INTRODUCTION TO LINEAR ALGEBRA

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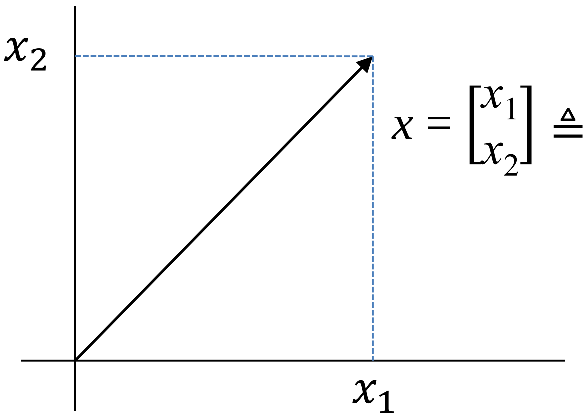
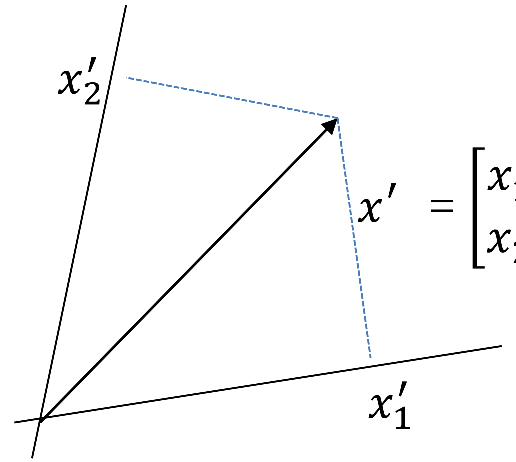
Vector

- *Vector Representation*



$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq (x_1, x_2)$$



$$x' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} \triangleq (x'_1, x'_2)$$

- *Inner Product, Dot Product*

$$x^T y = x \cdot y = < x, y > = [x_1 \ x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2$$

Matrix

- *Matrix Notation*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

→ *A* is an *m* x *n* matrix (*m* by *n* matrix)

m: number of rows; *n*: number of columns

- *Matrix Addition, scalar multiplication*

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 9 \\ 11 & 13 \end{bmatrix}, \quad 5\begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 \\ 10 & 15 & 10 \end{bmatrix}$$

Matrix

- *Matrix Multiplication*

— *Matrix & Vector Multiplication*

$$Ax = \begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} x = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 5 + 6 \cdot 0 \\ 3 \cdot 2 + 0 \cdot 5 + 3 \cdot 0 \\ 1 \cdot 2 + 1 \cdot 5 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

— *Matrix & Matrix Multiplication*

$$AB = \begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 5 + 6 \cdot 0 & 1 \cdot 3 + 1 \cdot 4 + 6 \cdot 1 \\ 3 \cdot 2 + 0 \cdot 5 + 3 \cdot 0 & 3 \cdot 3 + 0 \cdot 4 + 3 \cdot 1 \\ 1 \cdot 2 + 1 \cdot 5 + 4 \cdot 0 & 1 \cdot 3 + 1 \cdot 4 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 7 & 13 \\ 6 & 12 \\ 7 & 11 \end{bmatrix}$$

Matrix

- Properties of matrix multiplication

  (1) Matrix multiplication is associative: $(AB)C = A(BC)$

  (2) Matrix operations are distributive:

  $A(B + C) = AB + AC$ and $(B + C)D = BD + CD$

  (3) Matrix multiplication is not commutative, i.e., $AB \neq BA$ in general

- *Def*: The identity matrix I is the matrix that leaves every matrix unchanged when multiplied to that matrix.

  → I is an n x n matrix with 1's on the diagonal and 0's everywhere else

  → I is commutative under matrix multiplication

Matrix

- The transpose of a matrix A is denoted by A^T
  - $(A^T)_{ij} = A_{ji}$
  - The ith row of A^T = The row vector from the ith column of A
  - A: m x n, then A^T: n x m
- Some important results
  - $(AB)^T = B^T A^T$
  - $(A^{-1})^T = (A^T)^{-1}$
- *Def*: A is called a symmetric matrix if $A^T = A$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & 7 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 4 \\ 3 & 5 \\ 2 & 7 \end{bmatrix}$$

Gaussian Elimination

- *Linear equation*

$$\begin{aligned} y+z&=2 \\ 2x+2y+5z&=9 \\ 2x+3y+4z&=9 \end{aligned}$$

- *Sol*:

$$\begin{bmatrix} 0 & 1 & 1 & | & 2 \\ 2 & 2 & 5 & | & 9 \\ 2 & 3 & 4 & | & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 5 & | & 9 \\ 0 & 1 & 1 & | & 2 \\ 2 & 3 & 4 & | & 9 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 2 & 5 & | & 9 \\ 0 & 1 & 1 & | & 2 \\ 0 & 1 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 5 & | & 9 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & -2 & | & -2 \end{bmatrix}$$

