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457.643 Structural Random Vibrations

In-Class Material: Class 01

0. Introduction

Grigoriu, M. (2004) Research Perspective in Stochastic Mechanics. *Engineering Design Reliability Handbook*, edited by E. Nikolaidis, D.M. Ghiocel, and S. Singhal, CRC Press, Boca Raton, FL., Chap. 6

The response and evolution ($\mathcal{X}(x,t)$) of mechanical, biological, and other systems subjected to an input $\mathcal{Y}(x,t)$ can be characterized by equations of the form

$$\mathcal{D}[\mathcal{X}(x,t)] = \mathcal{Y}(x,t), \qquad t \geq 0, \qquad x \in D \subset \mathcal{R}^d$$

where

$\mathcal{D}$: algebraic, integral, or differential operator with random or deterministic coefficients

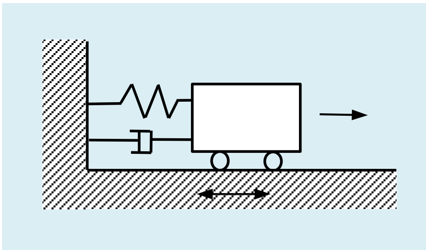
$\mathcal{Y}(x,t)$: random or deterministic input function

$\mathcal{X}(x,t)$: random or deterministic output (response) function

There are four classes of problems:

1. Deterministic systems and input (457.516 Dynamics of Structures)
2. Deterministic systems and stochastic input **(457.643 Structural Random Vibrations)**
3. Stochastic systems and deterministic input (457.646 Topics in Structural Reliability)
4. Stochastic systems and input

For example, consider an SDOF linear oscillator subject to earthquake ground motion:



E.O.M.:

Some results of "random vibration analysis":

- Mean and variance of $X(t)$:
- Instantaneous failure probability:
- First-passage failure probability:

See "Syllabus and Course Outline" handout for course objectives and contents.

I. Basic Elements

◎ Review on basic theories of probability

Self-review of "II. Basic Theory of Probability and Statistics" part of the course "457.646 Topics in Structural Reliability" is required.

Additional basic topics to review for this course:

◎ Characteristic function (L&S Chapter 3)

Alternative (complete/incomplete) description of random variable X

$M_X(\theta) \equiv \mathrm{E}_X[\exp(i\theta X)] = \int \qquad\qquad\qquad dx$ _____________ transform of ______

Therefore,

$f_X(x) = \frac{\quad}{\quad}\int \qquad\qquad\qquad d\theta$ ________ _________ transform of ______

※ See Appendix B of L&S for a brief review of Fourier transform (if necessary)

Note:

1) $M_X(\theta)$ always exists because the condition for the existence of a Fourier transform is $\int_{-\infty}^{\infty}|f_X(x)|dx < \infty$ ("absolutely integrable"), and we know that $\int_{-\infty}^{\infty}|f_X(x)|dx = \quad$.
2) Why use $M_X(\theta)$?

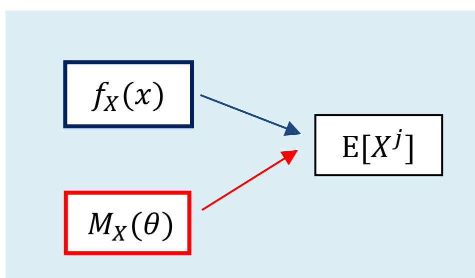
- ➢ Useful for analytical development or proof (will be shown later in the course)
- ➢ Especially useful for generating ___________

◎ M________ generating property of characteristic function

Remember $M_X(\theta) = \mathrm{E}_X[\exp(i\theta X)] = \int_{-\infty}^{\infty}\exp(i\theta X)\, f_X(x)dx$

$$\frac{d^j}{d\theta^j}M_X(\theta) = i^j\mathrm{E}_X[X^j\exp(i\theta X)]$$

$$\left.\frac{d^j}{d\theta^j}M_X(\theta)\right|_{\theta=0} = i^j\mathrm{E}_X[X^j]$$



Therefore,

$$\frac{1}{i^j}\left.\frac{d^j}{d\theta^j}M_X(\theta)\right|_{\theta=0} = \mathrm{E}_X[X^j] = \int$$

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※ McLauren series of $M_X(\theta)$

$$
\begin{aligned}
M_X(\theta) &= \sum_{k=0}^{\infty} \left.\frac{d^k M_X(\theta)}{d\theta^k}\right|_{\theta=0} \frac{\theta^k}{k!} \\
&= \sum_{k=0}^{\infty} i^k \mathrm{E}_X[X^k] \frac{\theta^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \mathrm{E}_X[X^k]
\end{aligned}
$$

- Could approximate the characteristic function using low-order moments?

※ "Moment generating function" $\mathrm{E}_X[\exp(-rX)] =$

- L_______ transform
- Moment generating equation more simple (because real-valued)
- May not exist mathematically for some probability density function (p. 86 L&S)

Example

1) Derive the characteristic function of $X \sim N(\mu, \sigma^2)$

2) Generate the first and second moment of X using the characteristic function to confirm.

◎ Log-characteristic function

$L_X(\theta) \equiv \ln M_X(\theta)$

nth order cumulant function $\kappa_n(X) \equiv \frac{1}{i^n}\frac{d^n L_X(\theta)}{d\theta^n}\Big|_{\theta=0}$

- $\kappa_1(X) = \frac{1}{i}\frac{dL_X(\theta)}{d\theta}\Big|_{\theta=0} =$
- $\kappa_2(X) = \mathrm{E}_X[(X-\mu)^2] =$
- $\kappa_3(X) = \mathrm{E}_X[(X-\mu)^3] =$
- $\kappa_4(X) = \mathrm{E}_X[(X-\mu)^4] - 3\sigma^4 =$

For Gaussian, $\kappa_3(X) =$ and $\kappa_4(X) =$. For $n \geq 3$, $\kappa_n(X) =$ because

- ➢ Cumulants are useful since they are related to "c_______" moments
- ➢ Another merit: $\kappa_n(X) \cong 0$ for higher order, so easier to approximate PDF (through log-characteristic function)

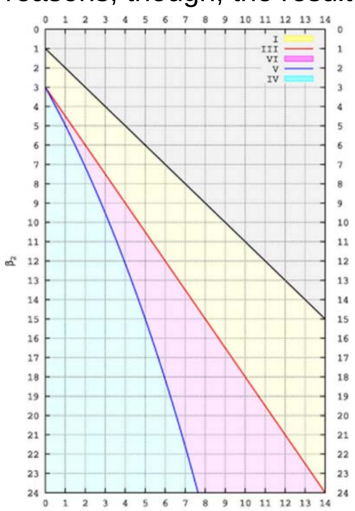
$$L_X(\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}\kappa_n(X)$$

Note: $\kappa_0(X) = 0$ (check by yourself)

◎ Importance of moment analysis (L&S 3.8)

"In many random variable problems (and in much of the analysis of stochastic processes), one performs detailed analysis of only the first and second moments of the various quantities, with occasional consideration of skewness and/or kurtosis. One reason for this is surely the fact that analysis of mean, variance, or mean squared value is generally much easier than analysis of probability distributions. Furthermore, in many problems, one has some idea of the shape of the probability density functions, so knowledge of moment information may allow evaluation of the parameters in that shape, thereby giving an estimate of the complete probability distribution. If the shape has only two parameters to be chosen, in particular, then knowledge of mean and variance will generally suffice for this procedure. In addition to these pragmatic reasons, though, the results in Eqs. 3.31, 3.32, 3.35, and 3.36 (i.e. McLauren series expansion of characteristic function and log-characteristic functions) give a theoretical justification for focusing attention on the low-order moments. Specifically, mean, variance, skewness, kurtosis, and so forth, in that order, are the first items in an infinite sequence of information that would give a complete description of the problem. In most situations, it is impossible to achieve the complete description, but it is certainly logical for us to focus our attention on the first items in the sequence."

…



For example, if one assumes a random quantity follows a Pearson distribution, the type is determined by the square of the skewness (β_1 in the left figure) and the kurtosis (β_2). The first four moments completely describe the parameters of the distribution.

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457.643 Structural Random Vibrations
In-Class Material: Class 02

I. Basic Elements (Contd.)

◎ Joint characteristic function

Alternative to ______ PDF

$$M_{\boldsymbol{X}}(\boldsymbol{\theta}) \equiv \mathrm{E}_{\boldsymbol{X}}\{\exp[i(\theta_1 X_1 + \theta_2 X_2 + \cdots + \theta_n X_n)]\} = \int \cdots \int \exp[i(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n)]\, f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$

→ m____variate F______ transform of _____ PDF

Therefore,

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(\qquad)^n} \int \cdots \int \exp[-i(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n)] \qquad\qquad d\boldsymbol{\theta}$$

One can show

$$\frac{1}{i^{m_1+\cdots+m_n}} \frac{\partial^{m_1+\cdots+m_n}}{\partial\theta_1^{m_1} \cdots \theta_n^{m_n}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{0}} = \mathrm{E}\left[X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}\right]$$

Some observations:

1) Consistency rule: $M_{\boldsymbol{X}}(\theta_1, \cdots, \theta_k, 0, \cdots, 0) =$
2) For statistically independent random variables, $M_{\boldsymbol{X}}(\boldsymbol{\theta}) =$

◎ Joint log characteristic function

Remember $M_X(\theta) = \mathrm{E}_X[\exp(i\theta X)] = \int_{-\infty}^{\infty} \exp(i\theta X)\, f_X(x) dx$

$L_{\boldsymbol{X}}(\boldsymbol{\theta}) =$

$$\kappa(\boldsymbol{X}) = \frac{1}{i^n} \frac{\partial^n L_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial\theta_1 \cdots \theta_n}\bigg|_{\boldsymbol{\theta}=\boldsymbol{0}}$$

- $\kappa(X_i) =$
- $\kappa(X_i, X_j) =$
- $\kappa(X_i, X_j, X_k) =$

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Example: PDF or characteristic function of $Y = X_1 + X_2 + \cdots + X_n$

◎ Multivariate normal (Gaussian) distribution

Joint PDF:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{|\det \mathbf{\Sigma}|}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{M})^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\mathbf{M})\right]$$

- completely determined by _____ and ______ order moments
- denoted by $\boldsymbol{X} \sim N(\boldsymbol{M}, \boldsymbol{\Sigma})$

e.g. $n = 1$, $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

Can show

$$\mathrm{M}_{\mathbf{X}}(\boldsymbol{\theta}) = \exp\left(i\boldsymbol{M}^{\mathrm{T}}\boldsymbol{\theta} - \frac{1}{2}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{\Sigma}\boldsymbol{\theta}\right)$$

$$\mathrm{L}_{\mathrm{X}}(\boldsymbol{\theta}) =$$

- _________ function of $\boldsymbol{\theta}$
- Higher order ($n \geq$) cumulants are zero

Example: $\kappa(X_i, X_j)$ for bivariate normal random variables

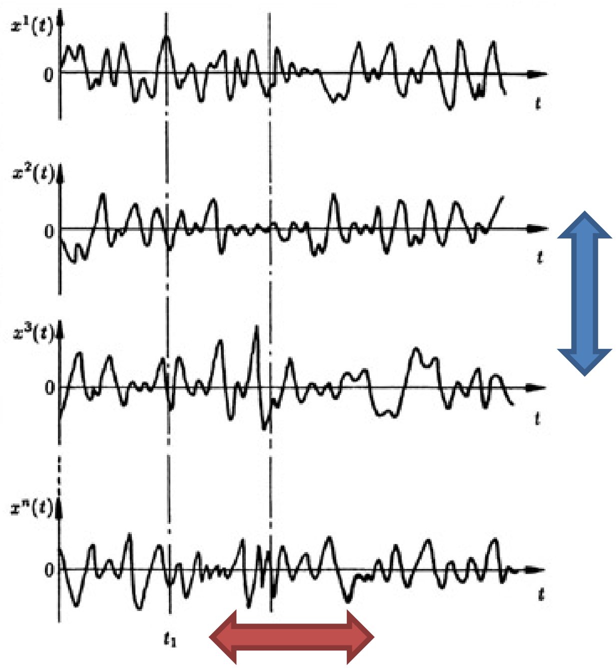
II. Introduction to Random Process

II-1. Random Process

◎ Definitions

Random (stochastic) process $\{X(t)\}$ or $X(t)$

e.g. earthquake ground motion



Definition 1: Random process is an "e_______" (collection) of possible t____ h______ $\{x^1(t), x^2(t), \ldots\}$

