Introduction to Photonics

The Vectorial Nature of Light (2)

Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Electromagnetic Boundary Conditions

Continuity conditions:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \;\rightarrow\; \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \cancel{\frac{\partial \mathbf{B}}{\partial t}} \cdot d\mathbf{s}$$

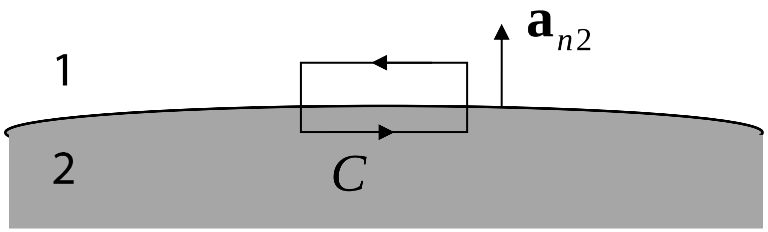
$$\rightarrow \boxed{E_{1t} = E_{2t} \quad (\mathrm{V/m})}$$

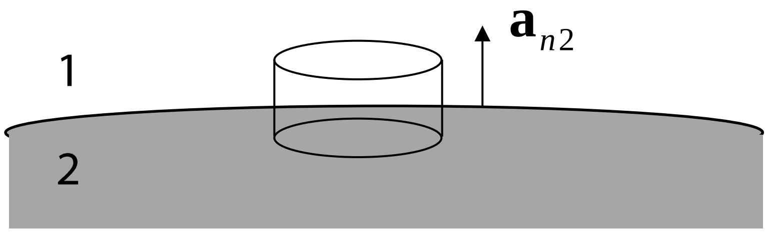
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \;\rightarrow\; \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}} \right) \cdot d\mathbf{s}$$

$$\rightarrow \boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\mathrm{A/m})}$$

$$\nabla \cdot \mathbf{D} = \rho \;\rightarrow\; \boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\mathrm{C/m^2})}$$

$$\nabla \cdot \mathbf{B} = 0 \;\rightarrow\; \boxed{B_{1n} = B_{2n} \quad (\mathrm{T})}$$





B.C. for a Plane Dielectric Interface

Lossless dielectric media: $\sigma_1 = \sigma_2 = 0$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \cancel{\mathbf{J}_s}$$

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \cancel{\rho_s}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

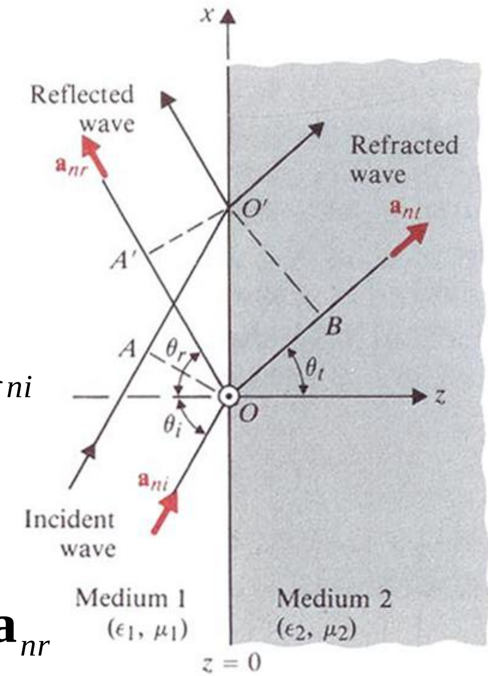
$$\rightarrow \quad \boxed{\begin{aligned} E_{1t} &= E_{2t} \\ H_{1t} &= H_{2t} \\ D_{1n} &= D_{2n} \\ B_{1n} &= B_{2n} \end{aligned}}$$

Incidence at a Plane Dielectric Boundary

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

For plane waves: $E,\ H\ \propto\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

(Isotropic media)



In medium 1 (z < 0):

Incident wave:

$$\mathbf{E}_i = \mathbf{E}_{i0} e^{i\mathbf{k}_i\cdot\mathbf{r}} \quad \rightarrow \mathbf{k}_i = k_1(\mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i) = k_1\mathbf{a}_{ni}$$

$$\mathbf{H}_i = \mathbf{H}_{i0} e^{i\mathbf{k}_i\cdot\mathbf{r}} = \frac{1}{\eta_1}\mathbf{a}_{ni} \times \mathbf{E}_{i0} e^{i\mathbf{k}_i\cdot\mathbf{r}}$$