$\therefore z = 1, y = 1, x = 1$

→ This is the usual variable elimination

Gaussian Elimination

- *Linear equation* $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 5 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ 9 \end{bmatrix}$

$$\begin{aligned} y + z &= 2 \\ 2x + 2y + 5z &= 9 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

- *Sol*: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 5 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

E_1: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

E_2: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & 2 & 5 & 9 \\ 2 & 3 & 4 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 5 & 9 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 4 & 9 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 5 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 5 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$\rightarrow E_2E_1A = U \rightarrow A = LU, \qquad L = (E_2E_1)^{-1}$$

Gaussian Elimination

- *Factorization* *A=LDU*

  For *A=LU'* where *U'* is invertible,

$$U' = \begin{bmatrix} d_1 & u_{12} & \cdots & u_{1n} \\ 0 & d_2 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & \cdots & u_{1n}/d_1 \\ 0 & 1 & \cdots & u_{2n}/d_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = DU$$

$\therefore$ We can obtain *A=LDU* , where *L* and *U* have 1's on the diagonal and *D* is the diagonal matrix

$$A = LU' = LDU, \qquad L = (E_2 E_1)^{-1}$$

$$Ax = y \rightarrow x = A^{-1}y, A^{-1} = U^{-1}D^{-1}L^{-1}$$

$$E_{ij} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & -l & 1 & \\ & & & & 1 \end{bmatrix} \Rightarrow E_{ij}^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & l & 1 & \\ & & & & 1 \end{bmatrix}$$

Gaussian Elimination

- *Gauss-Jordan method to find* A^{-1}

  $AA^{-1} = \mathrm{I}$

  We can perform elementary operations to A to get $E_n \cdots E_2 E_1 A = \mathrm{I}$

  By applying A^{-1} to both side, $E_n \cdots E_2 E_1 \mathrm{I} = A^{-1}$

- *One simple way to find* A^{-1}

  By applying elementary operations to $[A \mid \mathrm{I}\,] \rightarrow [\,\mathrm{I} \mid A^{-1}]$

Gaussian Elimination

- Example $A=\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$. Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -5/8 & -3/4 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -5/16 & -3/8 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}$$

Gaussian Elimination

- Computational Complexity of A^{-1}

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}. \text{ Find } A^{-1}.$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

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$2n$ multiplications
$2n$ additions
$\rightarrow 4n$ flops or 2n flops

n times operations for U
n times operations for L
$\rightarrow 4n(n)(n)$ flops
$\approx 4n^3 \rightarrow O(n^3)$

Vector Space

- *Linear Combination*

  For *vectors* **v**, **w** $\in V$ and *scalars* $\alpha, \beta \in F\ (R\ or\ C)$,

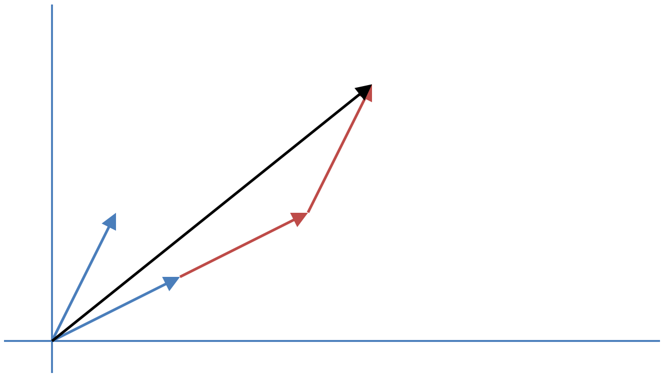
  $\alpha\mathbf{v} + \beta\mathbf{w}$ is a linear combination of **v** and **w**.

- *Vector space*

  For any *vectors* **v**, **w** $\in V$ and *any scalars* $\alpha, \beta \in F\ (R\ or\ C)$,

  if $\alpha\mathbf{v} + \beta\mathbf{w} \in V$, V becomes a vector space.

$$2\begin{bmatrix}2\\1\end{bmatrix}+1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix} \in V$$



Vector Space

- *Vector space*

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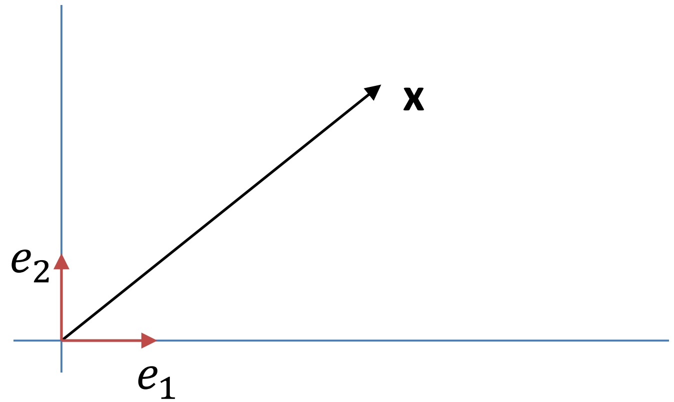

For any *vector* $\mathbf{x} \in V$, $\mathbf{x}$ can be spanned by a linear combination of a basis $\{e_1, e_2\}$ which is not unique.

$$\mathbf{x} = 5\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} = 5e_1 + 4e_2 = 2\begin{bmatrix}2\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\end{bmatrix} = 2b_1 + 1b_2$$

Vector Space

For any *vector* **x** $\in$ *V*, **x** can be spanned by a linear combination of a basis $\{e_1, e_2\}$ which is not unique.