Definition 2: "Continuously indexed" r_______ v________, a family of random variables $\{X(0), \ldots X(t_k), \ldots X(t_m), \ldots\}$

Note: the concept of random process can be generalized

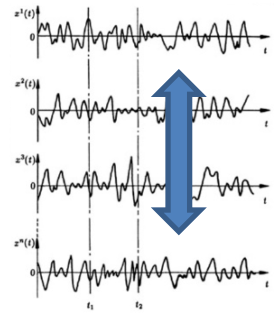
1) Random field $X(t, u, v)$ e.g. wind pressure at location (u, v) of the roof at time t
2) Vector random process:

$$\mathbf{X}(t) = \begin{Bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{Bmatrix} \quad \text{e.g. } \mathbf{X}_\mathrm{g}(t) = \begin{Bmatrix} x_g(t) \\ \dot{x}_g(t) \\ \ddot{x}_g(t) \end{Bmatrix}$$

3) Vector random field:

$$\mathbf{X}(t, u, v) = \begin{Bmatrix} X_1(t, u, v) \\ X_2(t, u, v) \\ \vdots \\ X_n(t, u, v) \end{Bmatrix}$$

◎ "Average" of random process

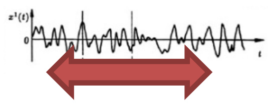


(a) "Ensemble" average: average over the ensemble

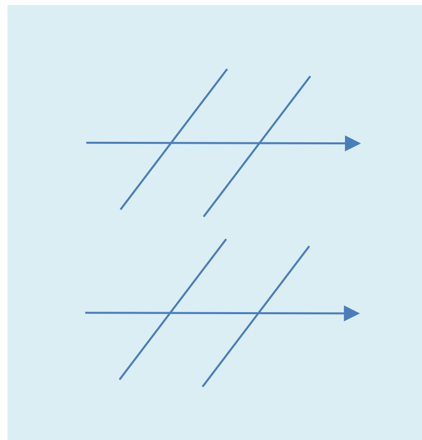
$$\mathrm{E}[X(t)] = \lim_{n\to\infty} \frac{\quad + \quad + \cdots + \quad}{n} = \int_{-\infty}^{\infty} \qquad\qquad dx$$

(b) "Temporal" average (for a specific time history)

$$\langle X(t)\rangle = \frac{\quad}{\quad}\int \quad x(t)dt$$



- Temporal average is a r_____ v______



◎ Specification of a random process

(a) By probabilistic distribution function

- $f_{X(t)}(x,t)$: 1st order "m_________" PDF
- $f_{X(t_1)X(t_2)}(x_1,t_1;x_2,t_2)$: 2nd order joint PDF
- $\vdots$
- $f_{X(t_1)\cdots X(t_n)}(x_1,t_1;\cdots;x_n,t_n)$: nth order joint PDF

Theoretically, one needs the ____th order joint PDF for complete description of a r.p.

(b) By characteristic function

- $M_{X(t)}(\theta,t)$: 1st order characteristic function
- $\vdots$
- $M_{X(t_1)\cdots X(t_n)}(\theta_1,t_1;\cdots;\theta_n,t_n)$: nth order joint characteristic function

(c) By moment functions (i.e. partial descriptors)
→ most common (because of lack of i________)

- $\mathrm{E}[X(\mathrm{t})] = \mu_X(t)$ or $\mu(t)$: _______ function
- $\mathrm{E}[X(t_1)X(t_2)] = \phi_{XX}(t_1,t_2)$ or $\phi(t_1,t_2)$: auto ___________ function
- $\mathrm{E}\{[X(t_1)-\mu(t_1)][X(t_2)-\mu(t_2)]\}$: auto _____________ function

(d) By a function of random variables

- $X(t) = At + B$
- $X(t) = \sum_{i=1}^{n} A_i\cos(\omega_i t + \Theta_i)$

(e) Others: log-characteristic function, cumulants, ARMA, etc.

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457.643 Structural Random Vibrations
In-Class Material: Class 03

◎ Central limit theorem (I. Basic Elements)

Consider $Z = X_1 + X_2 + \cdots + X_n$ where $X_i, i = 1, \dots, n$ are statistically independent, identically distributed (SIID) random variables. Try

$$Z' = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)$$

where μ and σ respectively denote the common mean and standard deviation of X_i's.

Let $Y_i = \frac{X_i - \mu}{\sigma}$ Then, $Z' = \frac{1}{\sqrt{n}} \sum_{i=0}^{n} Y_i$

The characteristic function of Z' is then derived as

$$\mathrm{M}_Z'(\theta) = \mathrm{E}[\exp(i\theta Z')] = \mathrm{E}\left\{ \exp\left[\frac{\mathrm{i}\theta}{\sqrt{n}} \sum_{j=1}^{n} Y_j \right] \right\} = \mathrm{E}\left[\prod_{j=1}^{n} \exp\left(\frac{\mathrm{i}\theta Y_j}{\sqrt{n}} \right) \right]$$

$$= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(\frac{\mathrm{i}\theta Y_j}{\sqrt{n}} \right) \right] = \prod_{j=1}^{n} M_{Y_j}\left(\frac{\theta}{\sqrt{n}} \right) = \left[M_Y\left(\frac{\theta}{\sqrt{n}} \right) \right]^n$$

statistically independent   Identically distributed

Let us consider the characteristic function of Y. Note that its mean is zero and standard deviation is one. From the moment generating property of the characteristic function,

$$\left. \frac{dM_Y(\theta)}{d\theta} \right|_{\theta=0} = \mathrm{i}\mathrm{E}[Y] = 0$$

$$\left. \frac{d^2 M_Y(\theta)}{d\theta^2} \right|_{\theta=0} = \mathrm{i}^2 \mathrm{E}[Y^2] = -(\sigma_Y^2 + \mu_Y^2) = -1$$

Therefore, the characteristic function $M_Y(\theta)$ can be constructed by a Taylor series:

$$M_Y(\theta) = 1 - \frac{\theta^2}{2} + o(\theta^2)$$

$$M_Y\left(\frac{\theta}{\sqrt{n}} \right) = 1 - \frac{\theta^2}{2n} + o\left(\frac{\theta^2}{n} \right)$$

$$M_{Z'}(\theta) = \left[1 - \frac{\theta^2}{2n} + o\left(\frac{\theta^2}{n} \right) \right]^n$$

From $\lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = e^x$

$$\lim_{n \to \infty} M_{Z'}(\theta) = \exp\left(-\frac{\theta^2}{2} \right)$$

The end result is the characteristic function of the standard normal distribution. Thus, we hereby prove that Z' asymptotically follows the standard normal distribution as $n \to \infty$. Since Z is a linear function of Z', Z also asymptotically follows a normal distribution.

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II-1. Random Process

◎ First & second order moment functions

$\mathrm{E}[X(\mathrm{t})] = \mu_X(t)$ or $\mu(\mathrm{t})$: Mean function

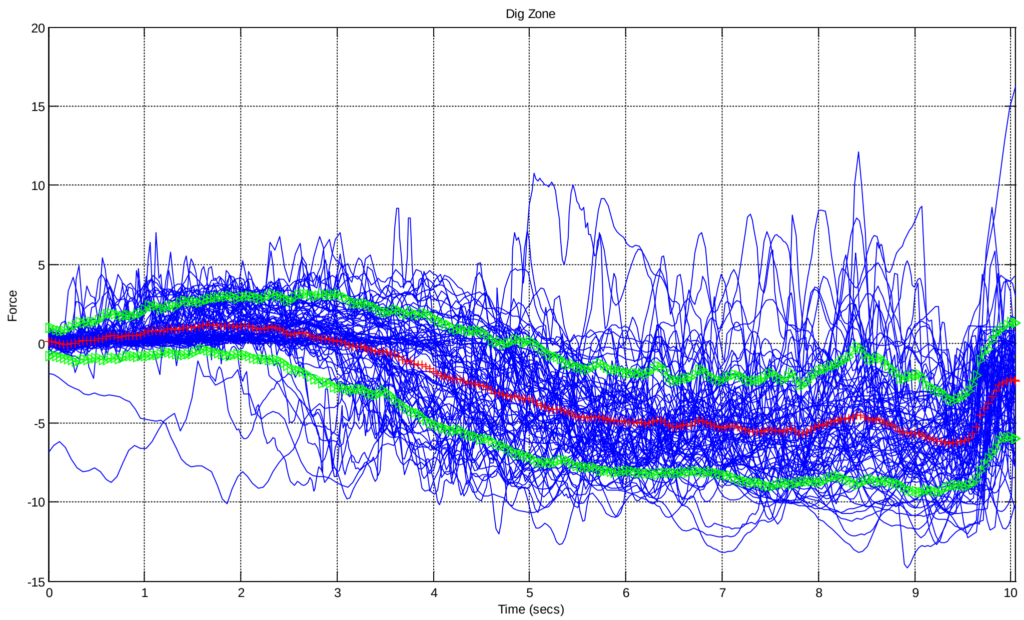
$\mathrm{E}[X(t_1)X(t_2)] = \phi_{XX}(t_1, t_2)$ or $\phi(t_1, t_2)$: Auto-correlation function

⋮

$\mathrm{E}\{[X(t_1) - \mu(t_1)][X(t_2) - \mu(t_2)]\} = \phi_{XX}(t_1, t_2) - \mu(t_1)\mu(t_2)$
$= \kappa_{XX}(t_1, t_2)$: Auto-covariance function

$\sigma_X(t) = \sqrt{\qquad}$: Standard deviation function

$\rho_{XX}(t_1, t_2) = \frac{\quad}{\quad}$: Auto-correlation-coefficient function



Example: 77 force time histories during "digging" tasks and their moment functions

Note:

If $\mu_X(t) = 0$ (zero-mean process),

$\phi_{XX}(t_1, t_2)$ $\qquad\qquad$ $\kappa_{XX}(t_1, t_2)$

One can transform a random process to a zero-mean process by

$Y(t) = X(t) -$

Why?

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• For a complex-valued random process,

$\phi_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X^*(t_2)]$

$\kappa_{XX}(t_1, t_2) = \mathrm{E}[(X(t_1) - \mu(t_1))(X^*(t_2) - \mu^*(t_2)]$

Note that $\phi_{XX}(t, t)$ and $\kappa_{XX}(t, t)$ are always ______-valued.

