Introduction to Photonics

The Vectorial Nature of Light (1)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Source-Free Wave Equations

Isotropic nonconducting media: $\rho = 0$ & $\mathbf{J} = 0$

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \cancel{\rho}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_o}\mathbf{B} - \mathbf{M}$$

→

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$= \nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E}$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$u = 1/\sqrt{\mu\varepsilon}$$

→ Phase velocity

$$\rightarrow \boxed{\nabla^2 \mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

$$\rightarrow \boxed{\nabla^2 \mathbf{H} - \frac{1}{u^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0}$$

→ Homogeneous wave equations

Poynting Theorem and Conservation Laws

Ampère's law:

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad \rightarrow \; \mathbf{J}\cdot\mathbf{E} = \mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}$$

$$\leftarrow \; \nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H})$$

$$\rightarrow \; \mathbf{J}\cdot\mathbf{E} = -\nabla\cdot(\mathbf{E}\times\mathbf{H}) - \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}$$

Energy density and Poynting vector:

$$U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

Poynting theorem (Conservation of energy):

$$\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}$$

Outward power flow

Internal heat dissipation

Complex-Function Formalism (1)

Complex exponential expression for a plane harmonic wave:

$$\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$$

Time derivative:

$$\frac{\partial}{\partial t}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)] = -i\omega\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$$

Spatial derivative:

$$\nabla\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)] = i\mathbf{k}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$$

Operator relations:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t} &\rightarrow -i\omega \\ \nabla &\rightarrow i\mathbf{k}\end{aligned}}$$

Maxwell Eqs:

$$\boxed{\begin{aligned}\nabla\times\mathbf{E} &= -\mu\frac{\partial\mathbf{H}}{\partial t} \\ \nabla\times\mathbf{H} &= \varepsilon\frac{\partial\mathbf{E}}{\partial t} \\ \nabla\cdot\mathbf{E} &= 0 \\ \nabla\cdot\mathbf{H} &= 0\end{aligned}} \rightarrow \boxed{\begin{aligned}\mathbf{k}\times\mathbf{E} &= \mu\omega\mathbf{H} \\ \mathbf{k}\times\mathbf{H} &= -\varepsilon\omega\mathbf{E} \\ \mathbf{k}\cdot\mathbf{E} &= 0 \\ \mathbf{k}\cdot\mathbf{H} &= 0\end{aligned}}$$

Complex-Function Formalism (2)

Field component:

Angular frequency

$$a(t) = |A|\cos(-\omega t + \alpha)$$

Amplitude *Phase*

Consider a complex amplitude:

$$A = |A|e^{i\alpha}$$

$$\rightarrow\ a(t) = \mathrm{Re}[Ae^{-i\omega t}]$$

Product of two sinusoidal functions:

$$a(t) = |A|\cos(-\omega t + \alpha) \qquad b(t) = |B|\cos(-\omega t + \beta)$$

$$= \mathrm{Re}[Ae^{-i\omega t}] \qquad = \mathrm{Re}[Be^{-i\omega t}]$$

$$\rightarrow\ a(t)b(t) = \frac{1}{2}|AB|[\cos(-2\omega t + \alpha + \beta) + \cos(\alpha - \beta)]$$

$$\rightarrow\ Ae^{-i\omega t}Be^{-i\omega t} = |AB|e^{i(-2\omega t + \alpha + \beta)}$$

$$\rightarrow\ \boxed{\mathrm{Re}[x]\mathrm{Re}[y] \neq \mathrm{Re}[xy]}$$

Time-Averaging Sinusoidal Products

Time average of the product of two sinusoidal functions of the same freq.:

$$\langle a(t)b(t)\rangle = \frac{1}{T}\int_0^T |A|\cos(-\omega t+\alpha)|B|\cos(-\omega t+\beta)dt$$

$$= \frac{1}{2}|AB|\cos(\alpha-\beta)$$

$$\rightarrow \langle a(t)b(t)\rangle = \frac{1}{2}\mathrm{Re}[AB^*]$$

$$\rightarrow \langle \mathrm{Re}[x(t)]\mathrm{Re}[y(t)]\rangle = \frac{1}{2}\mathrm{Re}[x(t)y(t)^*]$$

Time-averaged Poynting vector and energy density:

$$\mathbf{S} = \frac{1}{2}\mathrm{Re}[\mathbf{E}\times\mathbf{H}^*]$$

$$U = \frac{1}{4}\mathrm{Re}[\mathbf{E}\cdot\mathbf{D}^* + \mathbf{B}\cdot\mathbf{H}^*]$$