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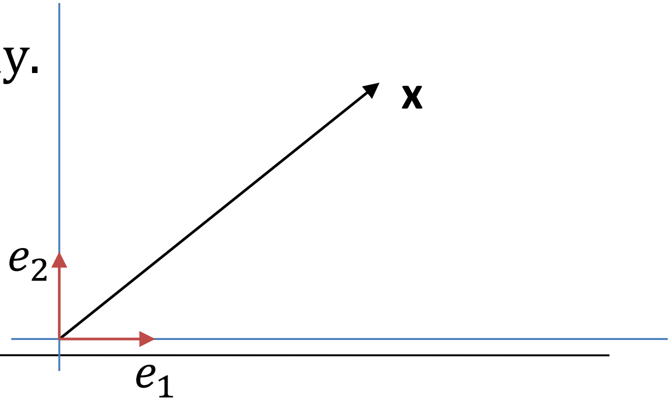
$$\mathbf{x} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix} = [e_1 \quad e_2]\begin{bmatrix}5\\4\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = [b_1 \quad b_2]\begin{bmatrix}2\\1\end{bmatrix}$$

$$\mathbf{x} = E\begin{bmatrix}5\\4\end{bmatrix} = B\begin{bmatrix}2\\1\end{bmatrix}$$

$\begin{bmatrix}5\\4\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ are the vector representations of a vector **x** with respect to the basis of $\{e_1, e_2\}$ and $\{b_1, b_2\}$, respectively.



Vector Space

$\begin{bmatrix}5\\4\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ are the vector representations of a vector **x** with respect to the basis of $\{e_1, e_2\}$ and $\{b_1, b_2\}$, respectively.

Basis is defined by a set of maximum number of linearly independent vectors in a vector space V.

Linearly independent:

For *vectors* $\boldsymbol{v_i} \in$ *V*, $i = 1, \ldots, n$ and *scalars* $\alpha_i \in F$, $\{\boldsymbol{v_i}\}$ are L.I when a linear combination $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v_2} + \cdots + \alpha_n \boldsymbol{v_n} = \boldsymbol{0}$ if only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \boldsymbol{0}$. Otherwise, $\{\boldsymbol{v_i}\}$ are linearly dependent.

Dimension of a vector space *V* : maximum number of linearly independent vectors in a vector space V.

Vector Space

Linearly independent:

For *vectors* $\boldsymbol{v_i} \in V$, $i = 1, \dots, n$ and *scalars* $\alpha_i \in F$, , $\{\boldsymbol{v_i}\}$ are L.I when a linear combination $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v_2} + \cdots + \alpha_n\boldsymbol{v_n} = \mathbf{0}$ if only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \mathbf{0}$.

Example:

Are (3 0 0), (4 1 0), (2 5 2) L.I. ?

$$\alpha_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = 0 \rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0} \rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0}$$

Vector Space

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$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0} \rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0} \rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0}, \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}, \; \dots \dots$$

$$\rightarrow solution\ set = \left\{ \alpha \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \middle| \; \alpha \in R \right\} \rightarrow Null\ space\ of\ A = \{x | Ax = 0\}$$

Vector Space

Example:

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0} \rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mathbf{0}, \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}, \ \ldots\ldots$$

$$\rightarrow solution\ set = \left\{ \alpha \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \mid \ \alpha \in R \right\} \rightarrow Null\ space\ of\ A = \{x | Ax = 0\}$$

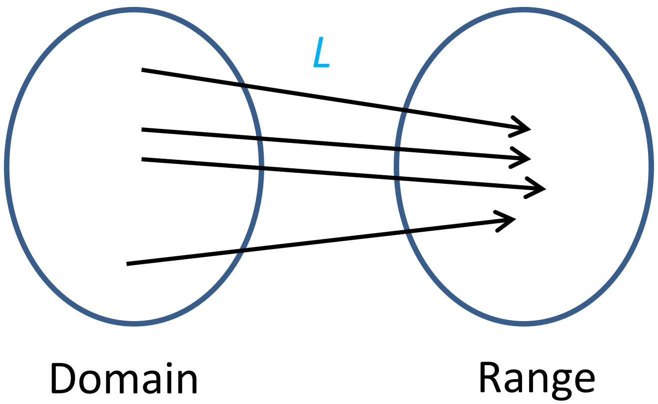
$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 2 \end{bmatrix} \neq \mathbf{0}, \quad \text{for} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin N(A)$$

$$\alpha_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = (\alpha_1 + 2\alpha_2) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\rightarrow set = \{b | \ Ax = b\} \rightarrow Range\ space\ of\ A = R(A)$$

