· approx. solution for convection. — integral method.

case 1) $Pr >> 1$ $\delta_T / \delta << 1$

	$\frac{\delta}{x}\sqrt{Re_x}$	$C_f\sqrt{Re_x}$	$Nu/(\sqrt{Re_x}\cdot\sqrt[3]{Pr})$
Exact	5.0	0.664	0.332
y/δ	3.46	0.578	0.289
$\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	4.64	0.646	0.323

assumed u/u_∞ and θ/θ_∞.

case 2) $Pr < 1$. or $\delta_T > \delta$. (rare, liquid metal, ...)

→ Need to integrate the eqn in two parts ($0 \sim \delta$, $\delta \sim \delta_T$).

- According to Eckert,

$$Nu = \frac{\sqrt{Re \cdot Pr}}{1.55\sqrt{Pr} + 3.09\sqrt{0.372 - 0.15 Pr}}.$$

STABILITY (laminar BL)

- laminar flow: unstable, transition to turb. ($Re > Re_c$)
- most engineering flows are turb.
- given a perturbation, does the state return to its original one?
  - yes: stable
  - NO: unstable → disturbance grows in time and/or space.

In Curve,

- perturbation analysis.
  1. Basic solution for stable state, Q_0.
  2. $Q = Q_0 + Q'$. $(Q' \ll Q_0)$
     ↳ perturbation.
  3. Linearize the eqn in Q'.
  4. Eigenvalue prob.

- Stability of BL flow (parallel flows)
  - Assume that the streamwise velocity $V(x,y) \simeq V(y)$, then introduce perturbation as

$$u(x,y,t) = V(y) + u'(x,y,t), \qquad (u' \ll V)$$

$$v(x,y,t) = 0 + v'(x,y,t) \qquad (v' \ll V)$$

$$p(x,y,t) = P_0 + p'(x,y,t), \qquad (p' \ll P_0)$$

→ into the gov. eqns.

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 .$$

$$\frac{\partial u'}{\partial t} + (V + \cancel{u'})\frac{\partial u'}{\partial x} + v'\left(\frac{dV}{dy} + \cancel{\frac{\partial u'}{\partial y}}\right) = -\frac{1}{\rho}\left(\cancel{\frac{dP_0}{dx}} + \frac{\partial p'}{\partial x}\right) + \nu\left(\frac{\partial^2 u'}{\partial x^2} + \cancel{\frac{d^2 V}{dy^2}} + \frac{\partial^2 u'}{\partial y^2}\right)$$

$$\frac{\partial v'}{\partial t} + (V + \cancel{u'})\frac{\partial v'}{\partial x} + \cancel{v'\frac{\partial v'}{\partial y}} = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right)$$

· when, no-perturbation, $0 = -\frac{1}{\rho}\frac{dP_0}{dx} + \nu \frac{d^2 V}{dy^2}$ → Linearize.

small u', v', p'

let's introduce perturbation stream function.

$$u' = \frac{\partial \psi}{\partial y}, \quad v' = -\frac{\partial \psi}{\partial x}.$$ → into linearized eq.

$$\rightarrow \frac{\partial^2 \psi}{\partial y \partial t} + V \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \cdot \frac{dV}{dy} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu \left(\frac{\partial^3 \psi}{\partial^2 x \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right) \quad \text{—①}$$

$$-\frac{\partial^2 \psi}{\partial x \partial t} - V \frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{\rho}\frac{\partial p'}{\partial y} - \nu \left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial x \partial y^2} \right) \quad \text{—②}$$

$\hookrightarrow \frac{\partial}{\partial y}(①)$ & $\frac{\partial}{\partial x}(②) \rightarrow \frac{\partial^2 p'}{\partial x \partial y}$ in common.

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}\right) - \frac{d^2 V}{dy^2}\cdot\frac{\partial \psi}{\partial x} = \nu\left(\frac{\partial^4 \psi}{\partial y^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial x^4}\right) \quad - ※$$

(4^{th}-order, linear PDE).

since, the perturbation is arbitrary

↳ "Fourier-series" representation

in "x"-direction.

$$\psi(x,y,t) = \int_0^\infty \underbrace{\Phi(y)}_{\text{amplitude}}\, \underbrace{e^{i\alpha(x-ct)}}_{\text{time-variant part}} \cdot d\alpha.$$

↳ amplitude.