Reflected wave:

$$\mathbf{E}_r = \mathbf{E}_{r0} e^{i\mathbf{k}_r\cdot\mathbf{r}} \quad \rightarrow \mathbf{k}_r = k_1(\mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r) = k_1\mathbf{a}_{nr}$$

$$\mathbf{H}_r = \mathbf{H}_{r0} e^{i\mathbf{k}_r\cdot\mathbf{r}} = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_{r0} e^{i\mathbf{k}_r\cdot\mathbf{r}}$$

In medium 2 (z > 0):

Transmitted wave:

$$\mathbf{E}_t = \mathbf{E}_{t0} e^{i\mathbf{k}_t\cdot\mathbf{r}} \quad \rightarrow \mathbf{k}_t = k_2(\mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t) = k_2\mathbf{a}_{nt}$$

$$\mathbf{H}_t = \mathbf{H}_{t0} e^{i\mathbf{k}_t\cdot\mathbf{r}} = \frac{1}{\eta_2}\mathbf{a}_{nt} \times \mathbf{E}_{t0} e^{i\mathbf{k}_t\cdot\mathbf{r}}$$

Snell's Law & Total Internal Reflection

Boundary conditions at z = 0: $\mathbf{E}_{1t} = \mathbf{E}_{2t}, \quad \mathbf{H}_{1t} = \mathbf{H}_{2t}$

$$\mathbf{E}_{i0t} e^{ik_1 \sin\theta_i x} + \mathbf{E}_{r0t} e^{ik_1 \sin\theta_r x} = \mathbf{E}_{t0t} e^{ik_2 \sin\theta_t x}$$

$$\mathbf{H}_{i0t} e^{ik_1 \sin\theta_i x} + \mathbf{H}_{r0t} e^{ik_1 \sin\theta_r x} = \mathbf{H}_{t0t} e^{ik_2 \sin\theta_t x}$$

To hold for any *x* values:

$$\rightarrow e^{ik_1 \sin\theta_i x} = e^{ik_1 \sin\theta_r x} = e^{ik_2 \sin\theta_t x}$$

$$\rightarrow k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

$$\rightarrow \theta_i = \theta_r$$ $\rightarrow$ Snell's law of reflection

$$\rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t$$ $\rightarrow$ Snell's law of refraction

For $n_1 > n_2$:

Critical angle

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \rightarrow \sin\theta_t > 1 \;\; if \;\; \theta_i > \theta_c \rightarrow \theta_c = \sin^{-1}(\frac{n_2}{n_1})$$

$\rightarrow$ Total internal reflection

Recall: $\mathbf{k}_t = k_2(\mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t)$

$$\rightarrow \cos\theta_t = \sqrt{1-\sin^2\theta_t} \rightarrow i\sqrt{\sin^2\theta_t - 1}$$ $\rightarrow$ Evanescent wave in *z*-axis

Continuity of E-and H-Fields

Boundary conditions at z = 0: $\mathbf{E}_{1t} = \mathbf{E}_{2t}, \quad \mathbf{H}_{1t} = \mathbf{H}_{2t}$

$$\mathbf{E}_{i0t}\cancel{e^{ik_1 \sin\theta_i x}} + \mathbf{E}_{r0t}\cancel{e^{ik_1 \sin\theta_r x}} = \mathbf{E}_{t0t}\cancel{e^{ik_2 \sin\theta_t x}} \quad \rightarrow \mathbf{E}_{i0t} + \mathbf{E}_{r0t} = \mathbf{E}_{t0t}$$

$$\mathbf{H}_{i0t}\cancel{e^{ik_1 \sin\theta_i x}} + \mathbf{H}_{r0t}\cancel{e^{ik_1 \sin\theta_r x}} = \mathbf{H}_{t0t}\cancel{e^{ik_2 \sin\theta_t x}} \quad \rightarrow \mathbf{H}_{i0t} + \mathbf{H}_{r0t} = \mathbf{H}_{t0t}$$