Linear Transformation

- Transformation = mapping = function



- *Def*: Let *L* be a transformation on a vector space *V*. Then *L* is called a *linear transformation* if $L(cx + dy) = cL(x) + dL(y)$ for all $x, y \in V$ and all numbers c and d.
- Any linear transformation maps 0 to 0

Linear Transformation

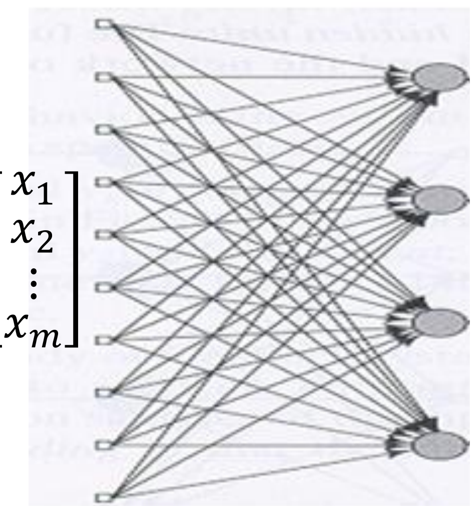
- $n \times m$ matrix $A \in R^{n\times m}$ is a linear operator for linear transformation from R^m to R^n, i.e., $A: R^m \rightarrow R^n$.
- For $y \in R^n, x \in R^m$, $y = Ax$.

  $n \times 1 = \quad n \times m \quad \cdot \quad m \times 1$

- $$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

  $$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$



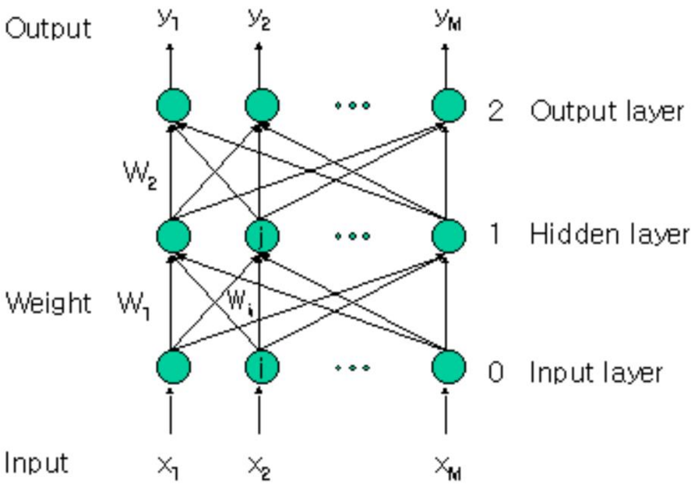
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

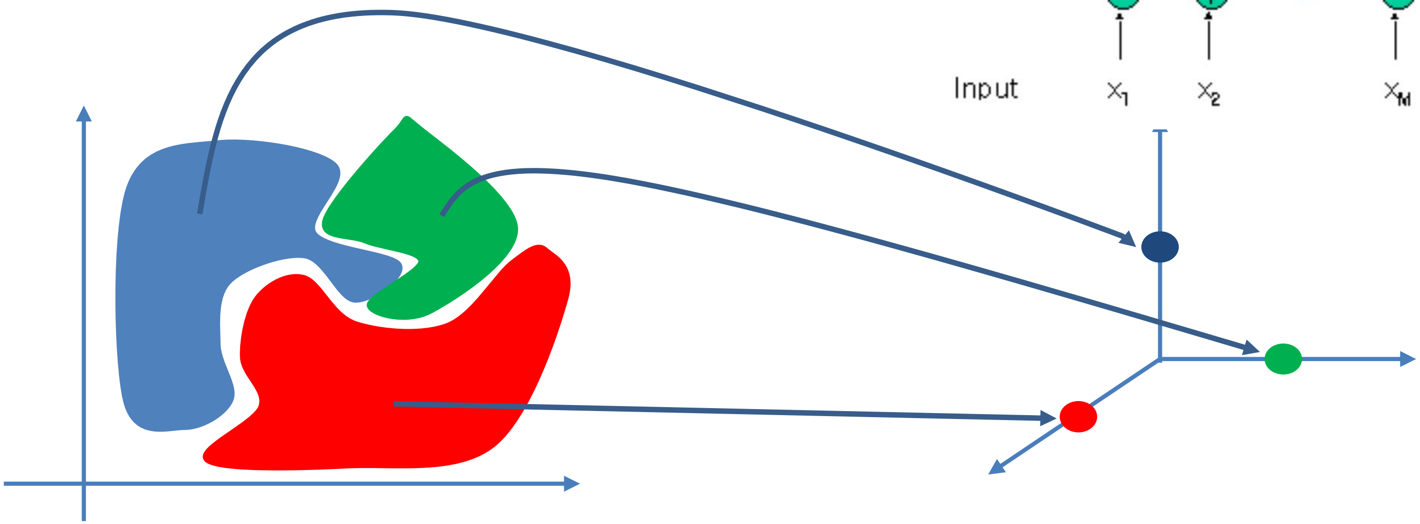
- Neural network: $y = \sigma(Ax) \leftarrow non - linear\ transformation$

