Precision Metrology 3

Error propagation for random error (cont'd)

Worst case, most extreme case, or most conservative case of error propagation:

Random error,

RA

$=RX\cdot|\partial F/\partial X| + RY\cdot|\partial F/\partial Y| + 2\rho_{xy}\sqrt{RX}\sqrt{RY}\ |\partial F/\partial X\cdot\partial F/\partial Y|$

RX, RY are the random error of X,Y.

Example:

Rectangle: $X=(1+0.01\pm0.005)$m, $Y=(1+0.01\pm0.005)$m

$A=F(X,Y)=XY$

What about for Area ΔA? RA?

Assume X,Y are independent, or 10% correlated.

Systematic part upto 2nd order:

$\partial F/\partial X=Y$, $\partial F/\partial Y=X$, $\partial^2F/\partial X^2=0$, $\partial^2F/\partial Y^2=0$, $\partial^2F/\partial X\partial Y=1$

For Systematic Part, ΔA

$=\Delta X \cdot \partial F/\partial X + \Delta Y \cdot \partial F/\partial Y +$

$+\Delta X^2 \cdot \partial^2 F/\partial X^2 + \Delta Y^2 \cdot \partial^2 F/\partial Y^2 + 2\Delta X \Delta Y \cdot \partial^2 F/\partial X \partial Y$

$=(0.01)(1)+(0.01)(1)+2(0.01)(0.01)(1)$

$=0.0202$

For Random part, RA

σ^2_F

$= \sigma^2_x (\partial F/\partial X)^2 + \sigma^2_y (\partial F/\partial Y)^2 + 2\rho_{xy}\sigma_x\sigma_y(\partial F/\partial X)(\partial F/\partial Y)$

$=(0.005/3)^2(1)+(0.005/3)^2(1)+2(0.1)(0.005/3)(0.005/3)(1)$

$=(2.2)\ (0.005/3)^2$ (if 10% correlated), and

$=(2)\ (0.005/3)^2$ (if independent)

Thus, $RA=3\sigma_F$

$=(0.005)\sqrt{2.2}$ (if 10% correlated), and

$=(0.005)\sqrt{2}$ (if independent)

Therefore, A_T, area after error propagation, is

$A_T=1+0.0202\pm(0.005)\sqrt{2.2}$ (if 10% correlated), and

$A_T=1+0.0202\pm(0.005)\sqrt{2}$ (if independent)

Worst case random error propagation is

$RA=RX\cdot|\partial F/\partial X| +RY\cdot|\partial F/\partial Y|+2\rho_{xy}\sqrt{RX}\sqrt{RY}\cdot|\partial F/\partial X\cdot\partial F/\partial Y|$

$=(0.005)(1)+(0.005)(1)+2(0.1)(1)(1)(0.005)$

$=(0.005)(2.2)$ (if 10% correlated), and

$=(0.005)(2)$ (if independent)

HW4) A function of planar machine's error is,

$F(X,Y,T)$

$=10^{-4}X^2+10^{-5}X\sin(2\pi X/5)+(0.5)10^{-4}XY+(0.24)10^{-4}TX^2$

(unit: m for displacement, K for temperature)

Point (X,Y,T) of Interest = nominal position=(1m,1m,1K)

Systematic error (ΔX, ΔY, ΔT)=(0.1m, 0.1m, 0.1K)

Random error (RX, RY, RT)=(0.02m, 0.02m, 0.05K), or

That is,

$X=(1+0.1\pm0.02)$m, $Y=(1+0.1\pm0.02)$m, $T=(1+0.1\pm0.05)$K

What about the machine error at the nominal position?