• More than one random process involved

$\phi_{XY}(t_1, t_2) = \mathrm{E}[X(t_1)Y^*(t_2)]$: ______ correlation function

$\kappa_{XY}(t_1, t_2) = \mathrm{E}[\big(X(t_1) - \mu(t_1)\big)\big(X^*(t_2) - \mu^*(t_2)\big)]$: ______ covariance function

$\rho_{XY}(t_1, t_2) = \overline{\qquad\qquad}$: _______ correlation coefficient function

• Importance of 1st and 2nd order moment functions

1) Most of the time, 1st and 2nd order moment functions are all one can get from data
2) For Gaussian, 1st and 2nd order moment functions are all you need for a complete description.
3) Using Chebyshev bounds, one can get upper bound estimate on the probability using moments

$$\mathrm{P}(|Z| > b) \leq \frac{\mathrm{E}[|Z|^c]}{\mathrm{b}^{\mathrm{c}}}$$

e.g. $c = 2$, $Z = X - \mu_X$

$$\mathrm{P}(|X - \mu_X| > b) \leq \frac{E[|X - \mu_X|^2]}{b^2} = \overline{\qquad}$$

◎ Five important properties of $\phi_{XY}(t_1, t_2)$ and $\kappa_{XY}(t_1, t_2)$

1) "Hermitian" ("Symmetric" for a real random process)

$\phi_{XY}(t_1, t_2) =$

$\kappa_{XY}(t_1, t_2) =$

2) Boundedness

Schwarz inequality $|\mathrm{E}[XY]| \leq \sqrt{\mathrm{E}[X^2]\mathrm{E}[Y^2]}$

Thus, $|\phi_{XY}(t_1, t_2)| \leq \sqrt{\phi_{XX}(\ ,\ )\phi_{YY}(\ ,\ )}$

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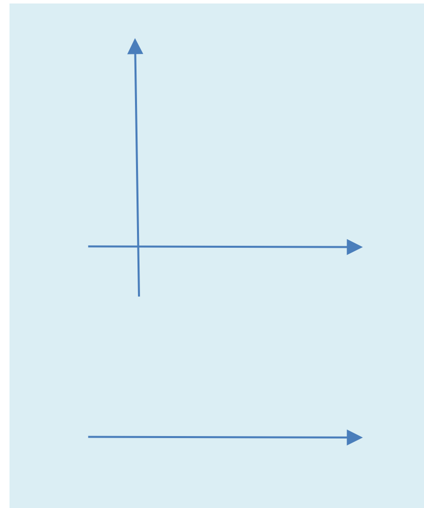
Also, $|\phi_{XX}(t_1, t_2)| \leq \sqrt{\phi_{XX}(\quad,\quad)\phi_{XX}(\quad,\quad)}$

Similarly, $|\kappa_{XY}(t_1, t_2)| \leq \sqrt{\kappa_{XX}(\quad,\quad)\kappa_{YY}(\quad,\quad)} = \sqrt{\sigma_X^2(\quad)\sigma_Y^2(\quad)}$



Note:

If $\mathrm{E}[X^2(t)]$ is bounded ($< \infty$) for $\forall t$,

$|\quad\;_{XX}(t, s)| < \infty$

If $\sigma_X^2(t)$ is bounded ($< \infty$) for $\forall t$,

$|\quad\;_{XX}(t, s)| < \infty$

$X(t)$ is a "________ _________" random process

if __________ is always finite

(Check L&S p.121. Later we will confirm that this means PSD exists)

3) Non-negative Definiteness

For an arbitrary function $h(t)$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_{XX}(t_i, t_j)h(t_i)h^*(t_j) \geq$$

Proof:

$$\begin{aligned}(\mathrm{LHS}) &= \{h(t_1)\ \cdots h(t_n)\}\left[\phi_{XX}(t_i, t_j)\right]_{n\times n}\{h^*(t_1)\ \cdots h^*(t_n)\}^{\mathrm{T}} \\ &= \mathbf{h}^{\mathrm{T}}\mathrm{E}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}]\mathbf{h}^* \\ &= \mathrm{E}[\mathbf{h}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\mathbf{h}^*] \\ &= \mathrm{E}[YY^*] \\ &= \mathrm{E}[\qquad]\qquad 0\end{aligned}$$

Why Important?

Fourier transform of non-negative definite function is _____________
(Lin 1967, p.42 – Bochner's theorem)

Lin, Y.K. (1967) *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, NY.

Note:
$\phi_{XY}(t_1, t_2)$: NOT non-negative definite

$\because E[\mathrm{XY}]$ can be __________
$\therefore$ Cross PSD can be ___________

457.643 Structural Random Vibrations
In-Class Material: Class 04

II-1. Random Process (contd.)

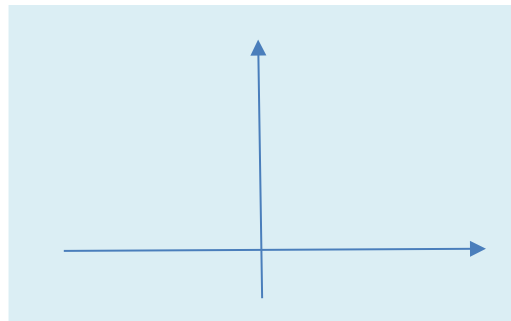
◎ **Five important properties of** $\phi_{XY}(t_1, t_2)$ and $\kappa_{XY}(t_1, t_2)$ **(contd.)**

4) For a process containing no periodic components,

__________ diminishes as $|t_1 - t_2| \to \infty$

$$\lim_{|t_1 - t_2| \to \infty} \kappa_{XX}(t_1, t_2) =$$

$$\lim_{|t_1 - t_2| \to \infty} \phi_{XX}(t_1, t_2) =$$



5) Continuity property

$\phi_{XY}(\cdot,\cdot)$ (or $\kappa_{XY}(\cdot,\cdot)$) must be continuous at (t_1, t_2) if $\phi_{XX}(\cdot,\cdot)$ and $\phi_{YY}(\cdot,\cdot)$ are continuous at (,) and (,) respectively.

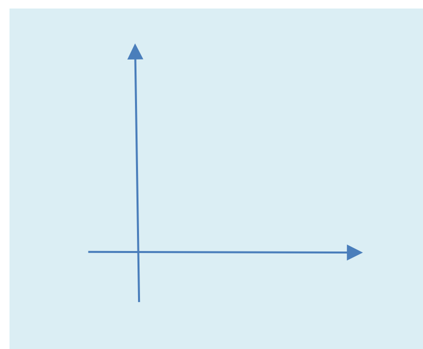
i.e.

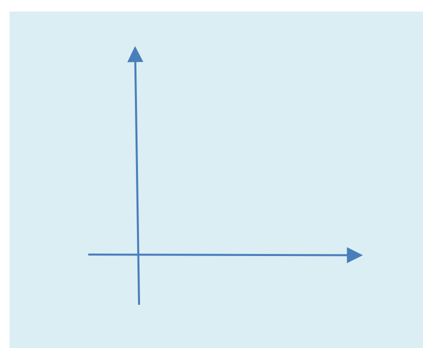
$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \phi_{XY}(t_1 + \epsilon_1, t_2 + \epsilon_2) =$$

if

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \phi_{XX}(t_1 + \epsilon_1, t_1 + \epsilon_2) = \qquad \text{and}$$

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \phi_{YY}(t_2 + \epsilon_1, t_2 + \epsilon_2) =$$





Therefore, if $\phi_{XX}(t_1, t_2)$ and $\phi_{YY}(t_1, t_2)$ are continuous at all points on the diagonal $t_1 = t_2$, $\phi_{XY}(t_1, t_2)$ is continuous at all points in the 2D domain (t_1, t_2)

Special case: $Y \to X$

$\phi_{XX}(\cdot,\cdot)$ (or $\kappa_{XX}(\cdot,\cdot)$) must be continuous at (t_1, t_2) if $\phi_{XX}(\cdot,\cdot)$ is continuous at (,) and (,).

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※ **Proof of "Continuity Property"**

Consider

$$\begin{aligned}\phi_{XY}(t_1+\epsilon_1,t_2+\epsilon_2)-\phi_{XY}(t_1,t_2) &= \mathrm{E}[X(t_1+\epsilon_1)Y(t_2+\epsilon_2)]-\mathrm{E}[X(t_1)Y(t_2)] \\ &= \mathrm{E}[\{X(t_1+\epsilon_1)-X(t_1)\}\{Y(t_2+\epsilon_2)-Y(t_2)\}] \\ &\quad +\mathrm{E}[\{X(t_1+\epsilon_1)-X(t_1)\}Y(t_2)] \\ &\quad +\mathrm{E}[X(t_1)\{Y(t_2+\epsilon_2)-Y(t_2)\}]\end{aligned} \tag{1}$$

Applying Schwarz's inequality to the first of the three expectations in Eq. (1), one can get

$$\begin{aligned}&|E[\{X(t_1+\epsilon_1)-X(t_1)\}\{Y(t_2+\epsilon_2)-Y(t_2)\}]| \\ &\qquad \le \sqrt{E[\{X(t_1+\epsilon_1)-X(t_1)\}^2]E[\{Y(t_2+\epsilon_2)-Y(t_2)\}^2]}\end{aligned}$$

The first term in the square root is expanded to

$$\phi_{XX}(t_1+\epsilon_1,t_1+\epsilon_1)-2\phi_{XX}(t_1+\epsilon_1,t_1)+\phi_{XX}(t_1,t_1)$$

This converges to zero if

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\phi_{XX}(t_1+\epsilon_1,t_1+\epsilon_2)=\phi_{XX}(t_1,t_2)$$

Therefore, the first expectation in Eq. (1) converges to zero.

Similarly, the other two expectations in Eq. (1) converge to zero if

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\phi_{XX}(t_1+\epsilon_1,t_1+\epsilon_2)=\phi_{XX}(t_1,t_1)$$

and

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\phi_{YY}(t_2+\epsilon_1,t_2+\epsilon_2)=\phi_{YY}(t_2,t_2)$$

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Example

$X(t) = A\cos\omega t + B\sin\omega t$

Given: $\mathrm{E}[A] = \mathrm{E}[B] = 0$, $\mathrm{E}[A^2] = \mathrm{E}[B^2] = \sigma^2$, $\mathrm{E}[AB] = \rho\sigma^2$

1) $\mathrm{E}[X(\mathrm{t})]$

2) $\phi_{XX}(t_1, t_2)$ and $\kappa_{\mathrm{XX}}(t_1, t_2)$

Does $\kappa_{XX}(t_1, t_2)$ diminish as $|t_1 - t_2| \to \infty$? Why or Why not?

3) $\sigma_X^2(t)$

4) $\rho_{XX}(t_1, t_2)$

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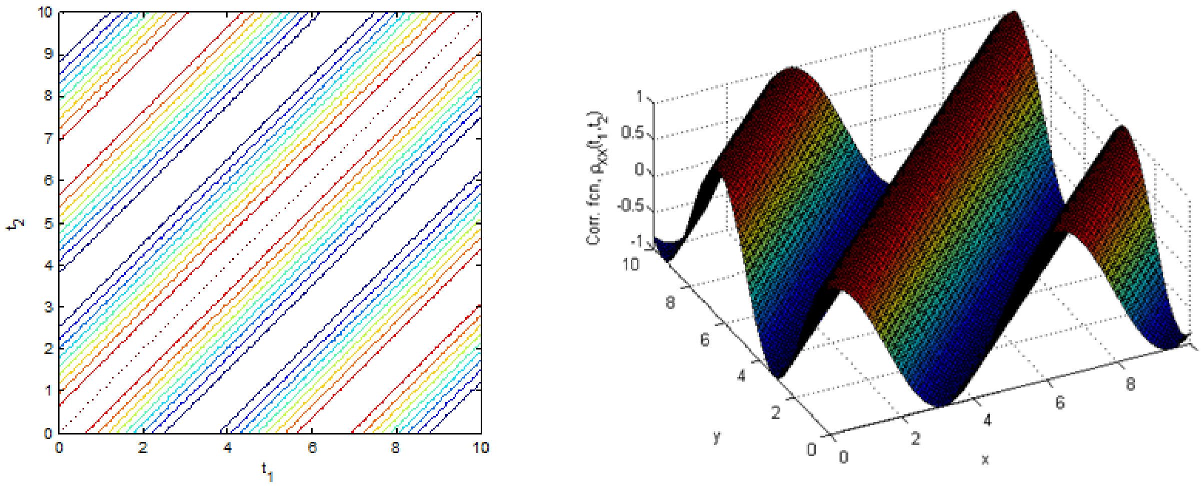
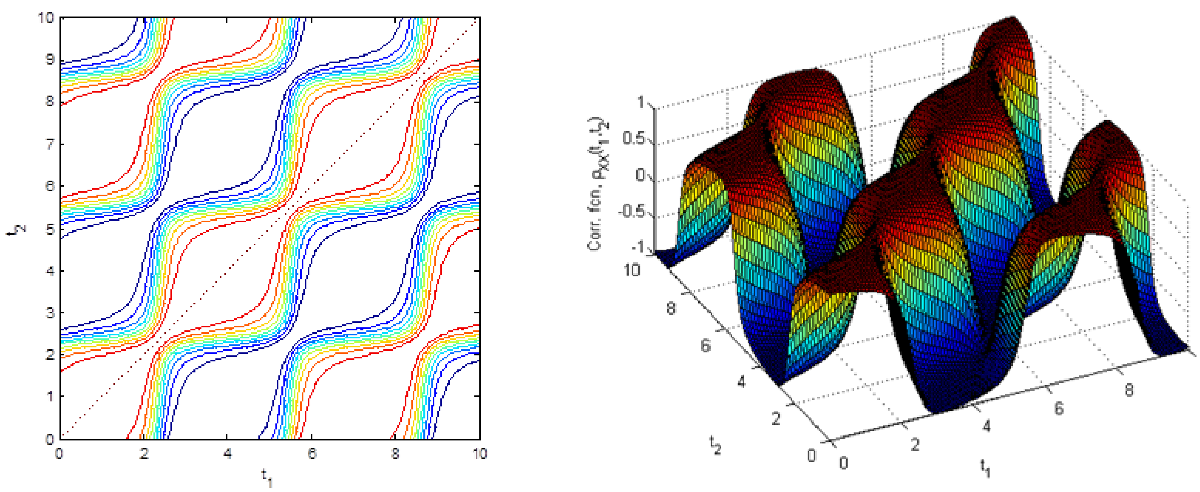
Correlation Coefficient Functions