→ time-variant part : $e^{-i\alpha ct}$.

α: wavenumber (real, positive)

into ⊛.

imaginary part.

$\mathrm{Im}(c) > 0$: dist. grows.

$\mathrm{Im}(c) < 0$: " decays

$\mathrm{Im}(c) = 0$: Neutrally stable.

$$(V-c)(\Phi''-\alpha^2\Phi)-V''\Phi = \frac{\nu}{i\alpha}\left(\Phi^{(4)}-2\alpha^2\Phi''+\alpha^4\Phi\right)$$

: Orr-Sommerfeld equation.

BC's : disturbance should vanish @ $y=0$, & @ $y\to\infty$.

$$\left\{\begin{array}{l} u'(x,0,t)=v'(x,0,t)=0 \\ u'(x,\infty,t)=v'(x,\infty,t)=0 \end{array}\right. \Rightarrow \left|\begin{array}{l}\Phi(0)=\Phi'(0)=0 \\ \Phi(\infty)=\Phi'(\infty)\to 0.\end{array}\right.$$

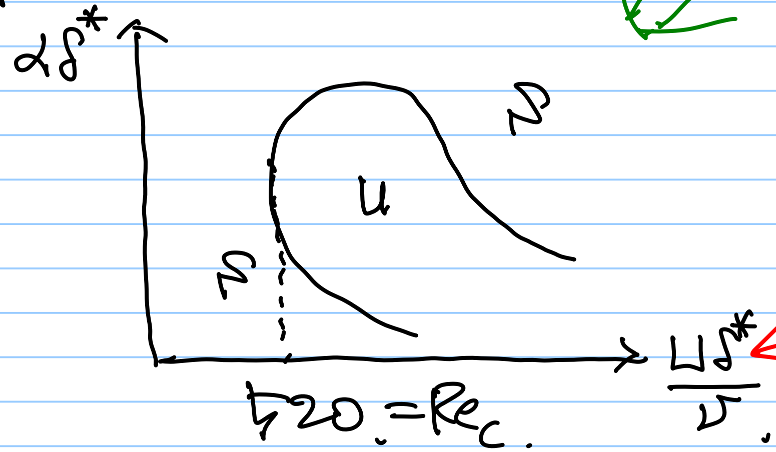
· Solution process of O-S eq.

- for a given $V(y)$ (undisturbed velocity) and α (wave number), solve O–S equation as an eigenvalue problem. $\rightarrow$ repeat for all $V(y)$

$0 < V(y) < U(x)$

free-stream.

· for Laminar BL

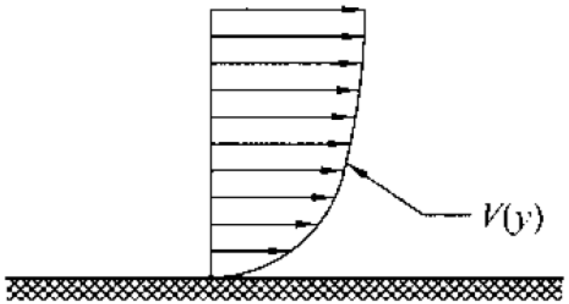


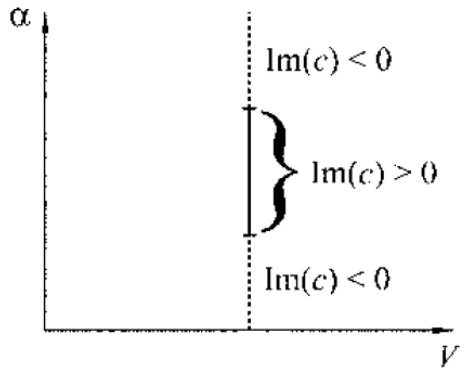
$\leftarrow$ displacement thickness.