1) Derive the systematic error at nominal position.
2) Evaluate the systematic error at nominal position.
3) Derive the random error at nominal position.
4) Evaluate the random error at nominal position.
5) Describe the total error at the nominal position

(assuming X,Y,T are independent)

Some useful statistical tests for metrology

"Small Sampling Theory" for small samples under $n<30$

1. Population Mean

: to be estimated from Sample mean and Sample std

For n Measurement Data: $X_1, X_2, X_3 \ldots X_n$

Sample Mean, $\underline{X} = \Sigma X_i / n$

Sample standard deviation, $s_{n-1} = \sqrt{\Sigma (X_i - \underline{X})^2 / n-1}$

Population Mean, μ, can be estimated with probability, that is,

Probability of μ to lie in the interval,

$\text{Prob} [\underline{X} - t \cdot s/\sqrt{n-1} \le \mu \le \underline{X} + t \cdot s/\sqrt{n-1}] = 1-\alpha$

where α is the significance level;

t is $t_{1-\alpha/2,\, n-1}$,

from the table of Student-t-distribution

(after W. G.Gosset's pseudonym 'student')

Ex) Estimate the Population Mean, μ, for sample measurement: $n=10$, $\underline{X}=14.7$, $s_{n-1}=0.1$

Probability 99%-> $\alpha=0.01$,

$t_{1-\alpha/2,n-1} = t_{0.995,9} = 3.25$

(from the Student's-t-distribution table)

Therefore Population mean lie in the interval,

$\underline{X}-3.25\cdot s_{n-1}/\sqrt{9} \leq \mu \leq \underline{X}+3.25\cdot s_{n-1}/\sqrt{9}$

$3.25\cdot s_{n-1}/\sqrt{9}=3.25(0.1)/3=0.108$

$14.7-0.108 \leq \mu \leq 14.7+0.108$

$\therefore 14.592 \leq \mu \leq 14.808$

2. Population Std

:To be estimated from the Sample Std, s

Population Std, σ, can be estimated with the Probability.

σ lies in the interval at the probability of

Prob [$s\sqrt{n-1}/\chi_{1-\alpha/2,\ n-1} \le \sigma \le s\sqrt{n-1}/\chi_{\alpha/2,\ n-1}$] = $1-\alpha$

where χ is from the Chi-square distribution table.

Ex) Estimate Population Std from the Sample Std

10 samples: $n=10$, $\underline{X}=14.7$, $s_{n-1}=0.1$, probability=99%

When $\alpha = 0.01$, from the Chi-square distribution table,

$\chi_{1-\alpha/2,\ n-1} = \chi_{0.995,\ 9} = \sqrt{23.6} = 4.858$

$\chi_{\alpha/2,\ n-1} = \chi_{0.005,\ 9} = \sqrt{1.73} = 1.315$

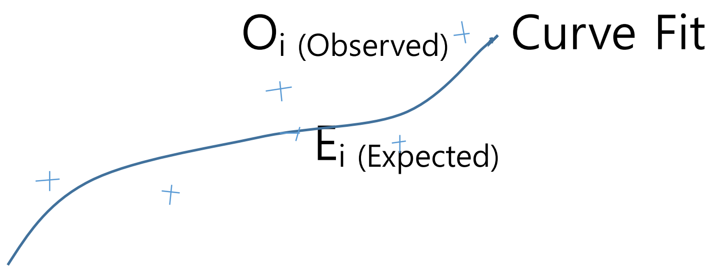
$s\sqrt{n-1}/\chi_{1-\alpha/2,\ n-1} = 0.1\sqrt{9}/4.858 = 0.0617$

$s\sqrt{n-1}/\chi_{\alpha/2,\ n-1} = 0.1\sqrt{9}/1.315 = 0.2281$

$\therefore 0.0617 \le \sigma \le 0.2281$

3. <u>Goodness of Fit</u>

With (1-α) Probability



$\chi^2_{\alpha/2,\ n-1}$ VS $\Sigma(O_i-E_i)^2/E_i$

χ^2 from Table, α/2 and (n-1) dof

O_i is observed (measured), and

E_i is expectation (fitted)

If $\Sigma(O_i-E_i)^2/E_i \leq \chi^2$; Accepted

If $\Sigma(O_i-E_i)^2/E_i \geq \chi^2$; Rejected

4. Test for Variance

:To test whether "Similar data" or "considerably Different data", using the F disdribution (named after R.A.Fisher) based on the variance.