Case I: $\rho = 0$



Case II: $\rho = 0.8$



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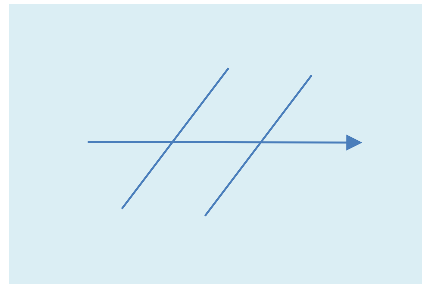
◎ Stationary process (cf. Homogeneous random field)

A R.P. is stationary if its ____________ description is invariant to a __________ in the parameter (time/space)

(Strictly Stationary)

$$f_{X\cdots X}(x_1,\cdots,x_n;t_1,\cdots,t_n)=f_{X\cdots X}(x_1,\cdots,x_n;t_1+h,\cdots,t_n+h)$$



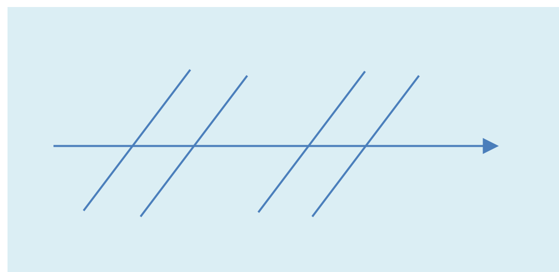
(1st Order Stationary)

$$f_X(x;t)=f_X(x;t+h)=$$

Therefore, $\mu_X(t)=\qquad$, $\sigma_X(t)=\qquad$, $\cdots$

(2nd Order Stationary)

$$f_{XX}(x_1,x_2;t_1,t_2)=f_{XX}(x_1,x_2;\qquad,\qquad)$$
$$=f_{XX}(x_1,x_2;\qquad)$$



Therefore,
$$\phi_{XX}(t_1,t_2)=\phi_{XX}(t_1+h,t_2+h)\quad\forall(t_1,t_2)$$
$$=R_{XX}(\tau)\quad\text{where } \tau=$$

$$\kappa_{XX}(t_1,t_2)=\kappa_{XX}(t_1+h,t_2+h)\quad\forall(t_1,t_2)$$
$$=\Gamma_{XX}(\tau)$$

"Weakly Stationary" or "Stationary in a Wide Sense" (Lin 1967)

When a random process satisfies

- $\mu_X(t)=$
- $\sigma_X(t)=$
- $\phi_{XX}(t_1,t_2)=$

Various Concepts of "Stationarity" in L&S

- Mean-value stationary
- Second-moment stationary
- *j*-th moment stationary
- *j*-th order stationary
- Strictly stationary

When (   ) and (   ) conditions above are satisfied, the random process is considered ________ ________

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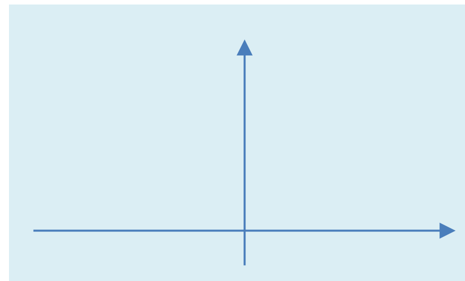
457.643 Structural Random Vibrations
In-Class Material: Class 05

II-1. Random Process (contd.)

◎ Properties of $R_{XX}(\tau)$ and $\Gamma_{XX}(\tau)$

(i.e. Properties of second motion functions of __________ process)

1) Hermitian (Symmetric)

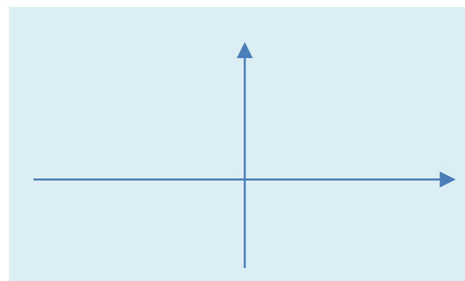


$$R_{XX}(\tau) = R_{XX}^{*}(-\tau)$$

$$\Gamma_{XX}(\tau) = \Gamma_{XX}^{*}(-\tau)$$

Real part, $\mathrm{Re}[R_{XX}(\tau)]$: _______ function

Imaginary part, $\mathrm{Im}[R_{XX}(\tau)]$: _______ function



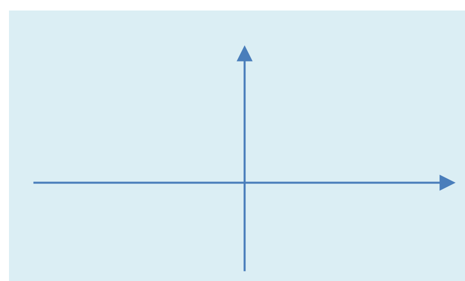
$$R_{XY}(\tau) =$$

$$\Gamma_{XY}(\tau) =$$

2) Boundedness

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(\quad)R_{YY}(\quad)}$$

$$|R_{XX}(\tau)| \leq \sqrt{R_{XX}(\quad)R_{XX}(\quad)} = R_{XX}(\quad) = \mathrm{E}[\qquad]$$



Similarly,

$$|\Gamma_{XY}(\tau)| \leq \sqrt{\Gamma_{XX}(\quad)\Gamma_{YY}(\quad)} =$$

$$|\Gamma_{XX}(\tau)| \leq$$

3) Non-negative Definiteness

$$\sum_i \sum_j R_{XX}(t_i - t_j)h(t_i)h^*(t_j) \geq 0$$

As the number of discretized points → ∞, the double summation becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1 - t_2)h(t_1)h^*(t_2)dt_1 dt_2 \geq 0$$

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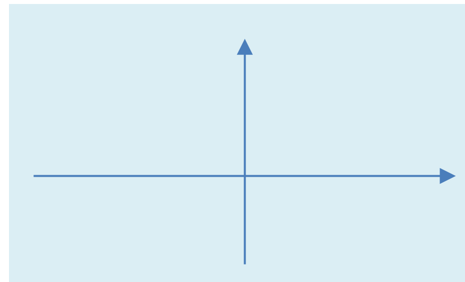
Substituting $t_1 = t_2 + \tau$, the integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau)h(t_2+\tau)h^*(t_2)\,d\tau dt_2 = \int_{-\infty}^{\infty} R_{XX}(\tau)H(\tau)d\tau \geq 0$$

4) Continuity



$R_{XX}(\tau)$ must be continuous at all τ

if $R_{XX}(\tau)$ is continuous at $\tau =$

5) $\kappa_{XX}(\tau)$ diminishes for r.p with no periodic components as $|\tau| \rightarrow$

$$\lim_{|\tau|\rightarrow\infty} \Gamma_{XX}(\tau) =$$

$$\lim_{|\tau|\rightarrow\infty} \mathrm{R}_{XX}(\tau) =$$

Example

$$R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{a} & 0 \leq |\tau| \leq ka \\ 0 & elsewhere \end{cases} \qquad 0 < k < 1,\ a > 0$$

Check if the auto-correlation model is valid in terms of the important properties.

Example

Recall $X(t) = A\cos\omega t + B\sin\omega t$ in an earlier example.

We derived $\phi_{XX}(t_1, t_2) = \sigma^2[\cos\omega(t_1 - t_2) + \rho\sin\omega(t_1 + t_2)]$ and $\mu_X(t) = 0$

1) Condition(s) to make $X(t)$ a weakly stationary process:

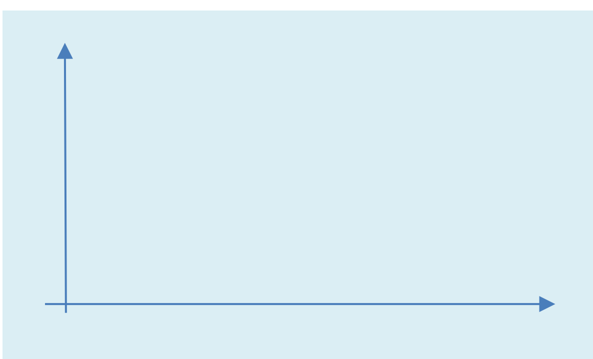
2) Suppose $\rho = 0$, and A and B are jointly Gaussian. Then, the process $X(t)$ is __________ ____________ ____________ process

◎ Poisson process

i) Example to demonstrate/review important concepts of random processes

ii) Introduction to an important class of random processes

$N(t)$: Number of _____________ in $(0, t]$



- C__________ index parameter
- D________-valued process
- Inherently stationary/non-stationary process
- Examples:

◎ Basic <u>assumptions</u> of Poisson random process

1) There exists m_____ o________ rate (or intensity function), defined as

$$\lim_{\Delta t \to 0} \frac{\text{Average No. of Occurrences in } (t, t + \Delta t)}{\Delta t} =$$

2) “Probability of two or more occurrences in Δt” $\ll$

Therefore,

$$
\begin{aligned}
&\text{Average No. of Ocurrences in } (t, t+\Delta t) \\
&= \nu(t)\cdot \\
&= \sum_{n=0}^{\infty} n \cdot P\big(n \text{ occurrences in } (t, t+\Delta t)\big) \\
&= 1 \cdot P\big(1 \text{ occurrence in } (t, t+\Delta t)\big) + 2 \cdot P\big(2 \text{ occurrences in } (t, t+\Delta t)\big) + \cdots \\
&\cong
\end{aligned}
$$

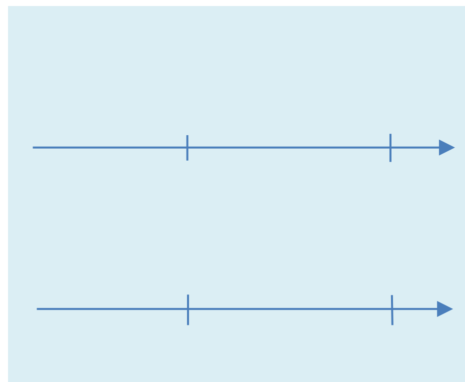
3) No. of occurrences in two non-overlapping intervals are _____________ _______________

◎ Probability functions and partial descriptors of Poisson process

1) Probability mass function (PMF) of $N(t)$

$$
\begin{aligned}
&P_{N(t)}(n;t) \equiv P(N(t) = \quad) \\
&= P_n(t) \\
&= P_n(t-\Delta t)\cdot(1-\nu\cdot\Delta t) + P_{n-1}(t-\Delta t)\cdot\nu\cdot\Delta t
\end{aligned}
$$

“scenario 1” “scenario 2”



Thus,

$$\frac{P_n(t) - P_n(t-\Delta t)}{\Delta t} + \nu \cdot P_n(t-\Delta t) = \nu \cdot P_{n-1}(t-\Delta t)$$

As $\Delta t \to 0$, we get a recursive ODE:

$$\frac{d}{dt}P_n(t) + \nu(t)\cdot P_n(t) = \nu(t)\cdot P_{n-1}(t)$$

Solution: $P_n(t)\cdot\exp\Big(\underbrace{\int_0^t \nu(t)dt}_{=m(t)}\Big) = \int_0^t \nu(t)\cdot P_{n-1}(t)\cdot\exp\left(\int_0^t \nu(t)dt\right)dt + C_n$

i) $n = 0$

$$P_0(t)\cdot e^{m(t)} = \int_0^t \nu(t)P_{-1}(t)e^{m(t)}\,dt + C_0$$

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$P_0(t) = C_0 \cdot e^{-m(t)}$

Initial condition $P_0(0) = 1$. Therefore, $C_0 =$

$P_0(t) =$

ii) $n = 1$

$P_1(t) \cdot e^{m(t)} = \int_0^t \nu(t) P_0(t) e^{m(t)}\, dt + C_1$

$=$

Initial condition $P_1(0) =$. Therefore, $C_1 =$

$P_1(t) =$

Solving recursively, one can get

$$P_n(t) = P_N(n; t) = \frac{[m(t)]^n \exp[-m(t)]}{n!}, \quad n = 0,1,2,\cdots$$

PMF of Poisson process $N(t)$ ("Poisson distribution" PMF)

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In-Class Material: Class 06

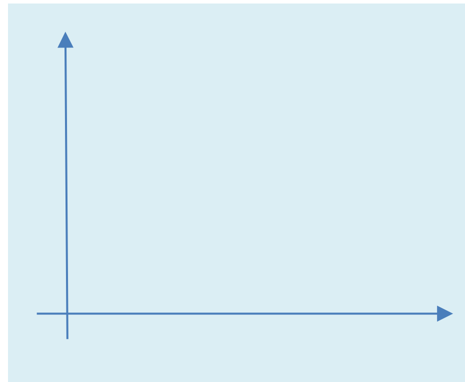
II-1. Random Process (contd.)

