dx}(U_0^2\theta) + \frac{dU_0}{dx}U_0\delta^* = \frac{\tau_w}{\rho}.$$

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U_0^2}.$$

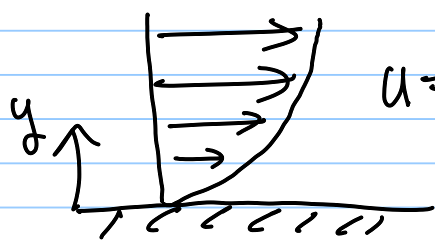
$$\rightarrow \frac{d\theta}{dx} + (2\theta+\delta^*)\frac{1}{U_0}\frac{dU_0}{dx} = \frac{\tau_w}{\rho U_0^2} = \frac{C_f}{2}$$

or $\frac{d\theta}{dx} + (2+H)\frac{\theta}{U_0}\frac{dU_0}{dx} = \frac{c_f}{2}$ $\rightarrow$ given, $\frac{\partial P}{\partial x}$.

$\hookrightarrow$ Boundary-layer integral equation!
(general mtm integral)

$\rightarrow$ One need velocity profile. $\rightarrow$ assume!

· for any assumed form of vel. profile, θ, δ^* and τ_w can be evaluated from GMI.



$u = u(y)$: assume it is 2ND order polynomial.

$\hookrightarrow \frac{u}{U_0} = 2\cdot\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$.

Integral (approx.)   Blasius (exact)

$\frac{\theta}{x}\sqrt{Re}$	0.73		0.664
$\frac{\delta}{x}\sqrt{Re}$	5.5		5.0
$\frac{\delta^*}{x}\sqrt{Re}$	1.83	$\xleftarrow{\text{with}}$ 10% $\rightarrow$	1.72
$C_f\sqrt{Re}$	0.73		0.664
$C_D\sqrt{Re}$	1.46		1.328

$*$ BC's to be satisfied by assumed u/U_0

① $u = 0$ @ $y = 0$

② $u = U_0$ @ $y = \delta$

minimum.

③ $\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = \tau_w/\rho$ (Newtonian fluid)

④ $\left.\dfrac{\partial u}{\partial y}\right|_{y=\delta} = 0$.

⑤ $\left.\dfrac{\partial^2 u}{\partial y^2}\right|_{y=0} = \dfrac{1}{\mu}\dfrac{\partial P}{\partial x}$. (from BL eqn)

↳ balance between pressure and viscous force in BL.

⑥ $\left.\dfrac{\partial^2 u}{\partial y^2}\right|_{y=\delta} = 0$. : No viscous force outside of BL.

⓪ Application of GMI to flows w/ $\partial P/\partial x$.

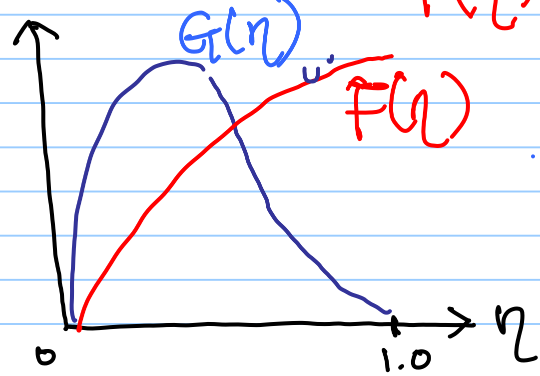
- Karman-Pohlhausen approximation (Currie)

$$\frac{u}{U_0} = a + b\eta + c\eta^2 + d\eta^3 + e\eta^4, \quad \eta = y/\delta.$$

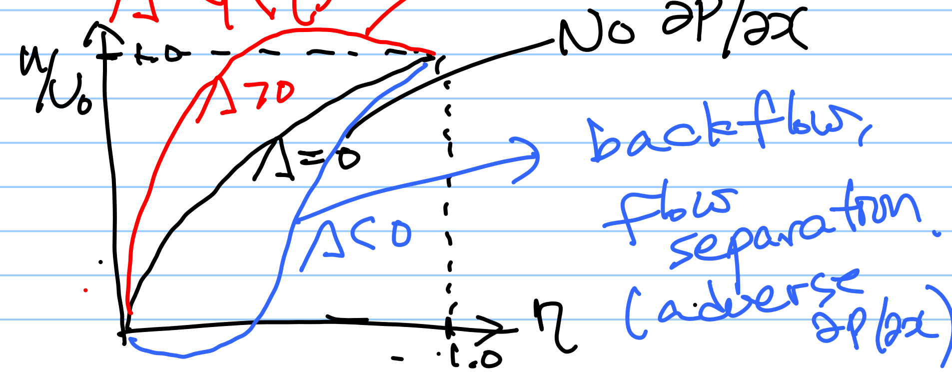
$$@\ \eta = 0: \quad \frac{u}{U_0} = 0, \quad \frac{\partial^2}{\partial\eta^2}\left(\frac{u}{U_0}\right) = -\frac{\delta^2}{\nu}\frac{dU_0}{dx} \equiv -\Lambda(x) \quad \left(\sim \frac{\partial p}{\partial x}\right)$$