Consider two orthogonal polarisations: $\perp$ & $//$

$$\mathbf{E}_{i0t} + \mathbf{E}_{r0t} = \mathbf{E}_{t0t} \quad \rightarrow \mathbf{E}_{i0t}^{\perp} + \mathbf{E}_{i0t}^{//} + \mathbf{E}_{r0t}^{\perp} + \mathbf{E}_{r0t}^{//} = \mathbf{E}_{t0t}^{\perp} + \mathbf{E}_{t0t}^{//}$$

$$\mathbf{H}_{i0t} + \mathbf{H}_{r0t} = \mathbf{H}_{t0t} \rightarrow \mathbf{H}_{i0t}^{\perp} + \mathbf{H}_{i0t}^{//} + \mathbf{H}_{r0t}^{\perp} + \mathbf{H}_{r0t}^{//} = \mathbf{H}_{t0t}^{\perp} + \mathbf{H}_{t0t}^{//}$$

Perpendicular polarisation (TE)

$$E_{i0t}^{\perp} + E_{r0t}^{\perp} = E_{t0t}^{\perp}$$

$$H_{i0t}^{\perp} + H_{r0t}^{\perp} = H_{t0t}^{\perp}$$

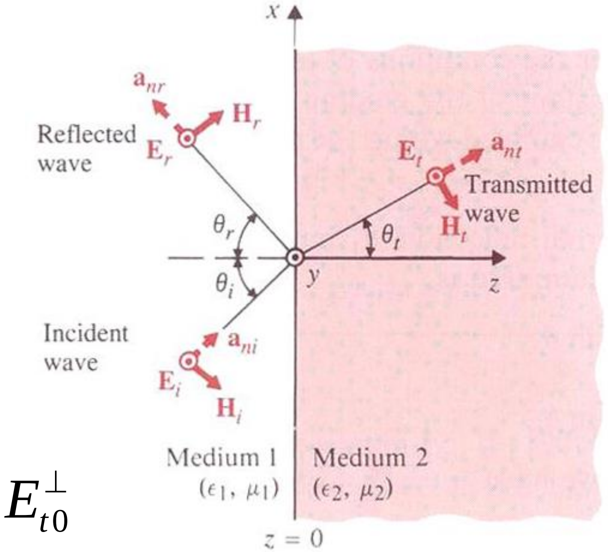
Parallel polarisation (TM)

$$E_{i0t}^{//} + E_{r0t}^{//} = E_{t0t}^{//}$$

$$H_{i0t}^{//} + H_{r0t}^{//} = H_{t0t}^{//}$$

Perpendicular Polarization: TE (1)

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.



Perpendicular polarisation (TE)

$$E_{i0t}^{\perp} + E_{r0t}^{\perp} = E_{t0t}^{\perp}$$

$$H_{i0t}^{\perp} + H_{r0t}^{\perp} = H_{t0t}^{\perp}$$

$$\mathbf{E}_{i0}^{\perp} = \mathbf{a}_y E_{i0}^{\perp}$$

$$\mathbf{H}_{i0}^{\perp} = \frac{1}{\eta_1}\mathbf{a}_{ni} \times \mathbf{E}_{i0}^{\perp} = \frac{E_{i0}^{\perp}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)$$

$$\mathbf{E}_{r0}^{\perp} = \mathbf{a}_y E_{r0}^{\perp}$$

$$\mathbf{H}_{r0}^{\perp} = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_{r0}^{\perp} = \frac{E_{r0}^{\perp}}{\eta_1}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)$$

$$\mathbf{E}_{t0}^{\perp} = \mathbf{a}_y E_{t0}^{\perp}$$

$$\mathbf{H}_{t0}^{\perp} = \frac{1}{\eta_2}\mathbf{a}_{nt} \times \mathbf{E}_{t0}^{\perp} = \frac{E_{t0}^{\perp}}{\eta_2}(-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t)$$

$$E_{i0}^{\perp} + E_{r0}^{\perp} = E_{t0}^{\perp}$$

$$\frac{1}{\eta_1}(E_{i0}^{\perp} - E_{r0}^{\perp})\cos\theta_i = \frac{1}{\eta_2}E_{t0}^{\perp}\cos\theta_t$$

Perpendicular Polarization: TE (2)

$$\boxed{\begin{aligned} &E_{i0}^{\perp} + E_{r0}^{\perp} = E_{t0}^{\perp} \\ &\frac{1}{\eta_1}(E_{i0}^{\perp} - E_{r0}^{\perp})\cos\theta_i = \frac{1}{\eta_2} E_{t0}^{\perp} \cos\theta_t \end{aligned}}$$

$$r_s = \frac{E_{r0}^{\perp}}{E_{i0}^{\perp}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