◎ Probability functions and partial descriptors of Poisson process (contd.)

2) "Homogeneous" Poisson process (HPP)

Definition: $\nu(t) =$

$$\therefore m(t) = \int_0^t \nu(t)dt =$$



Continuous change of ________
over time duration length *t*

PMF of HPP:

$$p_N(t) = \frac{[\quad]^n \exp(-\quad\quad)}{n!}$$

3) First-order characteristic function

$$M_N(\theta,t) = \mathrm{E}\left[\exp\left(\mathrm{i}\theta N(t)\right)\right]$$

$$= \sum_{\mathrm{n}=0}^{\infty} \exp(\mathrm{i}\theta n)\cdot\frac{[\quad\quad]^n\cdot\exp[\quad\quad]}{n!}$$

$$= \exp[\quad\quad]\sum_{\mathrm{n}=0}^{\infty}\frac{[\quad\quad\quad\quad\quad]^n}{n!}$$

Note: $\exp(\mathrm{x}) = \sum_{\mathrm{n}=0}^{\infty}\frac{x^n}{n!}$

$$= \exp[\quad\quad]\cdot\exp[\quad\quad\quad\quad\quad]$$

$$= \exp[-m(t)\cdot(1-\exp(\mathrm{i}\theta))]$$

4) Mean

$$\mathrm{E}[N(t)] = \frac{1}{i}\cdot\left.\frac{dM}{d\theta}\right|_{\theta=0}$$

$$\frac{dM}{d\theta} =$$

$$\mathrm{E}[N(t)] = \mu_N(t) = m(t) = \int_0^t \nu(t)dt$$

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5) Standard deviation

$$\mathrm{E}[N^2(t)] = \frac{1}{i^2}\cdot\frac{d^2M}{d\theta^2}\bigg|_{\theta=0}$$
$$= m(t) + m^2(t)$$

$$\therefore\ Var[N(t)] =$$

$$\therefore \sigma_{N(t)} = \sqrt{\qquad\qquad} = \sqrt{\qquad\qquad\qquad}$$

6) Mean and standard deviation for HPP

$$\mathrm{E}[N(t)] =$$

$$\mathrm{E}[N^2(t)] =$$

$$\sigma_{N(t)} =$$

Question: Is HPP a stationary process?

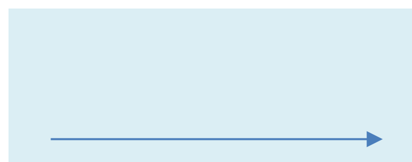
7) 2nd order joint PMF

$$P_{NN}(n_1, n_2; t_1, t_2) = P(\qquad\qquad\qquad\qquad\qquad)$$

$$= P(\qquad\qquad\qquad\qquad\qquad)$$

$$= P(N(t_2) = n_2) \times$$

$$= \frac{[m(t_2)]^{n_2}\cdot\exp[-m(t_2)]}{n_2!}$$
$$\times \frac{[\qquad\qquad]^{n_1-n_2}\cdot\exp[\qquad\qquad]}{(n_1-n_2)!}$$
$$= \frac{[m(t_2)]^{n_2}\cdot[m(t_1)-m(t_2)]^{n_1-n_2}\cdot\exp[-m(t_1)]}{n_2!\,(n_1-n_2)!}$$



Note: Set $t_1 > t_2$ and $n_1 \geq n_2$
The derivation depends on this convention

8) Joint characteristic function

$$M_{NN}(\theta_1, \theta_2; t_1, t_2) = \mathrm{E}\{\exp[\mathrm{i}(\theta_2 N(t_2) + \theta_1 N(t_1))]\}$$

→ This is not the same as the product of the two expectations, i.e. two marginal characteristic functions. Why?

$$= \mathrm{E}\{\exp[\mathrm{i}(\theta_1+\theta_2)N(t_2)]\cdot\exp[\mathrm{i}\theta_1(N(t_1)-N(t_2))]\}$$

$$= \mathrm{E}\{\exp[\mathrm{i}(\theta_1+\theta_2)N(t_2)]\}\cdot\mathrm{E}\{\exp[\mathrm{i}\theta_1(N(t_1)-N(t_2))]\}$$

→ Why?

$$= \exp[-m(t_2)\cdot(1-\exp(i(\theta_1+\theta_2)))]$$
$$\times\exp[-(m(t_1)-m(t_2))\cdot(1-\exp(i(\theta_1)))]$$

$$= \exp\{-m(t_2)[1-\exp(\mathrm{i}(\theta_1+\theta_2))] - (m(t_1)-m(t_2))(1-\exp(\mathrm{i}\theta_1))\}$$

9) Auto correlation function

$$
\begin{aligned}
\phi_{NN}(t_1, t_2) &= \mathrm{E}[N(t_1)\cdot N(t_2)] \\
&= \mathrm{E}\{N(t_2)\cdot[N(t_1) - N(t_2) + N(t_2)]\} \\
&= \mathrm{E}\{N(t_2)\cdot[N(t_1) - N(t_2)]\} + \mathrm{E}[N^2(t_2)] \\
&= \mathrm{E}[N(t_2)]\cdot \mathrm{E}[N(t_1) - N(t_2)] + \mathrm{E}[N^2(t_2)] \\
&= m(t_2)\cdot[m(t_1) - m(t_2)] + m^2(t_2) + m(t_2) \\
&= m(t_2) + m(t_1)\cdot m(t_2)
\end{aligned}
$$

→ Violating symmetry?

10) Auto covariance function

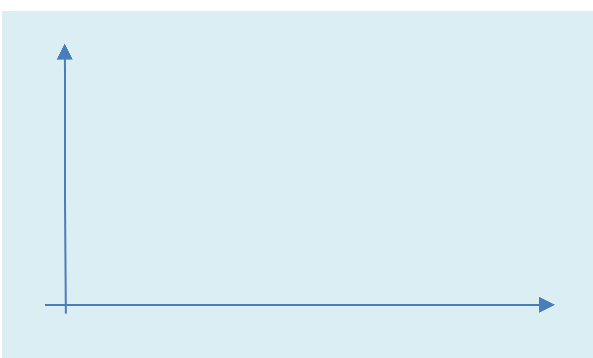
$$
\begin{aligned}
\kappa_{NN}(t_1, t_2) &= \phi_{NN}(t_1, t_2) - \\
&= m(t_2) + m(t_1)\cdot m(t_2) - \\
&=
\end{aligned}
$$

→ Violating symmetry?

11) Auto correlation coefficient function

$$\rho_{NN}(t_1, t_2) = \frac{\qquad}{\qquad} = \frac{\qquad}{\sqrt{\qquad}\cdot\sqrt{\qquad}} = \sqrt{\frac{\qquad}{\qquad}} \leq$$

◎ Waiting time until the n^{th} occurrence of a Poisson process W_n



W_n: Waiting time until the n^{th} occurrence

$T_n = W_n - W_{n-1}$: ___________ time

1) Probability ___________ function of W_n

$f_{W_n}(t)dt = P(t < W_n \leq t + dt)$ ~ Definition of PDF

Therefore,

$$f_{W_n}(t) = \lim_{\Delta t \to 0} \frac{P(t < W_n \leq t + \Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{P_N(n-1, t) \times \nu(t) \times \Delta t + o(\Delta t)}{\Delta t}$$

$$= \nu(t) \cdot \frac{[\qquad]^{n-1} \cdot \exp[\qquad]}{(\qquad)!}, \qquad t > 0$$

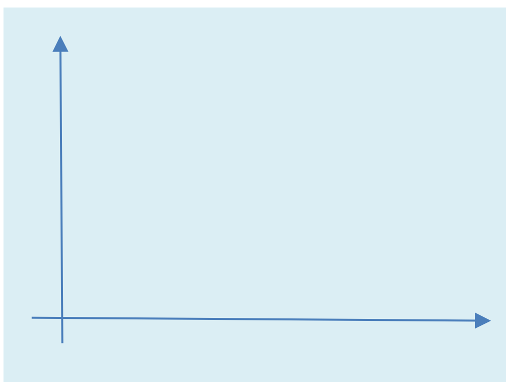
Note: $(n-1)$ occurrences up to time t and ________ occurrence during _____

2) PDF of W_n for HPP

$$f_{W_n}(t) = \text{—————————}, \quad t > 0$$

$$= \text{—————————}$$

" " **PDF:** n is a real number



$n = 1$: ____________________ distribution

$f_{W_1}(t) =$

PDF of waiting time until _____ occurrence $=$ PDF of ________ time, T_n (will be shown below)

3) Distribution functions of interarrival time $T_n = W_n - W_{n-1}$

CDF

$$F_{T_n}(t) = P(T_n \leq t)$$

$$= 1 - P(T_n > t)$$

$$= 1 - \int_0^\infty P(T_n > t | W_{n-1} = w) f_{W_{n-1}}(w) dw$$

Here,

$$P(T_n > t | W_{n-1} = w) = P(\qquad \text{events in } (w, w+t))$$

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Using Poisson distribution,

$$P(T_n > t|W_{n-1} = w) = \exp[-m(w+t) + m(w)]$$

Therefore,

$$F_{T_n}(t) = 1 - \int_0^{\infty} \exp[-m(w+t) + m(w)] \times \frac{\nu(w)[m(w)]^{n-2} \exp[-m(w)]}{(n-2)!} dw$$

$$= 1 - \int_0^{\infty} \frac{\nu(w) m(w)^{n-2} \exp[-m(w+t)]}{(n-2)!} dw$$

For HPP,

$$F_{T_n}(t) = 1 - \int_0^{\infty} \frac{\nu(\nu w)^{n-2} \exp[-\nu(w+t)]}{(n-2)!} dw = 1 - \exp(-\nu t)$$

$$f_{W_n}(t) = \frac{dF_{T_n}(t)}{dt} = \nu \cdot \exp(-\nu t) = f_{T_1}(t)$$

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457.643 Structural Random Vibrations
In-Class Material: Class 07

II-1. Random Process (contd.)

