

- like jets, most wakes are unstable and are to be turbulent.

HW#3. ETL.

BUOYANCY DRIVEN FLOWS (Natural convection)

Ch. 10 in Curme.

- forced vs. natural convection.
 - ↳ wind, pump, fan... → buoyancy effect.

② Forced convection.

- mtrm and temp. eqns are NOT coupled.

$$\left. \begin{array}{l} \text{continuity} \\ \text{mtrm eq} \end{array} \right\} \rightarrow \bar{u}, p. \Rightarrow \rho C_p \frac{DT}{dt} = \frac{Dp}{dt} + \nabla \cdot (k \nabla T) + \dot{Q} \Rightarrow T.$$

Steady in 2D \leftarrow incompressible flow

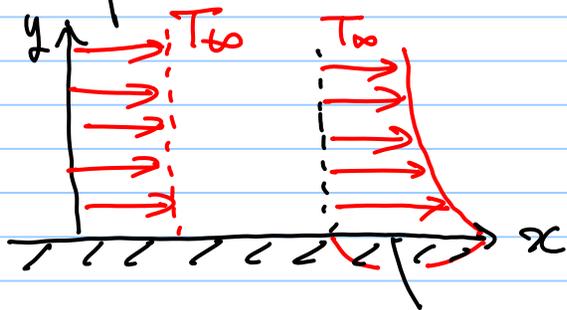
$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \left(u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} \right)$$

$$+ k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi$$

cf) in natural convection,

momentum & energy eqns are fully coupled.

* Concept of thermal BL (δ_T).



• flat-plate flow

• $\partial p / \partial x = 0$

• $\partial / \partial x \ll \partial / \partial y$

$T_0 (\neq T_\infty)$. $\Phi \ll 1$.

energy eq: $\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2}$ thermal diffusivity.

Let $\theta \equiv \frac{T_0 - T}{T_0 - T_\infty}$
 ↑ dimensionless temp.

$\alpha \frac{\partial^2 \theta}{\partial y^2} = u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y}, \alpha = \frac{k}{\rho C_p}$

W/ $\theta(0) = 0, \theta(\infty) = 1$.
 ↓ similar form to BL eqn.

from the stream eqn.

$\psi = \sqrt{\nu x U_\infty} \cdot f(\eta), \eta = y/\delta$

↑ Blasius solution.

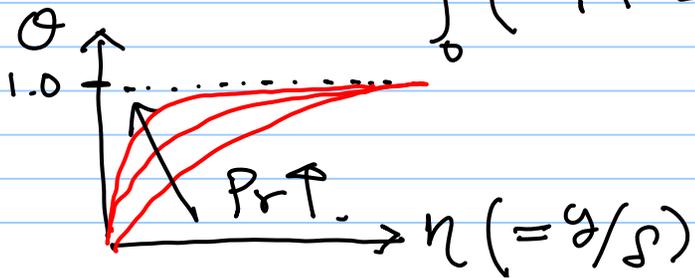
then, $\theta'' + \frac{Pr}{2} \cdot f \cdot \theta' = 0. \quad (' = d/d\eta)$

$(Pr \equiv \mu C_p / k)$

integrate: $\theta' = C_1 \cdot \exp\left[-\frac{Pr}{2} \int_0^\eta f \cdot d\eta\right]$

" : $\theta = \underline{C_1} \int_0^\eta \left[\exp\left[-\frac{Pr}{2} \int_0^\eta f \cdot d\eta\right] \right] d\eta + \underline{C_2}$

$\therefore \theta(\eta) = \frac{\int_0^\eta \left(\exp\left[-\frac{Pr}{2} \int_0^\eta f \cdot d\eta\right] \right) d\eta}{\int_0^\infty \left(\exp\left[-\frac{Pr}{2} \int_0^\eta f \cdot d\eta\right] \right) d\eta}, \quad \eta = y/\delta.$



→ Define δ_T as $y/\delta \big|_{\theta=0.99}$

also, $q_w = -k \frac{\partial T}{\partial y} \big|_{y=0}$

$$\text{Nu} \equiv \frac{hx}{k} \quad (h = \text{convective heat tr. coeff.})$$
$$\approx 0.332 \cdot \text{Pr}^{1/3} \cdot \text{Re}_x^{1/2} \quad (\text{laminar})$$

11/14, 11/26, 28 (No class) Final: 12/12 (15:30-)

⊙ Natural convection

Boussinesq approximation for buoyancy-driven flows.

$$\left\{ \begin{array}{l} \nabla \cdot \bar{u} = 0 \\ \rho \frac{D\bar{u}}{Dt} = -\nabla p + \mu \nabla^2 \bar{u} - \rho g \bar{e}_z \end{array} \right.$$

(unit directional vector in z-dir.)

↑
non-negligible body force.