Polarization of Monochromatic Plane Waves

Electric field vector:

$$\mathbf{E}(z,t) = \mathrm{Re}[\mathbf{A}e^{i(kz-\omega t)}] \quad \leftarrow \quad \mathbf{A} = \hat{\mathbf{x}}A_x e^{i\delta_x} + \hat{\mathbf{y}}A_y e^{i\delta_y}$$

$$\rightarrow \ E_x = A_x \cos(kz - \omega t + \delta_x)$$

$$\rightarrow \ E_y = A_y \cos(kz - \omega t + \delta_y)$$

Time-evolution locus:

← Polarization ellipse

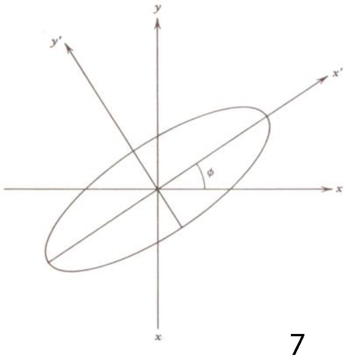
$$\rightarrow \left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\frac{\cos\delta}{A_x A_y}E_x E_y = \sin^2\delta \quad \leftarrow \quad \delta = \delta_y - \delta_x$$

In the principal coordinates system: $\rightarrow \begin{pmatrix} E_{x'} \\ E_{y'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$

$$\rightarrow \left(\frac{E_{x'}}{a}\right)^2 + \left(\frac{E_{y'}}{b}\right)^2 = 1 \qquad \rightarrow \ \tan 2\phi = \frac{2A_x A_y}{A_x^2 - A_y^2}\cos\delta$$

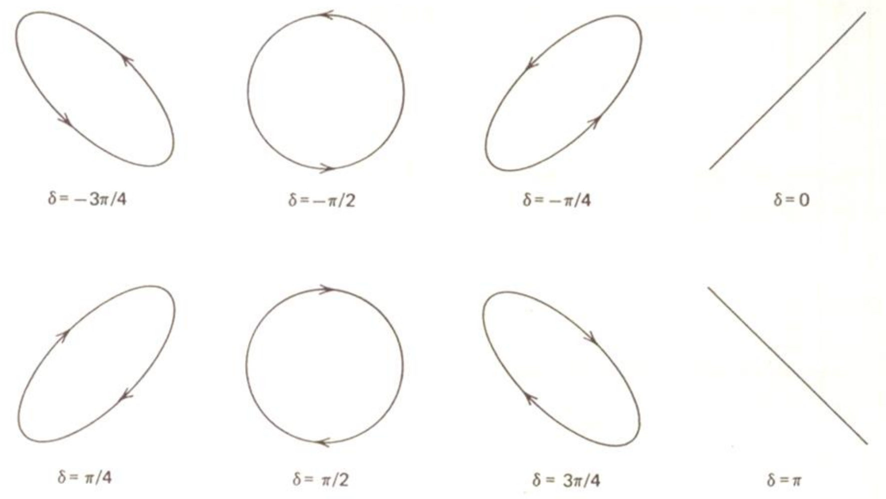
$$\leftarrow \ a^2 = A_x^2\cos^2\phi + A_y^2\sin^2\phi + 2A_x A_y\cos\delta\cos\phi\sin\phi$$

$$\leftarrow \ b^2 = A_x^2\sin^2\phi + A_y^2\cos^2\phi - 2A_x A_y\cos\delta\cos\phi\sin\phi$$



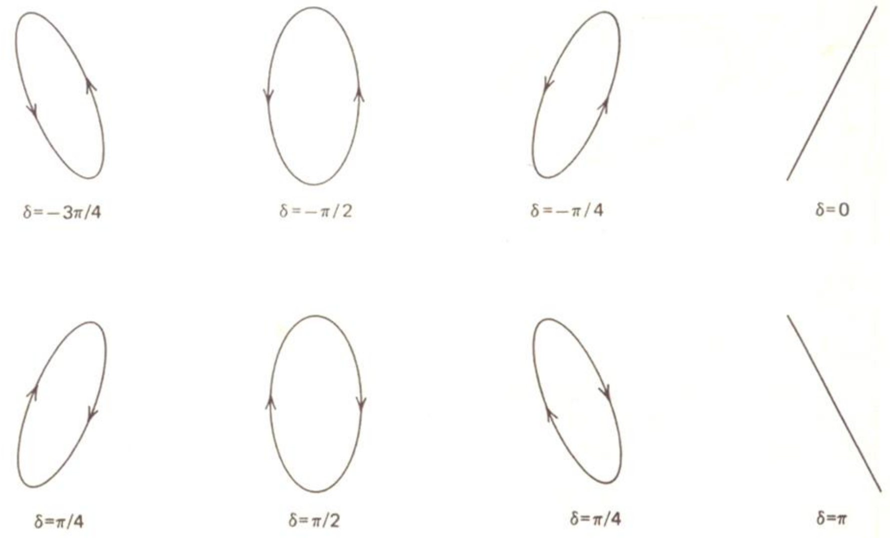
A. Yariv and P. Yeh, *Optical Waves in Crystals*, 1984