- O-S eqn
  - linearized disturbance relation for BL flows
  - can be solved for a variety of flow
  - predicts Tollmien-Schlitching waves. (T-S wave)
    - ↳ first indication of laminar flow instability (transition to turbulence)

Orr-Sommerfeld Equation

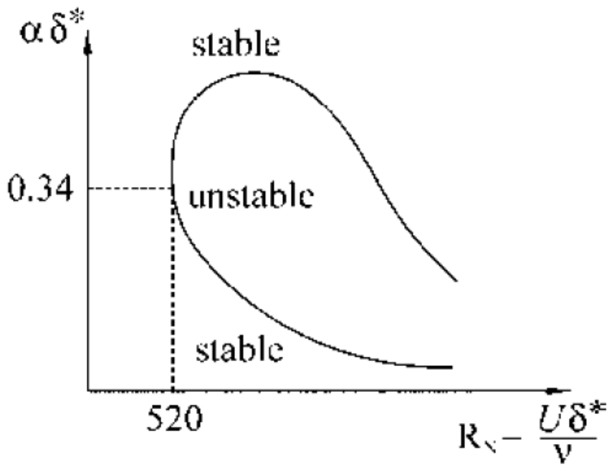
- Solution process of O-S equation
  1. For a given $V(y)$ (undisturbed velocity profile) and α (wavelength), solve the O-S equation as an eigenvalue problem (for the time constant c)





UNSTABLE

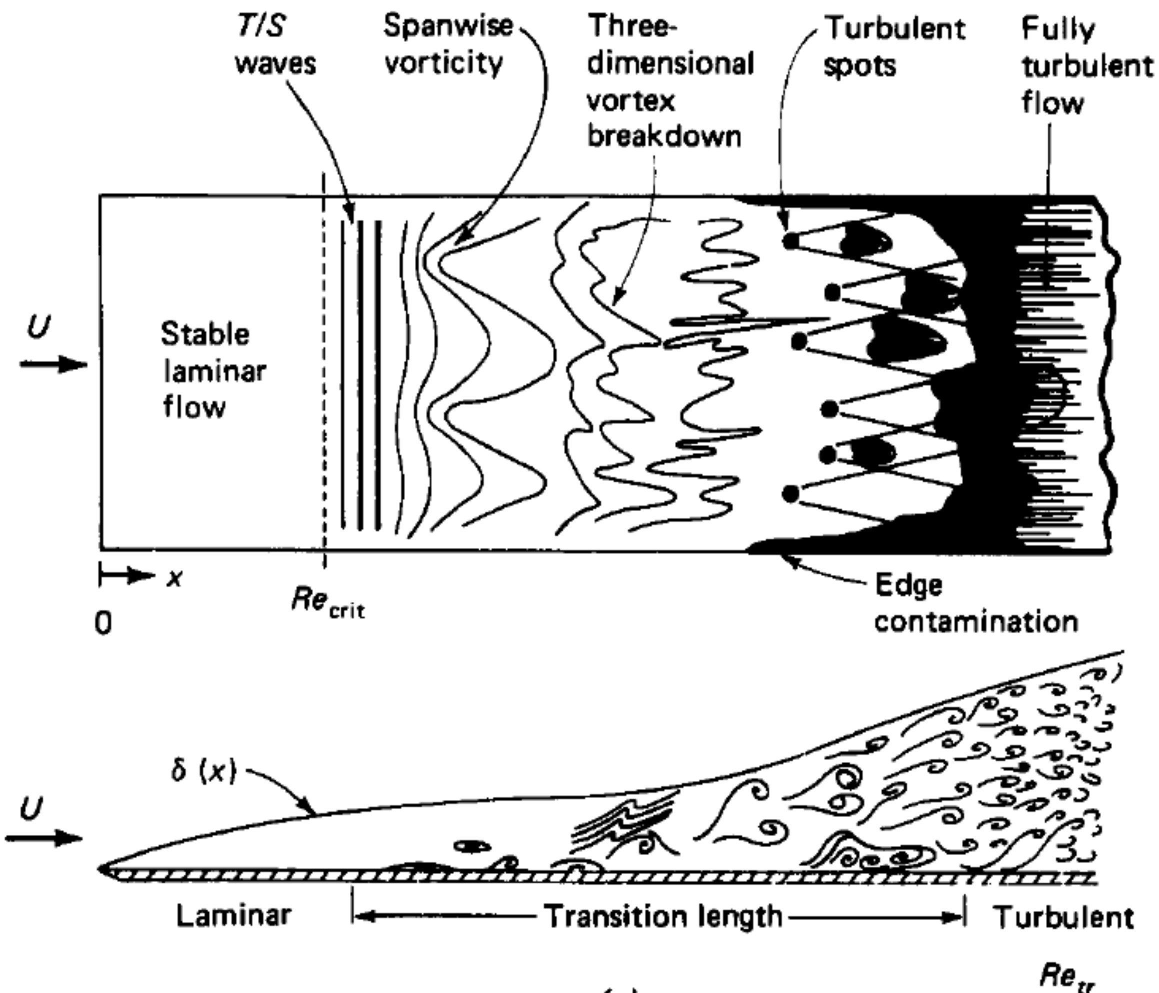
  2. Repeat (1) for all possible values of $V(y)$; $0 < V(y) < U(x)$ ← Free-stream vel.