Linear Transformation

- Neural network

  $y = \sigma(\cdots \sigma(W_2 \sigma(W_1 x)) \cdots)$





Vector Operations

- Norm: Length of a vector $x = [x_1 \quad \cdots \quad x_n]^T$

  $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$

  $\|x\|_\infty = \max_i |x_i|$

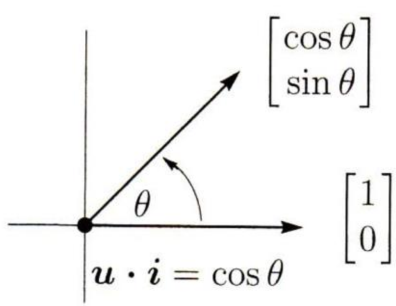
- Inner product (scalar product or dot product) of x and y :

$$x^T y = [x_1 \quad \cdots \quad x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

- x and y are orthogonal if and only if

  $x^T y = 0.$

  $x^T y = \|x\| \|y\| cos\theta$



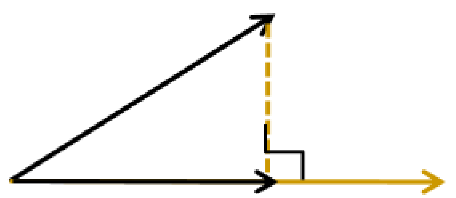
Vector Operations

- Projection of y onto x:

$$x \perp (y - ax) \leftrightarrow x^T(y - ax) = 0 \leftrightarrow x^T y = a x^T x$$

$$\therefore a = \frac{x^T y}{x^T x} = \frac{\|x\|\|y\| cos\theta}{\|x\|^2}$$

$$\therefore ax = \frac{x^T y}{x^T x} x = \frac{\|y\| cos\theta}{\|x\|} x$$



- Transpose of inner products:

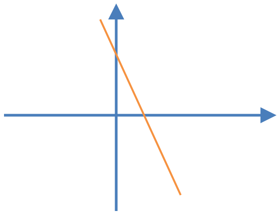
$$(Ax)^T y = x^T A^T y = x^T (A^T y)$$

$$(ABx)^T y = \qquad\qquad = x^T (B^T A^T) y$$

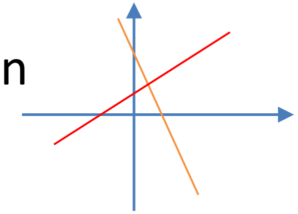
Least Squares

- Linear equation: $Ax = b$

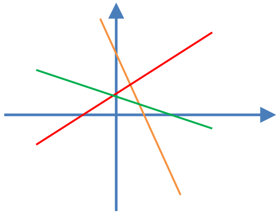
$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow 2x_1 + x_2 = 1$ →many solutions (부정)



$\begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ →unique solution



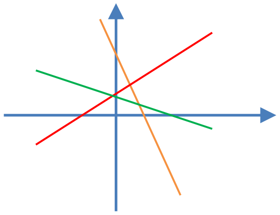
$\begin{bmatrix} 4 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ no solutions (불능)