$$@\ \eta = 1: \quad \frac{u}{U_0} = 1, \quad \frac{\partial}{\partial\eta}\left(\frac{u}{U_0}\right) = 0, \quad \frac{\partial^2}{\partial\eta^2}\left(\frac{u}{U_0}\right) = 0.$$

$$\Rightarrow \frac{u}{U_0} = \underbrace{1 - (1+\eta)(1-\eta)^3}_{F(\eta)} + \underbrace{\frac{\Lambda}{6}\eta(1-\eta)^3}_{\Lambda \cdot G(\eta)} = F + \Lambda G$$



→



$\rightarrow$ valid at $-12 < \Lambda < 12$.

· from u/U_0, $\longrightarrow \dfrac{\delta^*}{\delta} = \displaystyle\int_0^1 \left(1 - \frac{u}{U_0}\right) d\eta$, $\dfrac{\theta}{\delta} = \displaystyle\int_0^1 \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) d\eta$

$$\tau_w = \mu \frac{U_0}{\delta} \cdot \frac{d}{d\eta}\left(\frac{u}{U_0}\right)\Big|_{\eta=0}.$$

all fcn(Λ)

· then, Q is what is δ?

· from GMI, $\left(\dfrac{d\theta}{dx} + (2\theta + \delta^*)\dfrac{1}{U_0}\dfrac{dU_0}{dx} = \dfrac{\tau_w}{\rho U_0^2}\right) \times \dfrac{U_0 \theta}{\nu}$.

$$\rightarrow \frac{1}{2} U_0 \frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) + (2+H)\frac{\theta^2}{\nu}\cdot\frac{dU_0}{dx} = \frac{\tau_w \theta}{\mu U_0}.$$

$$\left(\Lambda \equiv \frac{\delta^2}{\nu}\frac{dU_0}{dx}\right) \qquad = \frac{\theta^2}{\delta^2}\Lambda = \text{fcn}(\Lambda) = K(x).$$

$$H \equiv \frac{\delta^*}{\theta} = f(K)$$

$$\frac{\tau_w \theta}{\mu U_0} = g(K)$$

$$\frac{1}{2} U_0 \frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) + \left(2 + f(K)\right)K = g(K)$$

let $\theta^2/\nu = Z$, $U_0 \frac{dZ}{dx} = 2\{g(K) - [2 + f(K)]K\} = H(K)$

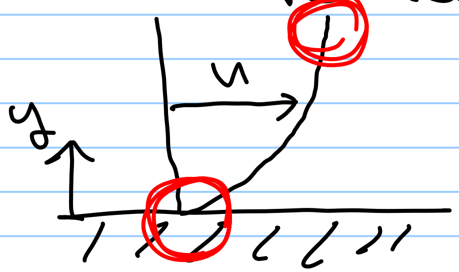
$\Rightarrow$ So, K and $H(K)$ can be obtained for any Λ.

- general procedure to solve.
  - determine $U_0(x)$ from the inviscid solution.
  - obtain θ^2. $\left(\theta^2 = \frac{0.45\nu}{U_0^6(x)} \int_0^x U_0^5(\xi)\, d\xi\right)$

- obtain Λ from θ^2, ν
- obtain δ from θ and Λ.
- calculate $\delta^*, u/U_0, \tau_w$ ..... from δ.

⊙ Boundary-layer separation.

along the $y=0$ (wall), the boundary-layer eqn reduces to $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$.



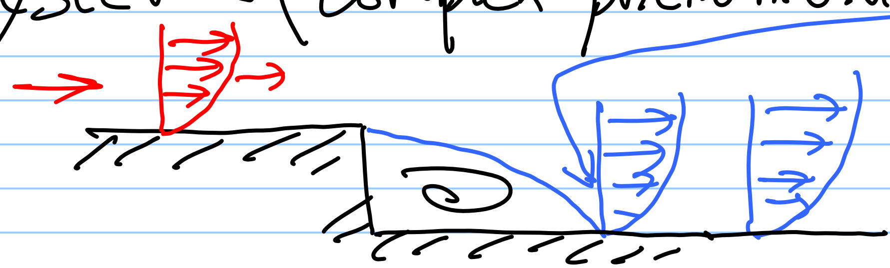
→ $\frac{\partial p}{\partial x} < 0$: curvature of vel. profile is negative → no sep.

$\frac{\partial p}{\partial x} > 0$: " positive but, $\frac{\partial^2 u}{\partial y^2} < 0$ at the edge of BL.