(for a flat-plate BLDY flow)

From Currie "Fundamental Mechanics of Fluids"

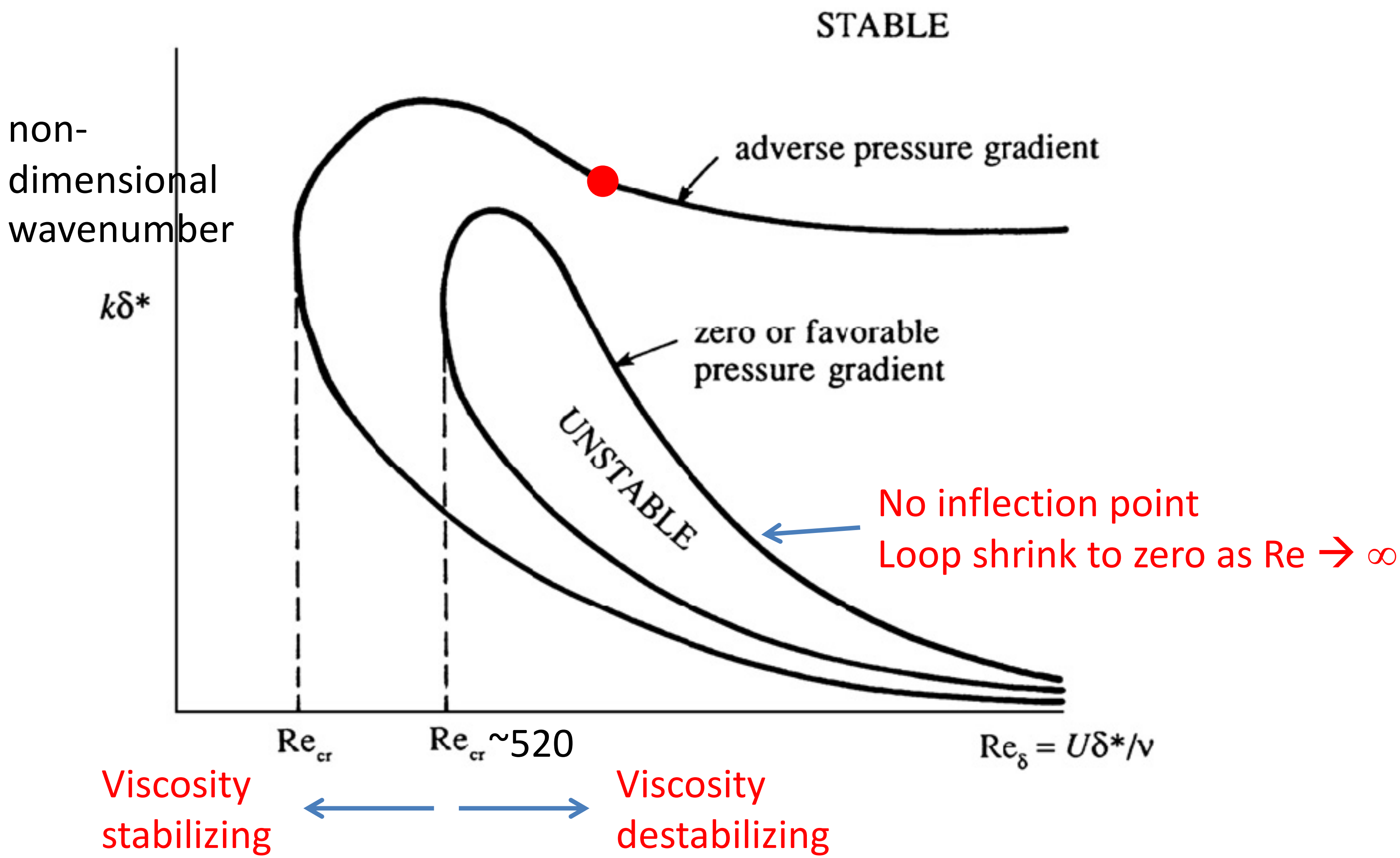
Description of the BDLY Transition Process