- static ($\bar{u} = 0$) : $0 = -\nabla p_0 - \rho_0 g \bar{e}_z$ (0: static)

- dynamic ($\bar{u} \neq 0$) : $p = p_0 + p^*$

$$\rho = \rho_0 + \rho^*$$

$$\bar{u} = 0 + \bar{u}^*$$

→ into cont. & intm eqns

$$\rightarrow \left\{ \begin{array}{l} \nabla \cdot \bar{u}^* = 0 \\ (\rho_0 + \rho^*) \frac{D\bar{u}^*}{Dt} = -\nabla p^* + \mu \nabla^2 \bar{u}^* - \rho^* g \bar{e}_z \end{array} \right.$$

"Boussinesq approx."

density variation causes the motion,
but not affect the acceleration term.
(reasonable when small ρ^* over
the moderate distance).

\rightarrow Q: how to generate ρ^* ?

in thermal convection,

ρ^* is caused by temperature variation

$$: \rho \approx \rho_0 \left[1 - \beta (T - T_0) \right]$$

\downarrow thermal exp. coeff.
 $(\approx 1/T_0 \text{ for ideal gas})$

$$\begin{aligned}
 \rightarrow \left\{ \begin{aligned}
 \nabla \cdot \bar{u} &= 0 \\
 \rho \frac{D\bar{u}}{Dt} &= -\nabla p + \mu \nabla^2 \bar{u} + \rho \beta (T - T_0) \bar{e}_z \\
 \rho c_p \frac{DT}{Dt} &= \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) + \Phi
 \end{aligned} \right. \left. \vphantom{\begin{aligned} \nabla \cdot \bar{u} &= 0 \\ \rho \frac{D\bar{u}}{Dt} &= -\nabla p + \mu \nabla^2 \bar{u} + \rho \beta (T - T_0) \bar{e}_z \\ \rho c_p \frac{DT}{Dt} &= \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) + \Phi \end{aligned}} \right\} \text{Coupled.}
 \end{aligned}$$

* Similarity solution for thermal BL

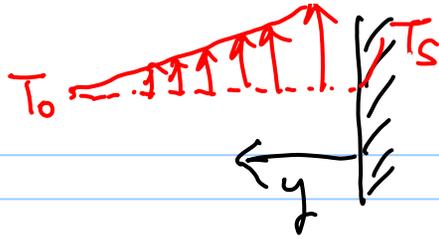
- Laminar flow on vertical isothermal wall



$$\cdot \partial p / \partial x = 0$$

$$\cdot \partial / \partial x \ll \partial / \partial y$$

$$\cdot \mathcal{O}(\rho) \approx \mathcal{O}(\rho T)$$



- moderate $\bar{u} \rightarrow$ negligible Φ
- incompressible, steady.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \beta (T - T_0)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

Assume, $\psi(x, y) = C_1 x^m f(\eta)$, $\eta = C_2 \frac{y}{x^n}$.

$$\theta(x, y) \equiv \frac{T - T_0}{T_s - T_0} = F(\eta)$$

\rightarrow substitute into momentum & energy eqns.

and find m & n to make the eqns
to be independent on "x" ← similarity sol.

$$\rightarrow m = 3/4, \quad n = 1/4.$$

→ choose C_1 and C_2 to make f, F, η to be
dimensionless.

$$C_1 = \frac{v}{4} \left(\frac{4g\beta(T_s - T_0)}{v^2} \right)^{1/4}, \quad C_2 = \left(\frac{4g\beta(T_s - T_0)}{v^2} \right)^{1/4}.$$

and we have

$$\left. \begin{aligned} f'''' + 3ff'' - 2(f')^2 + F &= 0 \\ F'' + 3Pr f F' &= 0 \end{aligned} \right\}$$

velocity
temp.

$$\text{w/ } f(0) = f'(0) = 0.$$

$$f(\infty) = 0$$

$$F(0) = 1, F(\infty) = 0$$

↳ to be solved numerically

Once f and F are obtained in terms of η

$$\psi(x, y) = \nu \cdot f \cdot 4 \left(\frac{Gr}{4} \right)^{1/4}, \quad \eta = \frac{y}{x} \left(\frac{Gr}{4} \right)^{1/4}$$

Gr (Grashof number) \equiv

$$\frac{g \beta (T_s - T_0) x^3}{\nu^2}$$

buoyancy
viscous f.

driving parameter for
natural convection.