◎ Normal (Gaussian) process (Read L&S 4.10)

$X(t)$ is a Gaussian process
if, for any n, and any $\{t_1, t_2, \cdots, t_n\}$,
the random variables $X(t_1), \cdots, X(t_n)$ are ___________ ____________

- The process is completely defined by specifying for $\forall t$ and $\kappa_{XX}(t,s)$ for $\forall(t,s)$
- For a Gaussian process, being "weakly stationary" implies stationarity in the ___________ sense
- Any linear function of Gaussian processes is a ___________ process
  e.g. $\dot{X}(t)$ is Gaussian if $X(t)$ is Gaussian (why?)
  e.g. $X(t)$ and $\dot{X}(t)$ are ___________ ____________
- Why useful?
  1) Convenient to handle
  2) _________ __________ theorem
- Hard to justify Gaussian process assumption if
  1) the distribution is not symmetric, or
  2) ___________ is not equal to 3
- Textbook focusing on non-Gaussian processes: M. Grigoriu (1995), *Applied Non-Gaussian Processes*

◎ Jointly Gaussian processes

$X_1(t), X_2(t), \cdots, X_m(t)$ are jointly Gaussian processes
if, for any n, and any $\{t_1, t_2, \cdots, t_n\}$, the random variables
$\{X_1(t_1), \cdots, X_1(t_n), X_2(t_1), \cdots, X_2(t_n), \cdots, X_m(t_1), \cdots, X_m(t_n)\}$ are
___________ ____________

- The processes are completely defined by specifying $\boldsymbol{M}_{\boldsymbol{X}}(t) = \{ \qquad \}^{\mathrm{T}}$ and $\boldsymbol{\Sigma}_{(t,s)} = [\qquad]$

II-2. Stochastic Calculus

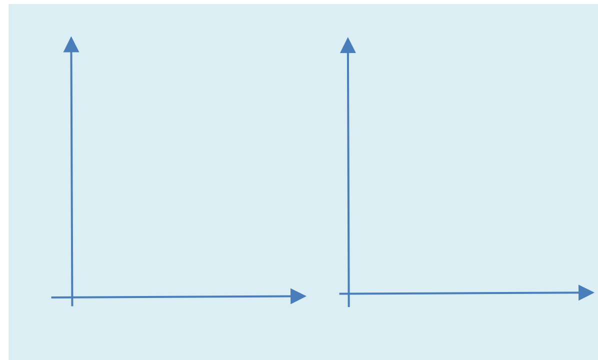
Lin, Y.K. (1967) *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, NY.

◎ Motivation

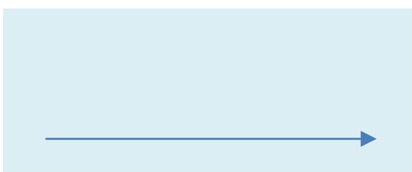
$$\frac{d}{dt}X(t) = \lim_{h\to 0}\frac{X(t+h) - X(t)}{h}$$

The conventional “limit” cannot be applied to random processes

Limit of a random process?



Need to consider the convergence of a sequence of random variables, i.e. $\lim_{n\to\infty}\{X_1, X_2, \cdots, X_n\}$



- ➔ Converging to the distribution of a random variable (not a particular value)
- ➔ “____________ Convergence”

◎ Definitions of stochastic convergence

1) Convergence with probability 1 (“almost sure” convergence)

$$P\left(\lim_{n\to\infty} X_n = X\right) =$$

2) Convergence in probability

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = \qquad , \forall\epsilon > 0$$

3) Convergence in distribution

$$\lim_{n\to 0} F_{X_n}(x) =$$

4) ** Convergence in the mean square

$$\lim_{n\to\infty} \mathrm{E}[|X_n - X|^2] =$$

→ requires $\mathrm{E}[X^2] < \infty$, i.e. “ ” process

Throughout this course, we use the fourth definition with the notation

$$\underset{t\to t_0}{\mathbf{l.i.m.}}\, \boldsymbol{X(t) = X}$$

to describe “**L**imit **I**n the **M**ean-square”

◎ Two theorems for limit in the mean square

Theorem 1:

If $\underset{t\to t_0}{\mathrm{l.i.m.}}\, X(t) = X$ and $\underset{s\to s_0}{\mathrm{l.i.m.}}\, Y(s) = Y$, then $\lim_{t\to t_0, s\to s_0} \mathrm{E}[X(t)\cdot Y(s)] =$

Proof:

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Using Theorem 1, we can show $\lim_{t\to t_0} \mathrm{E}[X(t)] = \mathrm{E}\left[\underset{t\to t_0}{\mathrm{l.i.m.}} X(t)\right]$

Namely, $E[\cdot]$ and $\mathrm{l.i.m.}$ are c_____________ or exchangeable

Proof:

Theorem 2:

$\underset{t\to t_0}{\mathrm{l.i.m.}} X(t) = X$ $\qquad \phi_{XX}(t,s)$ is continuous at (t_0, t_0) no matter how (t,s) approaches (t_0, t_0)

- See Ex 4.9 in L&S
- Of course, for "second-order process"

Proof:

◎ Mean-square derivative (derivative of r.p. in mean square sense)

Note: Deterministic: $\dot{x}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$

Definition of "mean-square" derivative of a random process:

$$\dot{X}(t) \equiv \underset{h \to 0}{\mathrm{l.i.m}} \frac{X(t+h) - X(t)}{h}$$

When is a random process "mean-square differentiable"? (or when does the limit exist in the mean square sense?)

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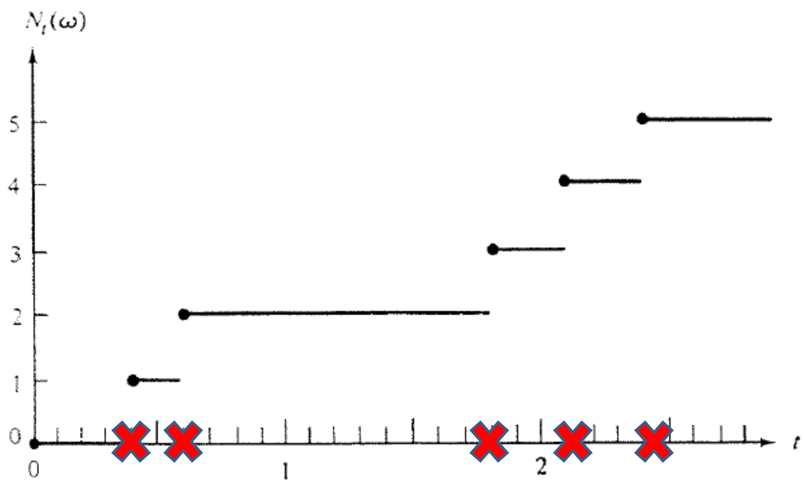
457.643 Structural Random Vibrations
In-Class Material: Class 08

II-1. Random Process (going back)

◎ Artificial generation of Poisson process



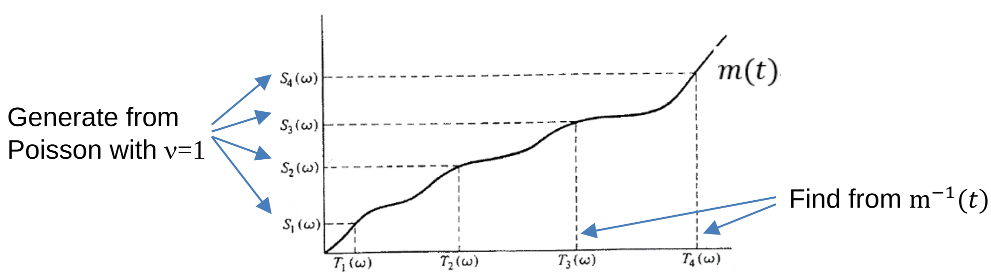
- Actually, we generate waiting times (arrival times) for $n = 1,2,3,\ldots$, i.e. $W_1, W_2, \ldots$
- For a homogeneous Poisson process, we can generate W_n using T_1
- We know T_1 follows ________ distribution
- In Matlab®, one can generate _________ random variables using `exprnd(μ,M,N)`

  1) μ: mean $= 1/\nu$

  2) M,N: size of the output matrix
- To generate non-homogeneous Poisson process, need to use a theorem,

  $W_i = m^{-1}(S_i), i = 1,2,\ldots$, are arrival times of the non-homogeneous Poisson process with $m(t)$ when $S_i, i = 1,2,\ldots$, are arrival times of the homogeneous Poisson process with $\nu = 1$

Cinlar, E. (1975). *Introduction to Stochastic Processes*, Dover Books (reprinted in 2013)



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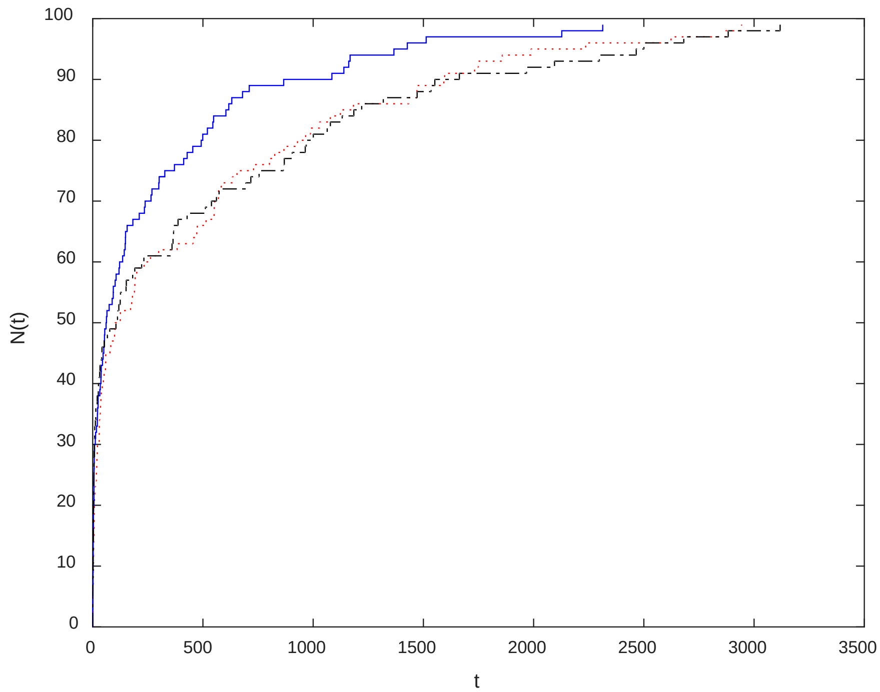
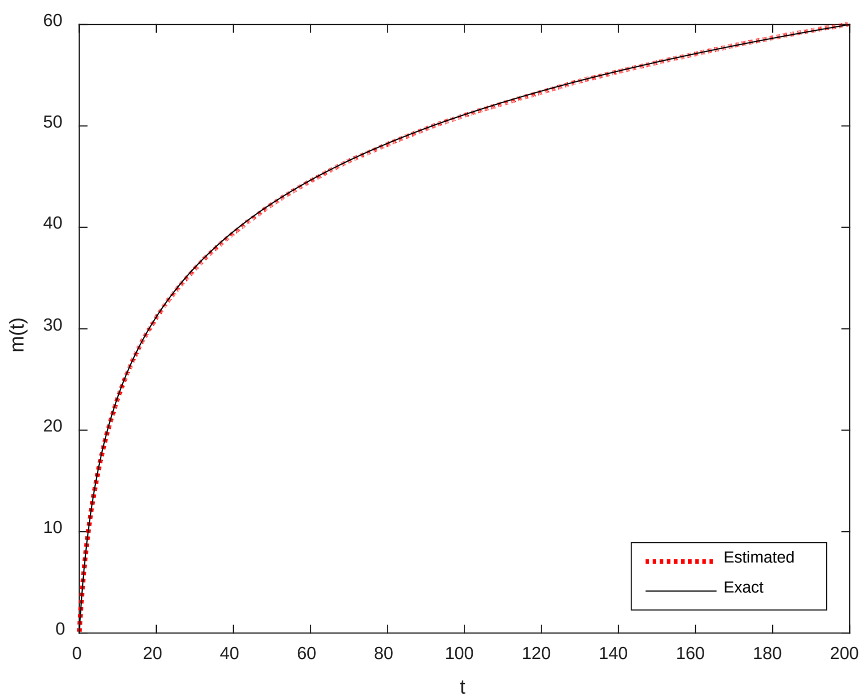
- **Example:** Generating NHPP with $m(t) = 13 \cdot \ln(0.5t + 1)$

  1) Three random samples:



  2) Comparison between exact $m(t)$ and estimated one using 1,000 samples



** Check “NHPoissonGenerationTest.m” at eTL website for details

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II-2. Stochastic Calculus (contd.)

◎ Mean-square derivative (derivative of r.p. in mean square sense) (contd.)

Definition of "mean-square" derivative of a random process:

$$\dot{X}(t) \equiv \underset{h\to 0}{\mathrm{l.i.m}} \frac{X(t+h)-X(t)}{h}$$

When is a random process "mean-square differentiable"? (or when does the limit exist in the mean square sense?)