Two samples: Similar? Or Different considerably?

n_1, $\underline{X}_1$, s_1 from populations having variance, $\sigma^2{}_1$

n_2, $\underline{X}_2$, s_2 from populations having variance, $\sigma^2{}_2$

At (1-α) probability

$F_{n1-1,\ n2-1\ ,1-\alpha} \equiv \{n_1 s_1{}^2/(n_1-1)\sigma^2{}_1\}/\{ n_2 s_2{}^2/(n_2-1)\sigma^2{}_2\}$

$=\{ n_1 s_1{}^2/(n_1-1)\}/\{ n_2 s_2{}^2/(n_2-1)\}$

($\because$ $\sigma^2{}_1= \sigma^2{}_2= \sigma^2$ from the same population)

$\fallingdotseq s_1{}^2/s_2{}^2$

($\because$ $n_1/(n_1-1) \fallingdotseq n_2/(n_2-1)$, if n_1, n_2 are sufficiently large, or $n_1=n_2$)

where the numerator is larger than the denumerator

If $s_1^2/s_2^2 > F$ then

Differ considerably (not from the same population)

If $s_1^2/s_2^2 \leq F$ then

Similar data (from the same population)

Ex) Two samples;

Sample1: 14.9;14.6;14.8;14.6;14.9

Sample2: 14.5;14.5;14.3;14.7;14.6

$n_1=5$, $\underline{X}_1=14.76$, $s_1=\sqrt{\Sigma(X_i-\underline{X}_1)^2/n_1-1}=0.1346$

$n_2=5$, $\underline{X}_2=14.58$, $s_2=\sqrt{\Sigma(X_i-\underline{X}_2)^2/n_2-1}=0.0836$

Probability=1-α=99% ∴α=0.001

(1) Population Mean

$\text{Prob}\ [\ \underline{X}-t\cdot s/\sqrt{n-1} \leq \mu \leq \underline{X}+t\cdot s/\sqrt{n-1}\ ] = 1-\alpha$

$t_{1-\alpha/2,\ n1-1} = t_{0.995,4} = 4.60 = t_{1-\alpha/2,\ n2-1}$

$ts_1/\sqrt{n-1}=4.60(0.1346)/\sqrt{4}=0.3096$;

$14.76-0.3096 \leq \mu_1 \leq 14.76+0.3096$

$\therefore 14.450 \leq \mu_1 \leq 15.070$

$ts_2/\sqrt{n}=4.60(0.0836)/\sqrt{4}=0.1923;$

$14.58-0.1923 \leq \mu_2 \leq 14.58+0.1923$

$\therefore 14.388 \leq \mu_2 \leq 14.772$

(2) Population Std

Prob [$s\sqrt{n-1}/\chi_{1-\alpha/2,\ n-1} \leq \sigma \leq s\sqrt{n-1}/\chi_{\alpha/2,\ n-1}$] = $1-\alpha$

$\chi^2_{1-\alpha/2,\ n1-1} = \chi^2_{0.995,4} = 14.9 = \chi^2_{1-\alpha/2,\ n2-1}$

$\chi^2_{\alpha/2,\ n1-1} = \chi^2_{0.005,4} = 0.207 = \chi^2_{\alpha/2,\ n2-1}$

$\therefore \chi_{1-\alpha/2,\ n-1}=\sqrt{14.9}=3.860,\ \chi_{\alpha/2,\ n-1}=\sqrt{0.207}=0.455$