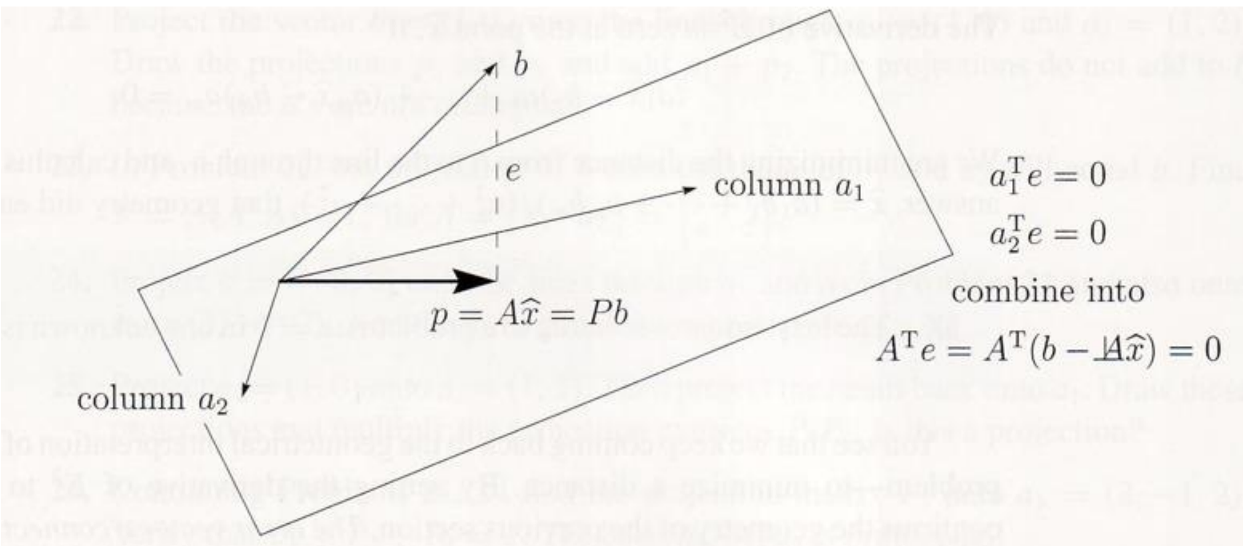


Least Squares

- Linear equation: $Ax = b$

$$\begin{bmatrix} 4 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ → no solutions (불능, inconsistent)

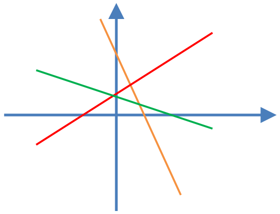




Least Squares

- Linear equation: $Ax = b$

$$\begin{bmatrix} 4 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ → no solutions (불능)



→ $Ax - b \neq 0$ → What is the best solution minimizing errors between Ax and b?

- Optimization and Gradient

$$\hat{x} = arg\min_{x} \|Ax - b\|_2^2 \triangleq E(x)$$

$$\nabla_x E(x) = \frac{dE(x)}{dx} = \frac{d}{dx}(Ax - b)^T(Ax - b) = 2A^T(Ax - b) = 0$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

Derivative of Multi-variable Function

- Two variable functional for $x = (x_1, x_2)$

$$f(x) = \|Ax\|_2^2 + b^T x, \text{ where } A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$f(x) = (Ax)^T (Ax) + b^T x = \begin{bmatrix} x_1 + 2x_2 \\ x_1 + x_2 \end{bmatrix}^T \begin{bmatrix} x_1 + 2x_2 \\ x_1 + x_2 \end{bmatrix} + x_1 + 2x_2$$

$$f(x) = 2x_1^2 + 6x_1x_2 + 5x_2^2 + x_1 + 2x_2$$

- Gradients

$$\nabla_x f(x) = \frac{d}{dx} f(x) = \begin{bmatrix} \partial f(x)/\partial x_1 \\ \partial f(x)/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 + 6x_2 + 1 \\ 6x_1 + 10x_2 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 2A^T Ax + b$$

Derivative of Multi-variable Function

- Two variable functional for $x = (x_1, x_2)$

$$f(x) = \|Ax\|_2^2 + b^T x, \text{ where } A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$f(x) = (Ax)^T (Ax) + b^T x = 2x_1^2 + 6x_1 x_2 + 5x_2^2 + x_1 + 2x_2$$

- Gradient

$$\nabla_x f(x) = \begin{bmatrix} \partial f(x)/\partial x_1 \\ \partial f(x)/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 + 6x_2 + 1 \\ 6x_1 + 10x_2 + 2 \end{bmatrix} = 2A^T A x + b$$

- Hessian or Hessian Matrix H : Second derivative of $f(x)$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix} = 2A^T A$$

Derivative of Multi-variable Function

- Gradient of $f(x)$: $\nabla_x f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$

- Hessian of $f(x)$: $H(f)$

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Derivative of Multi-variable Function

- Jacobian of $f(x)$: First derivative of a vector function $\mathbf{f}(x) = [f_1(x), \ldots, f_m(x)]^T$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

- Example

$$\mathbf{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 y \\ 5x + \sin y \end{bmatrix}$$

$$\mathbf{J}_\mathbf{f}(x,y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$$

$$\frac{d}{dx} Ax = ?$$

Derivative of Multi-variable Function

- Chain Rule of Derivatives

$$f\big(x(t), y(t)\big) = 2x(t)^2 + y(t)^2, x(t) = r\sin(t), y(t) = r\cos(t)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 4x(t)r\cos(t) - 2y(t)r\sin(t) = 2r\cos(t)\sin(t)$$

- Example

$$f(x) = \|Ax - b\|_2^2 + c^T x,\ f(x) = (Ax - b)^T (Ax - b) + c^T x$$

$$f(x) = p\big(g(x)\big) + h(x),$$

where, $p\big(g(x)\big) = g(x)^T g(x),\ g(x) = Ax - b,\ h(x) = c^T x$

$$\frac{df}{dx} = \frac{\partial g}{\partial x}\frac{\partial p}{\partial g} + \frac{\partial h}{\partial x} \rightarrow \nabla_x = J_x g \nabla_g p + \nabla_x h$$

$$= 2A^T(Ax - b) + c$$

Determinant

- Main uses of determinants

  1. Test of invertibility

     det $(A) \neq 0 \quad \Leftrightarrow \quad A$ is invertible

  2. det(A) equals the volume of a box in n-dimensional space.