(a)

From White “Viscous Fluid Flow”

Stability curves for laminar boundary layers



The addition of the inflection point in the adverse-pressure gradient case increases the parametric realm of instability.

⓪ Inviscid stability analysis for parallel flows.

· Orr-Sommerfeld eq.

$$(\alpha U-\omega)(\hat{v}''-\alpha^2\hat{v})-\alpha(U''\hat{v})+\frac{i}{Re}\left[\hat{v}^{(4)}-2\alpha^2\hat{v}''+\alpha^4\hat{v}\right]=0$$

undisturbed flow

amplitude of perturbation.

↪ Analytical solutions are difficult to obtain.

⇒ inviscid stability ($Re\to\infty$)

$$(\alpha U-\omega)(\hat{v}''-\alpha^2\hat{v})-\alpha U''\hat{v}=0$$: Rayleigh eq.

Let's consider temporal instability

$$\begin{cases}\alpha=\alpha_r+i\alpha_i=\alpha_r & (\alpha_i=0\text{, spatially stable})\\ \omega=\omega_r+i\omega_i & (\text{for } \boxed{\text{temporally unstable, } \omega_i>0})\end{cases}$$

$$\hat{v}'' = \alpha_r^2 \hat{v} + \frac{\alpha_r U'' \hat{v}}{\alpha_r U - \omega_r - i\omega_i} \quad \leftarrow \text{Introduce complex conjugate } (*)$$

$$\hat{v}''^{*} = \alpha_r^2 \hat{v}^{*} + \frac{\alpha_r U'' \hat{v}^{*}}{\alpha_r U - \omega_r + i\omega_i}$$

$$\rightarrow \hat{v}'' \hat{v}^{*} - \hat{v}''^{*} \hat{v} = \alpha_r U'' |\hat{v}|^2 \frac{2i\omega_i}{|\alpha_r U - \omega|^2} \quad \leftarrow \text{Integrate } \int_0^\infty dy.$$

@ $y = 0$, no-slip $\rightarrow \hat{u} = \hat{v} = 0$.

@ $y \to \infty$, $\hat{u} = \hat{v} = 0$ (no disturbance in potential flow regime)

$$2i\omega_i \int_0^\infty \frac{U'' |\hat{v}|^2}{\left|U - \left|\frac{\omega}{\alpha_r}\right|\right|^2} dy = 0.$$

since, $\omega_i \neq 0$ (temporally unstable)

$$U'' = \frac{\partial^2 U}{\partial y^2} = 0.$$ (necessary cond. for instability)

- Rayleigh criterion for instability
  : needs at least one inflection point in U.
- Fjortoft criterion for instability
  : magnitude of vorticity of base flow must have a max. within the flow.