$s_1\sqrt{(n_1-1)}/\ \chi_{1-\alpha/2,\ n-1}=(0.1346)(2)/3.860=0.0697$

$s_1\sqrt{(n_1-1)}\ /\ \chi_{\alpha/2,\ n-1}=(0.1346)(2)/0.455=0.5916$

$\therefore 0.0697 \leq \sigma_1 \leq 0.5916$

$s_2\sqrt{(n_2-1)}\ /\ \chi_{1-\alpha/2,\ n-1}=(0.0836)(2)/3.860=0.0433$

$s_2\sqrt{(n_2-1)}$ / $\chi_{\alpha/2,\ n-1}$=(0.0836)(2)/0.455=0.3674

∴ $0.0433 \leq \sigma_2 \leq 0.3674$

(3) F test for Similarity

$F_{n1-1,\ n2-1\ ,1-\alpha} \fallingdotseq s_1^2/s_2^2$

$F_{n1-1,n2-1,1-\alpha}=F_{4,4,0.99}=16.0$

$s_1^2/s_2^2=(0.1346/0.0836)^2=2.592 \leq F_{4,4,0.99}$

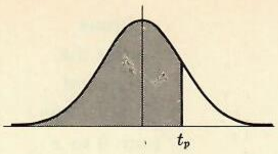
∴ Two samples are the Similar data

(from the same population)

HW5) Prepare two sets of sample measurement for your dedicated application, such as length measurement using scales. The sample sizes are   minimum 20 at two different time schedules. Estimate and discuss for the followings at 99% probability.

1) Population Mean
2) Population Std
3) Similarity Test

Percentile Values (t_p)
for
Student's t Distribution
with ν Degrees of Freedom
(shaded area = p)

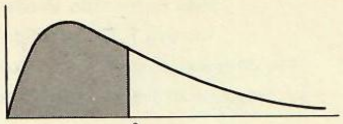


ν	$t_{.995}$	$t_{.99}$	$t_{.975}$	$t_{.95}$	$t_{.90}$	$t_{.80}$	$t_{.75}$	$t_{.70}$	$t_{.60}$	$t_{.55}$
1	63.66	31.82	12.71	6.31	3.08	1.376	1.000	.727	.325	.158
2	9.92	6.96	4.30	2.92	1.89	1.061	.816	.617	.289	.142
3	5.84	4.54	3.18	2.35	1.64	.978	.765	.584	.277	.137
4	4.60	3.75	2.78	2.13	1.53	.941	.741	.569	.271	.134
5	4.03	3.36	2.57	2.02	1.48	.920	.727	.559	.267	.132
6	3.71	3.14	2.45	1.94	1.44	.906	.718	.553	.265	.131
7	3.50	3.00	2.36	1.90	1.42	.896	.711	.549	.263	.130
8	3.36	2.90	2.31	1.86	1.40	.889	.706	.546	.262	.130
9	3.25	2.82	2.26	1.83	1.38	.883	.703	.543	.261	.129
10	3.17	2.76	2.23	1.81	1.37	.879	.700	.542	.260	.129
11	3.11	2.72	2.20	1.80	1.36	.876	.697	.540	.260	.129
12	3.06	2.68	2.18	1.78	1.36	.873	.695	.539	.259	.128
13	3.01	2.65	2.16	1.77	1.35	.870	.694	.538	.259	.128
14	2.98	2.62	2.14	1.76	1.34	.868	.692	.537	.258	.128
15	2.95	2.60	2.13	1.75	1.34	.866	.691	.536	.258	.128
16	2.92	2.58	2.12	1.75	1.34	.865	.690	.535	.258	.128
17	2.90	2.57	2.11	1.74	1.33	.863	.689	.534	.257	.128
18	2.88	2.55	2.10	1.73	1.33	.862	.688	.534	.257	.127
19	2.86	2.54	2.09	1.73	1.33	.861	.688	.533	.257	.127
20	2.84	2.53	2.09	1.72	1.32	.860	.687	.533	.257	.127
21	2.83	2.52	2.08	1.72	1.32	.859	.686	.532	.257	.127
22	2.82	2.51	2.07	1.72	1.32	.858	.686	.532	.256	.127
23	2.81	2.50	2.07	1.71	1.32	.858	.685	.532	.256	.127
24	2.80	2.49	2.06	1.71	1.32	.857	.685	.531	.256	.127
25	2.79	2.48	2.06	1.71	1.32	.856	.684	.531	.256	.127
26	2.78	2.48	2.06	1.71	1.32	.856	.684	.531	.256	.127
27	2.77	2.47	2.05	1.70	1.31	.855	.684	.531	.256	.127
28	2.76	2.47	2.05	1.70	1.31	.855	.683	.530	.256	.127
29	2.76	2.46	2.04	1.70	1.31	.854	.683	.530	.256	.127
30	2.75	2.46	2.04	1.70	1.31	.854	.683	.530	.256	.127
40	2.70	2.42	2.02	1.68	1.30	.851	.681	.529	.255	.126
60	2.66	2.39	2.00	1.67	1.30	.848	.679	.527	.254	.126
120	2.62	2.36	1.98	1.66	1.29	.845	.677	.526	.254	.126
∞	2.58	2.33	1.96	1.645	1.28	.842	.674	.524	.253	.126