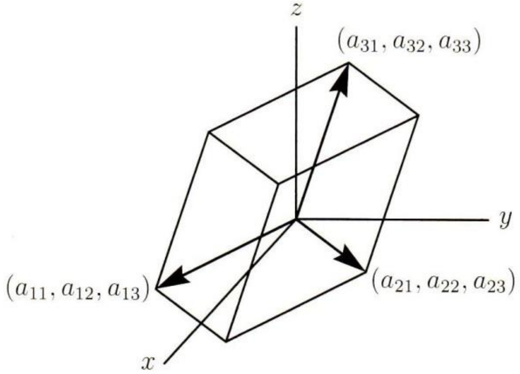


Figure 4.1 The box formed from the rows of A: volume $= |\,\text{determinant}\,|$.

Determinant

- Calculation rule

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix}+\begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32})+a_{12}(a_{23}a_{31}-a_{21}a_{33})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

$$= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}$$

Eigenvalues and Eigenvectors

- Eigenvectors and eigenvalues of a matrix:

$$\boldsymbol{A}\boldsymbol{v_i} = \lambda_i \boldsymbol{v_i}, \qquad \boldsymbol{i = 1, 2, \dots, N}$$

$[\boldsymbol{A}-\lambda_i \mathbf{I}]\boldsymbol{v_i} = 0$ $\Longrightarrow \exists\, N$ linearly independent vectors $\{\boldsymbol{v_i}\}$ if $\rho[\boldsymbol{A} - \lambda_i \mathbf{I}] = N - 1$

$\Longrightarrow N$ eigenvectors $\{\boldsymbol{v_i}\}$

$\Longrightarrow \{\boldsymbol{v_i}\}$ becomes a basis of the eigenspace of $\boldsymbol{A}$

$\Longrightarrow$ span the eigenspace of $\boldsymbol{A}$

$\Longrightarrow |\boldsymbol{A} - \lambda_i \mathbf{I}| = 0$ (determinent)

Eigenvalues and Eigenvectors

- Example $A=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\rightarrow |A-\lambda I|=(2-\lambda)(4-\lambda)-3$

$$\rightarrow \lambda^2-6\lambda+5=0 \ \rightarrow \lambda=1,5$$

$$\lambda=1:\ (A-\lambda I)x=0 \Leftrightarrow \begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=0$$

$$\therefore\ \text{eigenvector } x \text{ associated with eigenvalue 1 is } c\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\lambda=5:\ (A-\lambda I)x=0 \Leftrightarrow \begin{bmatrix}-3 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=0$$

$$\therefore\ \text{eigenvector } x \text{ associated with eigenvalue 5 is } c\begin{bmatrix}1\\ 3\end{bmatrix}$$

Eigenvalues and Eigenvectors

- Diagonalization:

  Define $\boldsymbol{v}_{\boldsymbol{i}}^{\boldsymbol{T}} = [v_{i1} \ \cdots \ v_{ii} \ \cdots v_{iN}]$,
  $\boldsymbol{A}\boldsymbol{v}_{\boldsymbol{i}} = \lambda_i \boldsymbol{v}_{\boldsymbol{i}}$;

$$\Rightarrow \boldsymbol{A}[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_N] = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_N]\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \lambda_i & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \lambda_N \end{bmatrix}$$

$$\Rightarrow \boldsymbol{AV} = \boldsymbol{V\Lambda} \quad \Rightarrow \boldsymbol{A} = \boldsymbol{V\Lambda V^{-1}} \quad \Rightarrow \boldsymbol{\Lambda} = \boldsymbol{V^{-1}AV}$$

Eigenvalues and Eigenvectors

- Orthonormal eigenvectors of $\boldsymbol{A}$:

$$\boldsymbol{A}\boldsymbol{v_i} = \lambda_i \boldsymbol{v_i}; \;\; \boldsymbol{v_i^T}\boldsymbol{v_j} = \begin{cases} 1 & if\ i = j \\ 0 & otherwise \end{cases}$$