→ Fresnel's equations for S-pol.

$$t_s = \frac{E_{t0}^{\perp}}{E_{i0}^{\perp}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

Note: $\eta_2 \cos\theta_i - \eta_1 \cos\theta_t = 0 \quad \rightarrow \quad r_s = 0$

→ Brewster angle

Brewster Angle for S-Pol (TE) Waves

Condition for $r_s = 0$: $$\boxed{\eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t}$$

$$\rightarrow \cos^2\theta_{B\perp} = \frac{\eta_1^2}{\eta_2^2}\cos^2\theta_t \quad \rightarrow 1-\sin^2\theta_{B\perp} = \frac{\eta_1^2}{\eta_2^2}\left(1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B\perp}\right)$$

$$\rightarrow 1-\frac{\eta_1^2}{\eta_2^2} = \left(1-\frac{\eta_1^2}{\eta_2^2}\frac{n_1^2}{n_2^2}\right)\sin^2\theta_{B\perp}$$

$$\rightarrow \sin^2\theta_{B\perp} = \frac{1-\eta_1^2/\eta_2^2}{1-\eta_1^2 n_1^2/\eta_2^2 n_2^2} = \frac{1-\dfrac{\mu_1}{\varepsilon_1}\dfrac{\varepsilon_2}{\mu_2}}{1-\dfrac{\mu_1}{\cancel{\varepsilon_1}}\dfrac{\cancel{\varepsilon_2}}{\mu_2}\dfrac{\mu_1\cancel{\varepsilon_1}}{\mu_2\cancel{\varepsilon_2}}} = \frac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-(\mu_1/\mu_2)^2}$$

No Brewster angle for $\mu_1 = \mu_2$

or non-magnetic media!

In case: $\varepsilon_1 = \varepsilon_2,\ \mu_1 \neq \mu_2$

$$\rightarrow \sin\theta_{B\perp} = \frac{1}{\sqrt{1+\mu_1/\mu_2}} \qquad \rightarrow \tan\theta_{B\perp} = \sqrt{\frac{\mu_2}{\mu_1}} = \frac{n_2}{n_1}$$

$$\rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

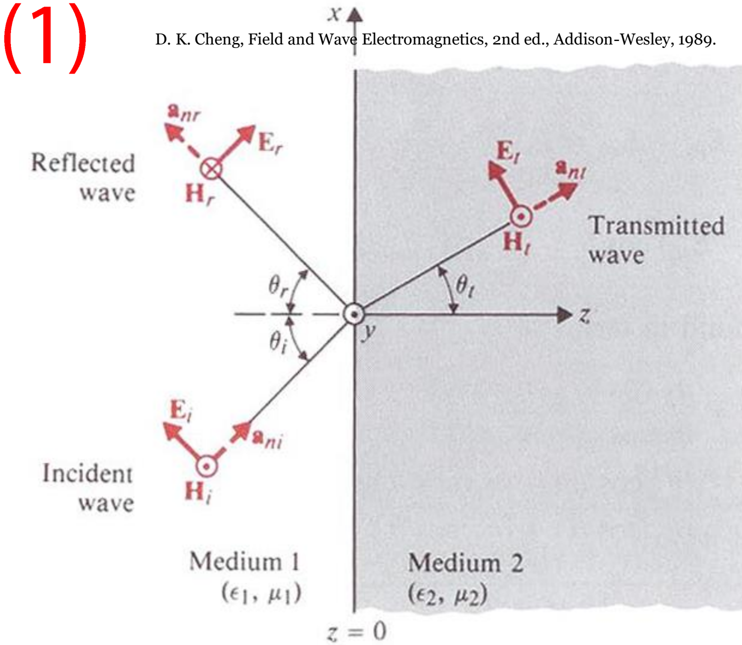
Parallel Polarization: TM (1)