Recall Theorem 2 with $X(t)$ replaced by $Y(t)$:

Theorem 2:

$\underset{t\to t_0}{\mathrm{l.i.m.}}\, Y(t) = Y$ iff $\phi_{YY}(t,s)$ is continuous at (t_0, t_0) no matter how (t,s) approaches (t_0, t_0)

Substituting $Y(t) = \frac{X(t+h)-X(t)}{h}$ above, we need to check the limit of $\phi_{YY}(t,s)$ at the diagonal, i.e. $t = s$. Consider

$$\begin{aligned}
\lim_{h\to 0, h'\to 0} \phi_{YY}(t,s) &= \lim_{h\to 0, h'\to 0} \mathrm{E}\left[\frac{X(t+h)-X(t)}{h}\cdot\frac{X(s+h')-X(s)}{h'}\right] \\
&= \lim_{h\to 0, h'\to 0} \frac{1}{h}\left[\frac{\phi_{XX}(t+h, s+h')-\phi_{XX}(t+h,s)}{h'} - \frac{\phi_{XX}(t,s+h')-\phi_{XX}(t,s)}{h'}\right] \\
&= \frac{\partial^2}{\partial \quad \partial \quad}
\end{aligned}$$

Therefore, $X(t)$ is mean-square differentiable iff $\phi_{XX}(t,s)$ is ______-_______ __________ at $t = s$

In summary,

- $\phi_{XX}(t,s)$ is continuous at $t = s = t_0$ iff $\underset{t\to t_0}{\mathrm{l.i.m}}\, X(t) = X$ (Theorem 2)
- $\phi_{XX}(t,s)$ is second-order differentiable at $t = s = t_0$ iff $\dot{X}(t)$ exists at $t = t_0$ (mean-square differentiable)

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◎ Properties of $\dot{X}(t)$

1) $\mathrm{E}\left[\dot{X}(t)\right] = \mathrm{E}\left[\underset{h\to 0}{\mathrm{l.i.m.}}\frac{X(t+h)-X(t)}{h}\right]$

$$= \lim_{h\to 0}\mathrm{E}\left[\frac{X(t+h)-X(t)}{h}\right]$$

$$= \lim_{h\to 0}$$

$$= \frac{d}{d}$$

The mean of the (mean-square) derivative of a r.p. is the derivative of the mean function

2) $\mathrm{E}\left[X(t)\cdot\dot{X}(s)\right] = \phi_{X\dot{X}}(t,s)$

$$= \mathrm{E}\left[X(t)\cdot\underset{h\to 0}{\mathrm{l.i.m.}}\frac{X(s+h)-X(s)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\quad}{\quad}\right]$$

$$= \frac{\partial}{\partial}$$

$$\therefore\ \mathrm{E}\left[\dot{X}(t)\cdot X(s)\right] = \phi_{\dot{X}X}(t,s) = \frac{\partial}{\partial}$$

3) $\mathrm{E}\left[\dot{X}(t)\cdot\dot{X}(s)\right] = \phi_{\dot{X}\dot{X}}(t,s)$

$$= \mathrm{E}\left[\underset{h_1\to 0}{\mathrm{l.i.m.}}\frac{X(t+h_1)-X(t)}{h_1}\cdot\underset{h_2\to 0}{\mathrm{l.i.m.}}\frac{X(s+h_2)-X(s)}{h_2}\right]$$

$$= \frac{\partial^2}{\partial\ \partial}$$

◎ Mean-square derivative $\dot{X}(t)$ for a stationary r.p. $X(t)$

- $\mu_X(t) = \mu$
- $\phi_{XX}(t,s) = R_{XX}(\tau),\ \ \tau = t - s$

1) $X(t)$ is mean-square continuous iff $R_{XX}(\tau)$ is continuous at τ =

2) $X(t)$ is mean-square differentiable iff $\frac{\partial^2\phi_{XX}(t,s)}{\partial t\partial s} =$ is unique and finite at τ =

- $\frac{\partial\phi_{XX}(t,s)}{\partial s} =$
- $\frac{\partial^2\phi_{XX}(t,s)}{\partial t\partial s} =$

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3) $\mu_{\dot{X}}(t) = \mathrm{E}\left[\dot{X}(t)\right] = \frac{d}{d}\_\_\_ =$

The mean of the time rate of a stationary r.p. is

4) Cross correlation between $X(t)$ and $\dot{X}(t)$

$$R_{X\dot{X}}(\tau) = \frac{\partial \phi_{XX}(t,s)}{\partial} = -\_\_\_$$

$$R_{\dot{X}X}(\tau) = \frac{\partial \phi_{XX}(t,s)}{\partial} = \_\_\_$$

at $\tau = 0$

$$R_{X\dot{X}}(0) = \mathrm{E}[\quad \cdot \quad] = -\frac{d}{d\tau} R_{XX}(\tau)\Big|_{\tau=0}$$

$$R_{\dot{X}X}(0) = \mathrm{E}[\quad \cdot \quad] = \frac{d}{d\tau} R_{XX}(\tau)\Big|_{\tau=0}$$

$\therefore R_{X\dot{X}}(0) = R_{\dot{X}X}(0) =$

When $X(t)$ is stationary r.p. and mean-square differentiable,

- $X(t)$ and $\dot{X}(t)$ are _____________, i.e. $\mathrm{E}\left[X\dot{X}\right] = 0$
- $X(t)$ and $\dot{X}(t)$ are _____________ as well because

457.643 Structural Random Vibrations

In-Class Material: Class 09

II-2. Stochastic Calculus (contd.)

◎ Mean-square derivative $\dot{X}(t)$ for a stationary r.p. $X(t)$ (contd.)

5) $R_{\dot{X}X}(\tau) = \mathrm{E}[\dot{X}(t+\tau) \cdot X(t)]$

$R_{X\dot{X}}(-\tau) = \mathrm{E}\big[X(t-\tau) \cdot \dot{X}(t)\big] = \mathrm{E}[X(t) \cdot \dot{X}(t+\tau)]$

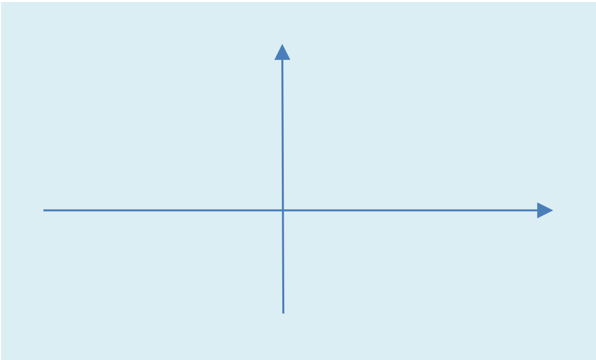
Therefore,

$R_{X\dot{X}}(-\tau) = R_{\dot{X}X}(\tau)$

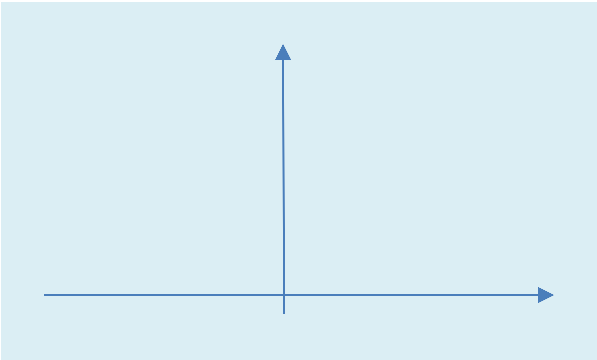
$= -R_{X\dot{X}}(\tau)$

Note: $R_{\dot{X}X}(\tau) = dR_{XX}(\tau)/d\tau$ and $R_{X\dot{X}}(\tau) = -dR_{XX}(\tau)/d\tau$

$R_{X\dot{X}}(\tau)$ is an ________ function (_____symmetric around $\tau =$)



◎ Example: $R_{XX}(\tau) = \frac{n\sigma^2}{2} \cdot \frac{sin\omega\tau}{\omega\tau}$



1) Is the random process $X(t)$ mean-square continuous?

$\lim_{\tau \to 0} R_{XX}(\tau) =$

$R_{XX}(\tau)$ is ______________ at $\tau =$. Therefore, $X(t)$ is ____________________

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2) Is the random process $X(t)$ mean-square differentiable?

$$\frac{dR_{XX}(\tau)}{d\tau} = \frac{n\sigma^2\omega}{2} \cdot \frac{\omega\tau \cdot \cos\omega\tau - \sin\omega\tau}{(\omega\tau)^2}$$

(Is $R_{\dot{X}X}(\tau)$ anti-symmetric around $\tau = 0$?)

$$\lim_{\tau\to 0}\frac{dR_{XX}(\tau)}{d\tau} = \lim_{\tau\to 0}\frac{n\sigma^2\omega}{2} \cdot \frac{\omega\tau\left(1-\frac{1}{2}(\omega\tau)^2+\cdots\right)-\left(\omega\tau-\frac{1}{6}(\omega\tau)^3+\cdots\right)}{(\omega\tau)^2} = \quad <$$

$$\frac{\mathrm{d}^2R_{XX}(\tau)}{d\tau^2} = -\frac{n\sigma^2\omega^2}{2} \cdot \frac{(\omega\tau)^2\sin\omega\tau + 2\omega\tau\cos\omega\tau - 2\sin\omega\tau}{(\omega\tau)^3}$$

$$\lim_{\tau\to 0}\frac{d^2R_{XX}(\tau)}{d\tau^2} = -\frac{n\sigma^2\omega^2}{6} <$$

Therefore, $X(t)$ is ______________________

◎ Integration of a random process

Deterministic

$$y = \int_a^b x(t)\,dt = \lim_{n\to\infty}\sum_{j=1}^{n} x_j \cdot \Delta t_j$$

Note: Integral is a limit. Therefore the integral of a random process needs to be defined as a stochastic limit.

Stochastic

$$Y = \int_a^b X(t)\,dt$$

$$Y_n = \sum_{j=1}^{n} X_j\,(t_{j+1} - t_j)$$

Mean-square convergence of the stochastic integral, denoted by $\underset{n\to\infty}{\mathrm{l.i.m.}}\, Y_n = Y$, is achieved when $\lim_{n\to\infty} \mathrm{E}[(Y_n - Y)^2] = 0$

From Theorem 2, if $\underset{n\to\infty}{\mathrm{l.i.m}}\, Y_n = Y$ exists, $\lim_{n\to\infty, m\to\infty} \mathrm{E}[Y_m Y_n]$ should exist.