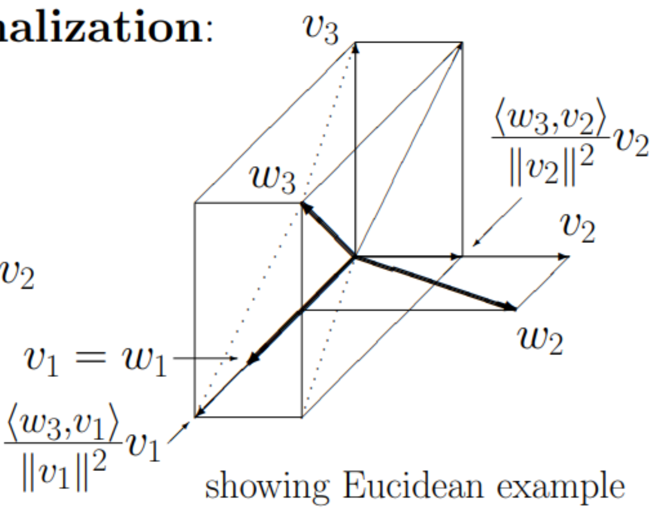
Gram-Schmidt orthogonalization:

$$v_1 = w_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\ldots$$



showing Eucidean example

$$\boldsymbol{V^T V} = \mathbf{I} \;\Rightarrow\; \boldsymbol{V^T} = \boldsymbol{V^{-1}} \;\Rightarrow\; \boldsymbol{A} = \boldsymbol{V \Lambda V^{-1}} = \boldsymbol{V \Lambda V^T}$$

Positive Definite Matrices

- Optimization problem

$$\min_{(x,y)} f(x,y), \quad \text{where } f(x,y) = 2x^2 + 4xy + 3y^2 - 2x$$

- Necessary condition

$$\nabla_{(x,y)} f(x,y) = \begin{bmatrix} 4x + 4y - 2 \\ 4x + 6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightarrow y = -1, y = \frac{3}{2}$$

- Necessary and Sufficient condition (Convexity)

$$H(f) = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} > 0 \quad \leftarrow \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4v_1^2 + 8v_1v_2 + 6v_2^2$$

$$\nabla_{(x,y)} f(x,y) = 0, \qquad = 4(v_1 + v_2)^2 + 2v_2^2 > 0$$

- A is positive definite matrix, i.e., $A > 0$ iff $v^T A v > 0$ for $\forall v \neq 0$.
- A is positive semi-definite matrix, i.e., $A \geq 0$ iff $v^T A v \geq 0$ for $\forall v \neq 0$.

Positive Definite Matrices

- Definitness of Matrix
  - If A is positive definite matrix, all $\lambda_i(A) > 0$
  - If A is positive semi-definite matrix, all $\lambda_i(A) \geq 0$
  - If A is negative definite matrix, all $\lambda_i(A) < 0$
  - If A is negative semi-definite matrix, all $\lambda_i(A) \leq 0$
  - If A is indefinite matrix, some $\lambda_i(A) > 0$, some $\lambda_i(A) < 0$

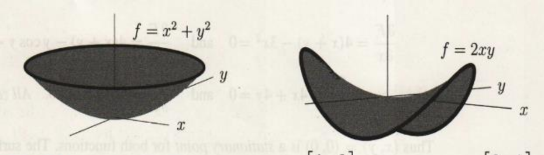


Figure 6.1 A bowl and a saddle: Definite $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and indefinite $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- Example

$$f = \frac{1}{2}(x^2 + y^2), \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; v^T H v = v_1^2 + v_2^2 > 0 \rightarrow H > 0$$

$$f = \frac{1}{2}(x^2 + 2xy + y^2), \; H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; v^T H v = (v_1 + v_2)^2 \geq 0$$

$\rightarrow H \geq 0 \rightarrow f = (x+y)^2$ has minimum at $x = -y$

$$f = xy, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; v^T H v = 2 v_1 \, v_2 \; \nleq \ngeq 0 \rightarrow H \; \nleq \ngeq 0$$

Exercise

- Solve using Gaussian elimination

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Exercise

- Find A^{-1} using Gaussian elimination
  $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

Exercise

- Find the null space and range space of A

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise

- Find eigenvalues and eigenvectors of A

  $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

Exercise

- Find $\|x\|_2$ and $x^T y$ for

$$x = [2 \quad 7 \quad 3]^T, \quad y = Ax, \quad A = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & -2 \\ -1 & 2 & 7 \end{bmatrix}$$

Exercise

- Find x that minimize $\|Ax - b\|_2$ for

$$A = \begin{bmatrix} 4 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Exercise

- For $f(x) = 21x_1^2 + 14x_1x_2 + 6x_2^2 + 2x_1 + x_2$
  findGradient $\nabla_x f$, Hessian $H(f)$.

Exercise

- For $f(x) = \|Ax\|_2^2 + b^T x$, where
  $$A = \begin{bmatrix} 4 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ b = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
  find Gradient $\nabla_x f$, Hessian $H(f)$.

Exercise

- For $f(x) = \begin{bmatrix} 2x_1^3 + x_3^2 \\ x_1 + 3x_2^5 \\ x_1^2 + e^{2x_2} + 3x_3^4 \end{bmatrix}$,
  find Jacobian $J(f)$.

Exercise

Find x, y such that $\min_{(x,y)} f(x, y)$, where $f(x, y) = x^2 + 4xy + 3y^2 + 3x$