Polarization Ellipses



A. Yariv and P. Yeh, *Optical Waves in Crystals, 1984*

$$\leftarrow \quad \begin{aligned} E_x &= \cos(kz - \omega t) \\ E_y &= \cos(kz - \omega t + \delta) \end{aligned}$$



A. Yariv and P. Yeh, *Optical Waves in Crystals, 1984*

$$\leftarrow \quad \begin{aligned} E_x &= \frac{1}{2}\cos(kz - \omega t) \\ E_y &= \cos(kz - \omega t + \delta) \end{aligned}$$

Linear and Circular Polarizations

Linear polarization:

$$\delta = \delta_y - \delta_x = m\pi \quad (m = 0, 1)$$

$$\rightarrow \frac{E_y}{E_x} = (-1)^m \frac{A_y}{A_x}$$

Circular polarization:

$$\delta = \delta_y - \delta_x = \pm\frac{1}{2}\pi \quad \& \; A_y = A_x$$

$$\rightarrow \delta = -\frac{1}{2}\pi$$ ← Right circularly polarized

$$\rightarrow \delta = +\frac{1}{2}\pi$$ ← Left circularly polarized

Elliptical polarization:

$$e = \pm\frac{b}{a}$$ ← Ellipticity of a polarization ellipse

Jones Vector

Column vector of complex amplitudes:

$$\mathbf{J} = \begin{pmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \end{pmatrix}$$

Normalized Jones vector:

$$\rightarrow \ \mathbf{J}^* \cdot \mathbf{J} = 1$$

Linearly polarized light:

$$\begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \perp \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}$$

$$\rightarrow \ \hat{\mathbf{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Circularly polarized light:

$$\hat{\mathbf{R}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \hat{\mathbf{L}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\rightarrow \ \hat{\mathbf{R}}^* \cdot \hat{\mathbf{L}} = 0$$

Jones Matrix

Superposition of polarizations:

$$\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} - i\hat{\mathbf{y}}\right)$$

$$\hat{\mathbf{L}} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + i\hat{\mathbf{y}}\right)$$

$$\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{R}} + \hat{\mathbf{L}}\right)$$

$$\hat{\mathbf{y}} = \frac{i}{\sqrt{2}}\left(\hat{\mathbf{R}} - \hat{\mathbf{L}}\right)$$

General elliptical polarization:

$$\chi = e^{i\delta}\tan\psi = \frac{A_y}{A_x} e^{i(\delta_y - \delta_x)}$$

$$\rightarrow \ \mathbf{J} = \begin{pmatrix} \cos\psi \\ e^{i\delta}\sin\psi \end{pmatrix}$$

Jones Matrix:

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ ← Polarizers

$$W_{\lambda/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W_{\lambda/4} = \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}$$ ← Waveplates

Partial Polarization

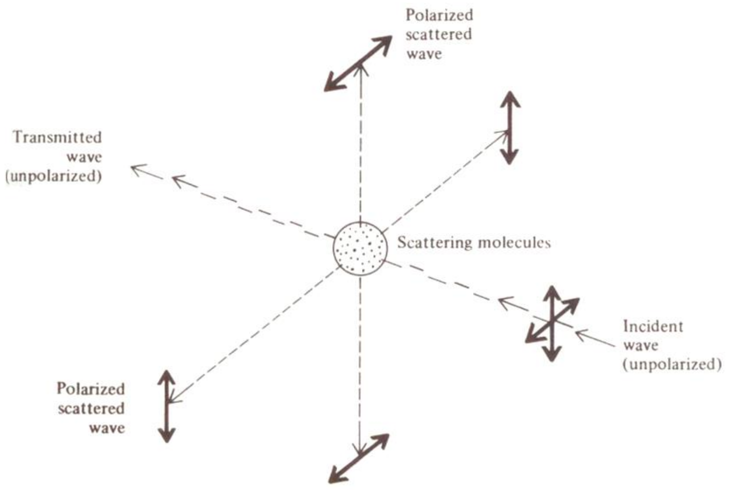
Degree of polarization:

$$P = \frac{I_{pol}}{I_{pol} + I_{unpol}}$$

For partial linear polarization:

$$\rightarrow \; P = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

Scattering and polarization:



G. R. Fowles, Introduction to Modern Optics, 1975.