We need an inflection point

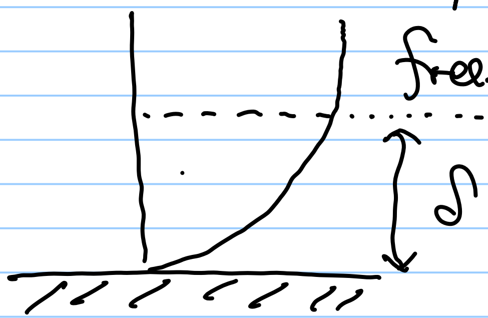
$\left(\frac{\partial^2 u}{\partial y^2} = 0 \right)$ @ $0 < y < \delta$, → separation is possible.

- Prediction of separation point is very difficult.
  1. No analytic solution for separated BL.
  2. Seriously affects the performance of flow system. (complex phenomena, reattachment, sep. bubble, ...)



③ Issues in numerical simulation.

① BL calculation w/ asymptotic expansion method.



obtain ① outer solution
② inner solution
③ match the inner and outer solutions asymptotically
→ composite solution.

* Friedrich's model of the BL.

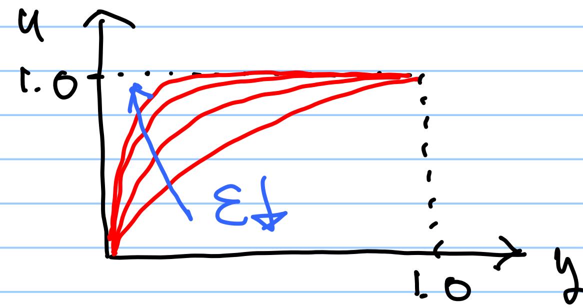
$$(*)\quad \varepsilon \cdot \frac{d^2u}{dy^2} + \frac{du}{dy} = a. \quad (\varepsilon \ll 1) \quad w/ \ u(0) = 0, \ u(1) = 1.$$

$\varepsilon = \frac{1}{Re}$

to solve the near-wall solution.

* compared to typical BL equation.

① exact solution : $u(y) = ay + (1-a)\dfrac{1-\exp(y/\varepsilon)}{1-\exp(-1/\varepsilon)}$.



② asymptotic sol.

as $\varepsilon \to 0$. $\dfrac{du}{dy} = a$. $u(y) = ay + C_1$ (linear)

(outer solution)

C_1? $u(1) = 1$. $\to C_1 = 1-a$.

$u(y) = ay + (1-a) \leftarrow$ does not

to find out inner solution,
define magnified variable, $Y \equiv y/\varepsilon$.
(more resolved)

$$U(Y) = u(y).$$

$$\therefore \; ⊛ : \frac{d^2U}{dY^2} + \frac{dU}{dY} = a\varepsilon \simeq 0 \quad ; \text{ inner solution.}$$

satisfy $U(0) = 0$.

$u \leftarrow U, \; y \leftarrow Y$.

satisfy $u(0) = 0$.

$$\rightarrow U_{inner} = C_2 \left(1 - e^{-Y}\right). \quad C_2 = 1 - a.$$

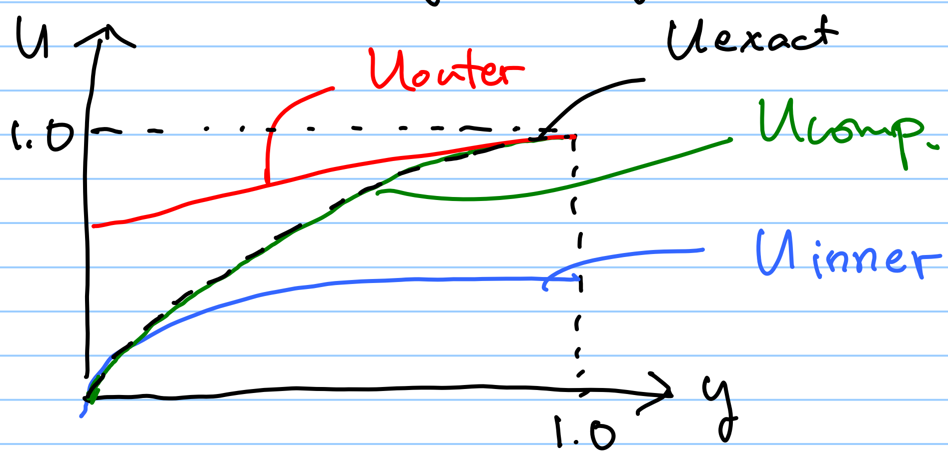
$$= (1-a)\left(1 - e^{-Y}\right).$$

→ Composite solution.

$$u_{comp} = u_{outer} + u_{inner} - (\text{common term})$$

$$= [ay + (1-a)] + [(1-a)(1-e^{-y/\varepsilon})] - (1-a)$$

$$= (1-a)(1-e^{-y/\varepsilon}) + ay.$$



for $\varepsilon \lesssim 0.2$ deviation is $\pm 1\%$.