Source: R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research* (5th edition), Table III, Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publishers.

Source: M.R.Spiegel, etal., Probability and Statistics, Schaum's outlines, p488, McGraw Hill

Percentile Values (χ^2_p)
for
the Chi-Square Distribution
with ν Degrees of Freedom
(shaded area = p)

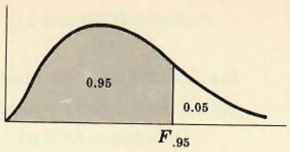


ν	$\chi^2_{.995}$	$\chi^2_{.99}$	$\chi^2_{.975}$	$\chi^2_{.95}$	$\chi^2_{.90}$	$\chi^2_{.75}$	$\chi^2_{.50}$	$\chi^2_{.25}$	$\chi^2_{.10}$	$\chi^2_{.05}$	$\chi^2_{.025}$	$\chi^2_{.01}$	$\chi^2_{.005}$
1	7.88	6.63	5.02	3.84	2.71	1.32	.455	.102	.0158	.0039	.0010	.0002	.0000
2	10.6	9.21	7.38	5.99	4.61	2.77	1.39	.575	.211	.103	.0506	.0201	.0100
3	12.8	11.3	9.35	7.81	6.25	4.11	2.37	1.21	.584	.352	.216	.115	.072
4	14.9	13.3	11.1	9.49	7.78	5.39	3.36	1.92	1.06	.711	.484	.297	.207
5	16.7	15.1	12.8	11.1	9.24	6.63	4.35	2.67	1.61	1.15	.831	.554	.412
6	18.5	16.8	14.4	12.6	10.6	7.84	5.35	3.45	2.20	1.64	1.24	.872	.676
7	20.3	18.5	16.0	14.1	12.0	9.04	6.35	4.25	2.83	2.17	1.69	1.24	.989
8	22.0	20.1	17.5	15.5	13.4	10.2	7.34	5.07	3.49	2.73	2.18	1.65	1.34
9	23.6	21.7	19.0	16.9	14.7	11.4	8.34	5.90	4.17	3.33	2.70	2.09	1.73
10	25.2	23.2	20.5	18.3	16.0	12.5	9.34	6.74	4.87	3.94	3.25	2.56	2.16
11	26.8	24.7	21.9	19.7	17.3	13.7	10.3	7.58	5.58	4.57	3.82	3.05	2.60
12	28.3	26.2	23.3	21.0	18.5	14.8	11.3	8.44	6.30	5.23	4.40	3.57	3.07
13	29.8	27.7	24.7	22.4	19.8	16.0	12.3	9.30	7.04	5.89	5.01	4.11	3.57
14	31.3	29.1	26.1	23.7	21.1	17.1	13.3	10.2	7.79	6.57	5.63	4.66	4.07
15	32.8	30.6	27.5	25.0	22.3	18.2	14.3	11.0	8.55	7.26	6.26	5.23	4.60
16	34.3	32.0	28.8	26.3	23.5	19.4	15.3	11.9	9.31	7.96	6.91	5.81	5.14
17	35.7	33.4	30.2	27.6	24.8	20.5	16.3	12.8	10.1	8.67	7.56	6.41	5.70
18	37.2	34.8	31.5	28.9	26.0	21.6	17.3	13.7	10.9	9.39	8.23	7.01	6.26
19	38.6	36.2	32.9	30.1	27.2	22.7	18.3	14.6	11.7	10.1	8.91	7.63	6.84
20	40.0	37.6	34.2	31.4	28.4	23.8	19.3	15.5	12.4	10.9	9.59	8.26	7.43
21	41.4	38.9	35.5	32.7	29.6	24.9	20.3	16.3	13.2	11.6	10.3	8.90	8.03
22	42.8	40.3	36.8	33.9	30.8	26.0	21.3	17.2	14.0	12.3	11.0	9.54	8.64
23	44.2	41.6	38.1	35.2	32.0	27.1	22.3	18.1	14.8	13.1	11.7	10.2	9.26
24	45.6	43.0	39.4	36.4	33.2	28.2	23.3	19.0	15.7	13.8	12.4	10.9	9.89