(cf) Re is for

from momentum eq. $\rightarrow \rho u \frac{\partial u}{\partial x} \sim \rho g \beta (T_s - T_0)$ (forced convection)

$$\Rightarrow u \sim \sqrt{g \beta (T_s - T_0) x}$$



$$Re_x = \frac{u x}{\nu} \sim Gr_x^{1/2}$$

$$\cdot \underline{q_w} \equiv -k \frac{\partial T}{\partial y} \Big|_{y=0} = -k(T_s - T_0) \underline{F'(0)} \frac{(\frac{1}{2}Gr)^{1/4}}{x}$$

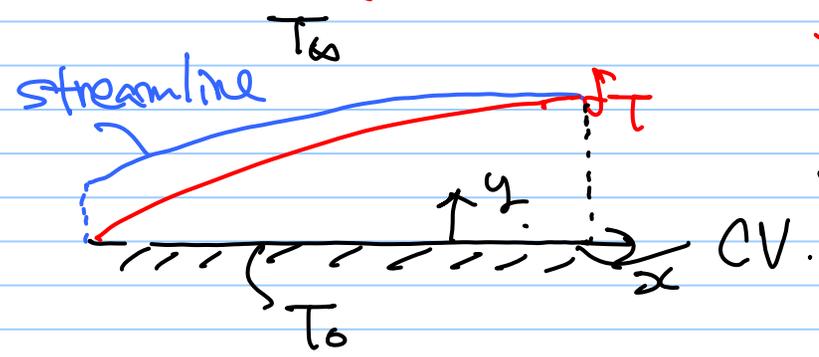
[wall]

→ h (convective heat-tr rate) $\equiv \frac{q_w}{T_s - T_0} \sim x^{-1/4}$

(in forced convection, $h \sim x^{-1/2}$)

$$\cdot Nu \equiv \frac{hx}{k} \sim Gr^{1/4}$$

⊙ Approximate methods for forced convection (thermal energy integral relation) (refer to Schlichting) write.



internal energy conservation.

if $dh = c_p \cdot dT$. $\rightarrow \frac{d}{dx} \left[\int_0^{\delta_T} (T_\infty - T) u \, dy \right] = \frac{k}{\rho c_p} \frac{\partial T}{\partial y} \Big|_{y=0}$ (*)

We need to assume "T" to solve (*) Streamwise velocity

Let $\frac{T - T_0}{T_\infty - T_0} (= \theta) = \frac{3}{2} \left(\frac{y}{\delta_T} \right) - \frac{1}{2} \left(\frac{y}{\delta_T} \right)^3$. w/ $\theta(0) = 0$
 $\theta(\delta_T) = 1$.

\rightarrow put into (*) and integrate.

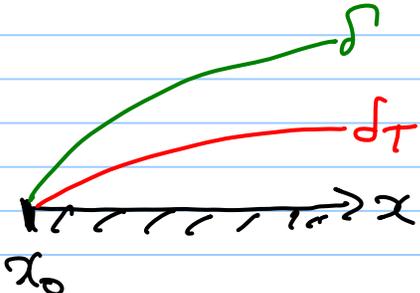
Case 1) $Pr \gg 1$. ($\rightarrow \delta_T \ll \delta$)

$$\frac{d}{dx} \left[\rho_\infty u_\infty \delta \left\{ \frac{3}{20} \left(\frac{\delta_T}{\delta} \right)^2 - \frac{3}{280} \left(\frac{\delta_T}{\delta} \right)^4 \right\} \right] = \frac{3}{2} \alpha \frac{\rho_\infty}{\delta} \left(\frac{\delta_T}{\delta} \right)$$

$$\rightarrow \left(\frac{\delta_T}{\delta} \right)^3 + \frac{4}{3} x \cdot \frac{d}{dx} \left(\frac{\delta_T}{\delta} \right) = \frac{13}{14} \cdot \frac{1}{Pr}$$

→ Assume that the solution $\frac{f_T}{f} = a + Cx^n$.

$$\rightarrow a = \frac{13}{14} Pr, \quad n = -3/4.$$

$$\therefore \frac{f_T}{f} = \frac{1}{1.026 \cdot \sqrt[3]{Pr}} \cdot \sqrt[3]{1 - \left(\frac{x_0}{x}\right)^{3/4}} \quad \text{--- (1)}$$


let $x_0 = 0$. (entire plate is heated)

$$\frac{f_T}{f} = \frac{1}{1.026 \cdot \sqrt[3]{Pr}}$$

$$h \equiv \frac{q_w}{T_0 - T_w} = \frac{-k \cdot \partial T / \partial y|_{y=0}}{T_0 - T_w} = \frac{3}{2} \left(\frac{k}{f_T} \right) \quad \text{--- (2)}$$

from the integral approx. for (mitan) BL

$$\frac{\delta}{x} \approx 4.64 \sqrt{\frac{\nu}{x U_0}} \quad \text{--- (3)}$$

①, ③ → ②.

$$h = 0.323 \frac{k}{x} \cdot \frac{\sqrt[3]{Pr} \cdot \sqrt{C_{fo} x / \nu}}{\sqrt[3]{1 - (x_0/x)^{3/4}}}$$

$$\left(\text{or } Nu \equiv hx/k = 0.323 \frac{\sqrt[3]{Pr} \cdot \sqrt{C_{fo} x / \nu}}{\sqrt[3]{1 - (x_0/x)^{3/4}}} \right)$$