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Parallel polarisation (TM)

$$E^{//}_{i0t} + E^{//}_{r0t} = E^{//}_{t0t}$$

$$H^{//}_{i0t} + H^{//}_{r0t} = H^{//}_{t0t}$$



$$\mathbf{E}^{//}_{i0} = E^{//}_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)$$

$$\mathbf{H}^{//}_{i0} = \mathbf{a}_y \frac{1}{\eta_1} E^{//}_{i0}$$

$$\mathbf{E}^{//}_{t0} = E^{//}_{t0}(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)$$

$$\mathbf{H}^{//}_{t0} = \mathbf{a}_y \frac{1}{\eta_2} E^{//}_{t0}$$

$$\mathbf{E}^{//}_{r0} = E^{//}_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)$$

$$\mathbf{H}^{\perp}_{r0} = -\mathbf{a}_y \frac{1}{\eta_1} E^{//}_{r0}$$

$$(E^{//}_{i0} + E^{//}_{r0})\cos\theta_i = E^{//}_{t0}\cos\theta_t$$

$$\frac{1}{\eta_1}(E^{//}_{i0} - E^{//}_{r0}) = \frac{1}{\eta_2} E^{//}_{t0}$$

Parallel Polarization: TM (2)

$$\boxed{\begin{aligned}&(E_{i0}^{//}+E_{r0}^{//})\cos\theta_i = E_{t0}^{//}\cos\theta_t\\&\frac{1}{\eta_1}(E_{i0}^{//}-E_{r0}^{//}) = \frac{1}{\eta_2}E_{t0}^{//}\end{aligned}}$$

$$r_p = \frac{E_{r0}^{//}}{E_{i0}^{//}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

→ Fresnel's equations for P-pol.

$$t_p = \frac{E_{t0}^{//}}{E_{i0}^{//}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

Note: $\eta_2\cos\theta_t - \eta_1\cos\theta_i = 0 \quad \rightarrow r_p = 0$

→ Brewster angle

Brewster Angle for P-Pol (TM) Waves

Condition for $r_p = 0$: $\boxed{\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B//}}$

$$\rightarrow \cos^2\theta_{B//} = \frac{\eta_2^2}{\eta_1^2}\cos^2\theta_t \quad \rightarrow 1-\sin^2\theta_{B//} = \frac{\eta_2^2}{\eta_1^2}\left(1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B//}\right)$$

$$\rightarrow 1-\frac{\eta_2^2}{\eta_1^2} = \left(1-\frac{\eta_2^2}{\eta_1^2}\frac{n_1^2}{n_2^2}\right)\sin^2\theta_{B//}$$

$$\rightarrow \sin^2\theta_{B//} = \frac{1-\eta_2^2/\eta_1^2}{1-\eta_2^2 n_1^2/\eta_1^2 n_2^2} = \frac{1-\dfrac{\mu_2}{\varepsilon_2}\dfrac{\varepsilon_1}{\mu_1}}{1-\dfrac{\mu_2}{\varepsilon_2}\dfrac{\varepsilon_1}{\mu_1}\dfrac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} = \frac{1-\mu_2\varepsilon_1/\mu_1\varepsilon_2}{1-(\varepsilon_1/\varepsilon_2)^2}$$

No Brewster angle for $\varepsilon_1 = \varepsilon_2$

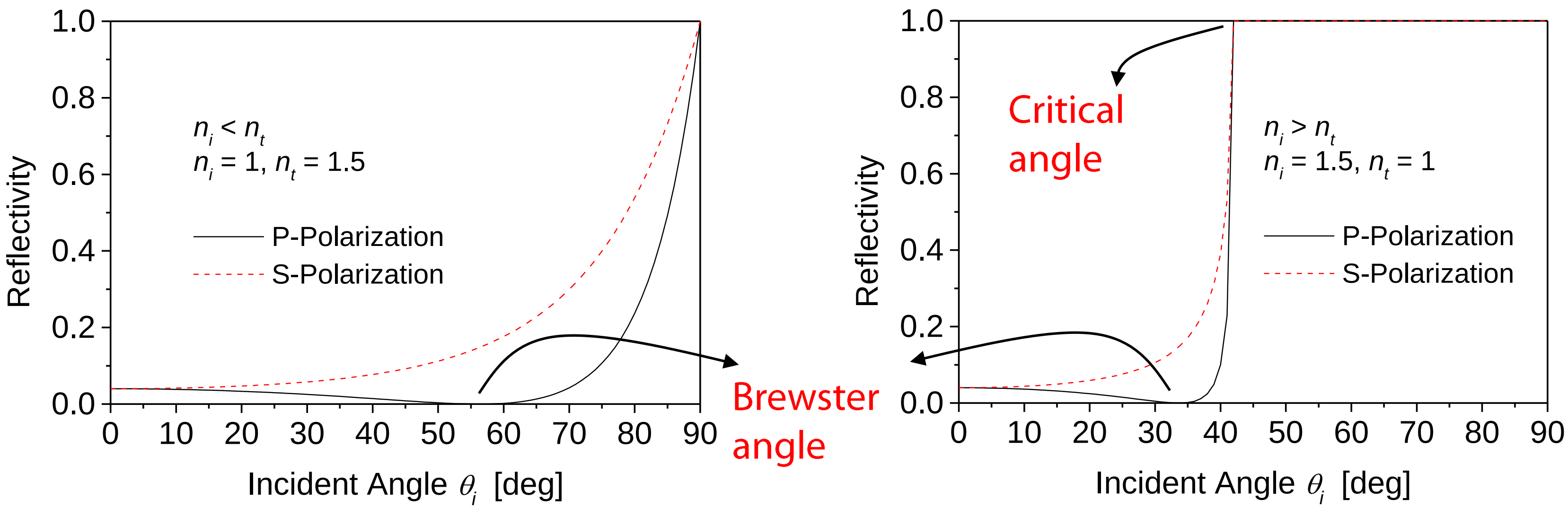
In case: $\varepsilon_1 \neq \varepsilon_2,\ \mu_1 = \mu_2 = \mu_0$