25	46.9	44.3	40.6	37.7	34.4	29.3	24.3	19.9	16.5	14.6	13.1	11.5	10.5
26	48.3	45.6	41.9	38.9	35.6	30.4	25.3	20.8	17.3	15.4	13.8	12.2	11.2
27	49.6	47.0	43.2	40.1	36.7	31.5	26.3	21.7	18.1	16.2	14.6	12.9	11.8
28	51.0	48.3	44.5	41.3	37.9	32.6	27.3	22.7	18.9	16.9	15.3	13.6	12.5
29	52.3	49.6	45.7	42.6	39.1	33.7	28.3	23.6	19.8	17.7	16.0	14.3	13.1
30	53.7	50.9	47.0	43.8	40.3	34.8	29.3	24.5	20.6	18.5	16.8	15.0	13.8
40	66.8	63.7	59.3	55.8	51.8	45.6	39.3	33.7	29.1	26.5	24.4	22.2	20.7
50	79.5	76.2	71.4	67.5	63.2	56.3	49.3	42.9	37.7	34.8	32.4	29.7	28.0
60	92.0	88.4	83.3	79.1	74.4	67.0	59.3	52.3	46.5	43.2	40.5	37.5	35.5
70	104.2	100.4	95.0	90.5	85.5	77.6	69.3	61.7	55.3	51.7	48.8	45.4	43.3
80	116.3	112.3	106.6	101.9	96.6	88.1	79.3	71.1	64.3	60.4	57.2	53.5	51.2
90	128.3	124.1	118.1	113.1	107.6	98.6	89.3	80.6	73.3	69.1	65.6	61.8	59.2
100	140.2	135.8	129.6	124.3	118.5	109.1	99.3	90.1	82.4	77.9	74.2	70.1	67.3

Source: Catherine M. Thompson, *Table of percentage points of the χ^2 distribution*, Biometrika, Vol. 32 (1941), by permission of the author and publisher.

Source: M.R.Spiegel, etal., Probability and Statistics, Schaum's outlines, p489, McGraw Hill

95th Percentile Values
for the *F* Distribution
(ν_1 degrees of freedom in numerator)
(ν_2 degrees of freedom in denominator)

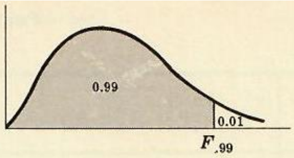


ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Source: E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. 2 (1972), Table 5, page 178, by permission.

Source: M.R.Spiegel, etal., Probability and Statistics, Schaum's outlines, p490, McGraw Hill

99th Percentile Values
for the F Distribution
(ν_1 degrees of freedom in numerator)
(ν_2 degrees of freedom in denominator)



ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4052	5000	5403	5625	5764	5859	5928	5981	6023	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.3	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.82	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Source: E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticians*, Vol. 2 (1972), Table 5, page 180, by permission.

Source: M.R.Spiegel, etal., Probability and Statistics, Schaum's outlines, p491, McGraw Hill