$$\rightarrow \sin\theta_{B//} = \frac{1}{\sqrt{1+\varepsilon_1/\varepsilon_2}} \qquad \rightarrow \tan\theta_{B//} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}$$

$$\rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

Brewster Angle & Critical Angle

Numerical examples:



Brewster Angle (Non-magnetic):

For P-polarization

$$\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right)$$

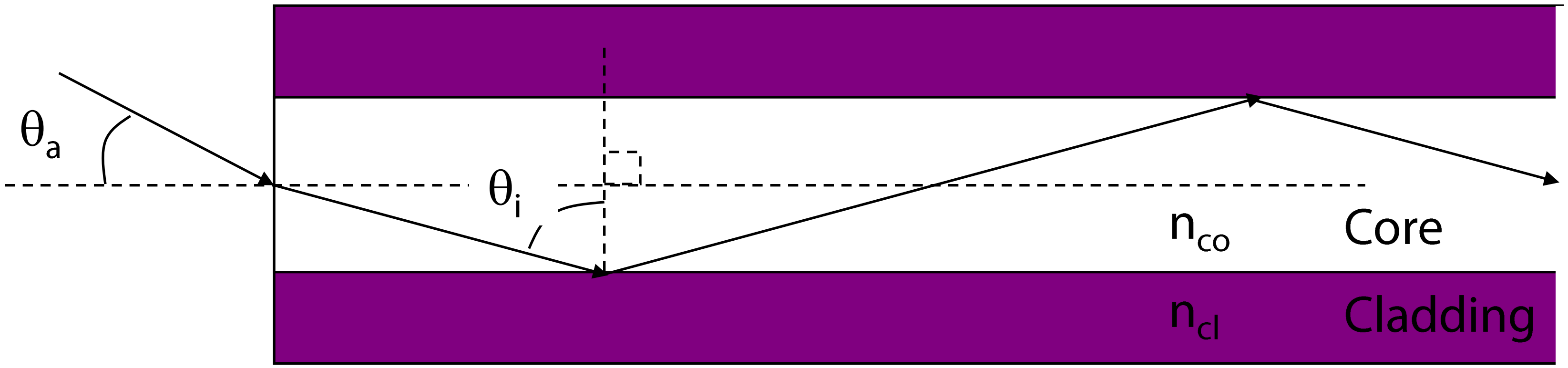
$$\rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

Critical Angle: Total internal reflection

$$\theta_c = \sin^{-1}\left(\frac{n_t}{n_i}\right)$$

$$\rightarrow n_i > n_t, \ \theta_t = \frac{\pi}{2}$$

Optical Waveguides



Total internal reflection (TIR):

$$\theta_i > \theta_c = \sin^{-1}\left(\frac{n_{cl}}{n_{co}}\right)$$

Numerical aperture:

$$NA = n_o \sin\theta_a \approx \theta_a = \sqrt{n_{co}^2 - n_{cl}^2}$$

Phase changes in TIR:

H.W.#3: Derive eqs. (2.70) & (2.71).