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That is, the following should exist.

$$\lim_{n\to\infty,m\to\infty} \mathrm{E}\left[\sum_{j=1}^{n} X_j(t_{j+1}-t_j)\sum_{k=1}^{m} X_k(t_{k+1}-t_k)\right] = \int_a^b \int_a^b \phi_{XX}(t_1,t_2)\,dt_1\,dt_2$$

In summary, the stochastic integral $Y = \int_a^b X(t)\,dt$ exists in the mean-square sense if $\int_a^b \int_a^b \phi_{XX}(t_1,t_2)\,dt_1\,dt_2$ exists

1) Mean of the stochastic integral

$$\mathrm{E}\left[\int_a^b X(t)\,dt\right] = \int_a^b \mathrm{E}[X(t)]dt$$

$$= \int_a^b \qquad dt$$

Note: Theorem 1 →

2) Mean square of the stochastic integral

$$\mathrm{E}\left\{\left[\int_a^b X(t)\,dt\right]^2\right\} = \int_a^b \int_a^b \mathrm{E}[X(t_1)X(t_2)]dt_1dt_2$$

$$= \int_a^b \int_a^b \phi_{XX}(t_1,t_2)dt_1dt_2$$

3) Generalization ("r.p. → r.v." or "r.p. → r.p.")

$$Y = \int_a^b X(t)\cdot f(t)dt$$

or

$$Z(\omega) = \int_a^b X(t)h(t,\omega)dt$$

- Example of $f(t)$: shape (envelope) function
- Example of $h(t,\omega)$: $\exp(\mathrm{i}\omega t)$ ~ Fourier transform

※ For the existence of $Z(\omega)$ in the mean-square sense, it should be satisfied that

$$< \int\int \phi_{XX}(t_1,t_2)h(t_1,\omega)h^*(t_2,\omega) <$$

Then,

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$$\mu_Z(\omega) = \mathrm{E}[Z(\omega)] = \int_a^b \qquad\qquad dt$$

$$\begin{aligned}\phi_{ZZ}(\omega) &= \mathrm{E}[Z(\omega_1)Z^*(\omega_2)] \\ &= \mathrm{E}\left[\int_a^b X(t_1)h(t_1,\omega_1)dt_1 \int_a^b X^*(t_2)h^*(t_2,\omega_2)dt_2\right] \\ &= \int_a^b \int_a^b \qquad\qquad h(t_1,\omega_1)h^*(t_2,\omega_2)dt_1 dt_2\end{aligned}$$

◎ Spectral decomposition of a random process

Characterization of a random process in ______________ domain

$$Z(\omega) = \int_a^b X(t)h(t,\omega)dt$$

How about using Fourier transform of the random process? That is, using the following filter in the above equation?

$$h(t,\omega) = \frac{1}{2\pi}\exp(-\mathrm{i}\omega t)$$

As a result,

$$\bar{X}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\exp(-\mathrm{i}\omega t)\,dt$$

Fourier transform of $X(t)$ → represents/describe $X(t)$ by harmonic components

Inverse relationship:

$$X(t) = \int_{-\infty}^{\infty} \bar{X}(\omega)\exp(\mathrm{i}\omega t)\,d\omega$$

※ $\bar{\mathrm{X}}(\omega)$ exists in the mean-square sense iff $\mathrm{E}[\bar{X}(\omega_1)\bar{X}^*(\omega_2)]$ exists.

$$\mathrm{E}[\bar{X}(\omega_1)\bar{X}^*(\omega_2)] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_{XX}(t_1,t_2)\exp[-\mathrm{i}(\omega_1 t_1 - \omega_2 t_2)]\,dt_1 dt_2 = \widehat{\Phi}(\omega_1,\omega_2)$$

➔ This is called "Generalized Power Spectral Density Function"

If this exists,

$$\phi_{XX}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{\Phi}(\omega_1,\omega_2)\exp[\mathrm{i}(\omega_1 t_1 - \omega_2 t_2)]\,d\omega_1 d\omega_2$$

However, $\bar{X}(\omega)$ does NOT exist when $X(t)$ is stationary, i.e. $\mathrm{E}[\bar{X}(\omega_1)\bar{X}^*(\omega_2)] = \widehat{\Phi}(\omega_1,\omega_2)$ blows up.

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457.643 Structural Random Vibrations
In-Class Material: Class 10

II-2. Stochastic Calculus (contd.)

◎ Spectral decomposition of a random process (contd.)

Fourier transform of $X(t)$, i.e. $\bar{X}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\exp(-\mathrm{i}\omega t)\,dt$ can be considered as a spectral decomposition of the random process.

It is noted that the stochastic integral $\bar{X}(\omega)$ exists in the mean square sense if and only if $\mathrm{E}[\bar{X}(\omega_1)\bar{X}^*(\omega_2)] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{XX}(t_1,t_2)\exp[-\mathrm{i}(\omega_1 t_1 - \omega_2 t_2)]\,dt_1 dt_2 = \hat{\Phi}_{XX}(\omega_1,\omega_2)$ exist.

However, the generalized PSD $\hat{\Phi}_{XX}(\omega_1,\omega_2)$ does not exist when $X(t)$ is a *stationary* random process. Therefore, $\bar{X}(\omega)$ is not useful for the purpose of spectral decomposition.

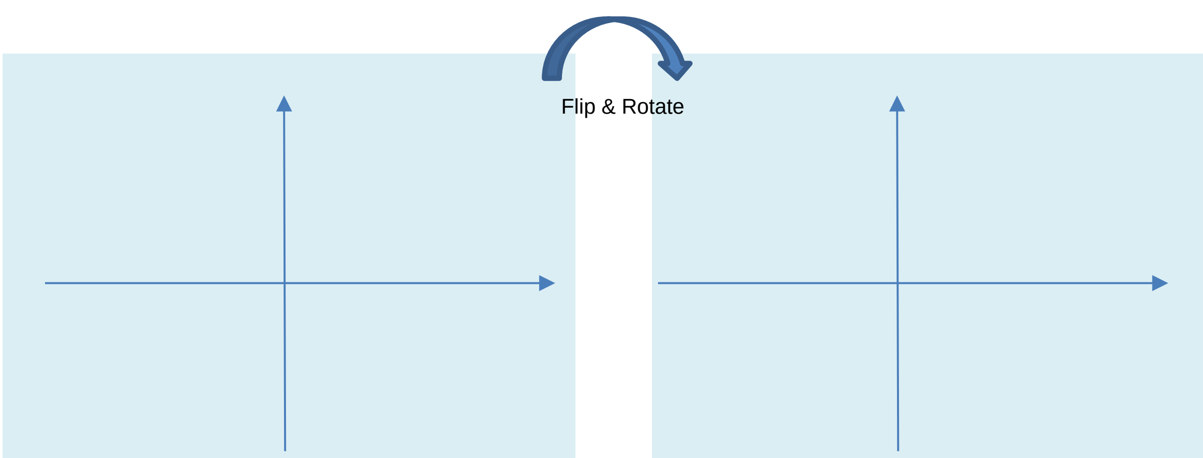
To show this, consider a "truncated" Fourier transform of a stationary process $X(t)$,

$$\bar{X}(\omega, T) = \frac{1}{2\pi}\int_{-T}^{T} X(t)\exp(-\mathrm{i}\omega t)\,dt$$

$$\mathrm{E}[\bar{X}(\omega_1,T)\bar{X}^*(\omega_2,T)] = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)\exp[-\mathrm{i}(\omega_1 t_1 - \omega_2 t_2)]\,dt_1 dt_2$$

Let us check the case $\omega_1 = \omega_2 = \omega$, i.e. (after changing variable $t_1 = \tau + t_2$)

$$\mathrm{E}[|\bar{X}(\omega,T)|^2] = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int \qquad R_{XX}(\tau)\exp(-i\omega\tau)\,d\tau dt_2$$



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$$\mathrm{E}[|\bar{X}(\omega,T)|^2] = \frac{1}{(2\pi)^2}\int_{-2T}^{0}\int_{-T-\tau}^{T} R_{XX}(\tau)\exp(-i\omega\tau)\,dt_2 d\tau$$
$$+\frac{1}{(2\pi)^2}\int_{0}^{2T}\int_{-T}^{T-\tau} R_{XX}(\tau)\exp(-i\omega\tau)\,dt_2 d\tau$$
$$= \frac{1}{(2\pi)^2}\int_{-2T}^{0}(\qquad)R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$
$$+\frac{1}{(2\pi)^2}\int_{0}^{2T}(\qquad)R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$

$$\therefore\ \mathrm{E}[|\bar{X}(\omega,T)|^2] = \frac{1}{(2\pi)^2}\int_{-2T}^{2T}(2T - \qquad)R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$

$$\mathrm{E}[\bar{X}(\omega_1)\bar{X}^*(\omega_2)] = \mathrm{E}[|\bar{X}(\omega)|^2] = \lim_{T\to\infty}\mathrm{E}[|\bar{X}(\omega,T)|^2]$$

This _________ because as $T \to \infty$, $T \cdot R_{XX}(\tau)$ ________ in general

Therefore, the generalized PSD $\hat{\Phi}_{XX}(\omega_1,\omega_2)$ may not converge to a finite value in general, and thus the mean-convergence of $\bar{X}(\omega)$ is not guaranteed.

Then, how about… introducing $\frac{\pi}{T}$ prior to taking the limit? That is,

$$\lim_{T\to\infty}\frac{\pi}{T}E[|\bar{X}(\omega,T)|^2] = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$

One can show that the limit above is equal to the following (Lin 1967):

$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$

This is a _________ transform of $R_{XX}(\tau)$. Thus, the integral exists when $R_{XX}(\tau)$ is “absolutely integrable” i.e. $\int_{-\infty}^{\infty}|R_{XX}(\tau)|d\tau < \infty$

Note: Being “absolutely integrable” is not the same concept as being a second-order process, i.e. $R_{XX}(0) = \mathrm{E}[X^2] < \infty$

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◎ Power spectral density (PSD) function of a stationary process $X(t)$

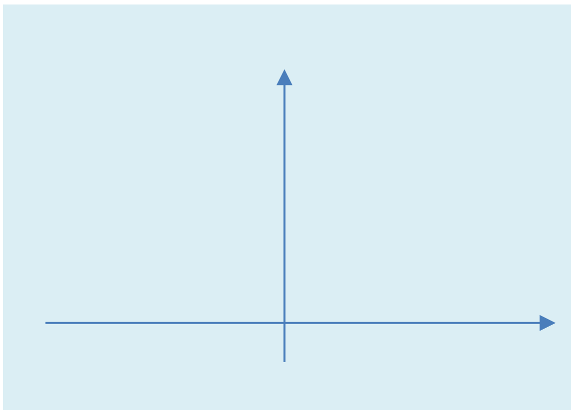
Fourier pair involving the power spectral density function and auto-correlation function:

$$\Phi_{XX}(\omega) \equiv \lim_{T\to\infty} \frac{\pi}{T} \mathrm{E}[|\bar{X}(\omega,T)|^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-i\omega\tau)\,d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\exp(i\omega\tau)\,d\omega$$

- This is often called “Wiener-Khintchine formula.”
- The PSD exists when the auto-correlation function is absolutely integrable
- $R_{XX}(0) = \int_{-\infty}^{\infty} \qquad d\omega = \mathrm{E}[\quad]$

  This indicates that $\Phi_{XX}(\omega)$ describes the distribution of “ ” process, i.e. X^2 over _________ domain. That is why it is called power spectral density function.



◎ Properties of PSD $\Phi_{XX}(\omega)$

1) Non-negative

   $\because\ \Phi_{XX}(\omega) \propto \mathrm{E}[|\quad|^2]$

   ($R_{XX}(\tau)$ is _______________)

2) Symmetric and Real

   $\Phi_{XX}(-\omega) =$

   $\because\ R_{XX}(\tau)$ is ____________ and $\Phi_{XX}(\omega) \propto \mathrm{E}[|\quad|^2]$

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3) Tail behavior of PSD tells us about whether the process is a 2nd order process or not.

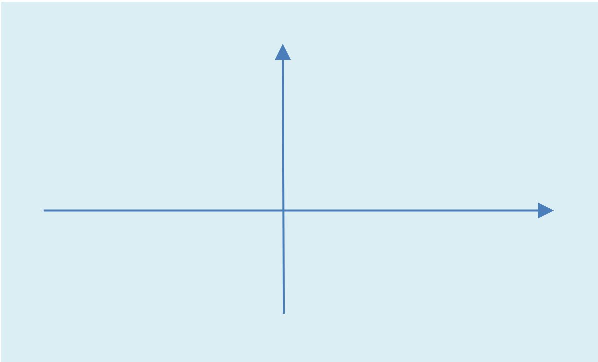
   If $\lim_{|\omega|\to\infty} |\omega| \cdot \Phi_{XX}(\omega) = 0$, the integral $\int_{-\infty}^{\infty} \Phi_{XX}(\omega) d\omega = \mathrm{E}[\quad]$ is ___________, thus $X(t)$ is a 2nd order process.

4) Behavior of PSD at $\omega = 0$: Note that

   $$\Phi_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau$$

   Therefore, $\Phi_{XX}(0)$ diverges if $\lim_{\tau\to\infty} R_{XX}(\tau) \neq 0$

   If the process has non-zero mean or include periodic component, the PSD diverges at $\omega = 0$



※ Alternative definition of PSD (e.g. L&S)

$$\Phi_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{XX}(\tau) \exp(-i\omega\tau)\, d\tau$$

(Reasoning of the alternative definition)

Since $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \mu^2$ (even without periodic components in the process), $\Phi_{XX}(0)$ diverges in general. By contrast, if $\Gamma_{XX}(\tau)$ is used in the definition of PSD, $\Phi_{XX}(0)$ may not diverge even if the process has non-zero mean μ.

Of course, there is no problem if $\mu = 0$ (since $R_{XX}(\tau) = \Gamma_{XX}(\tau)$)

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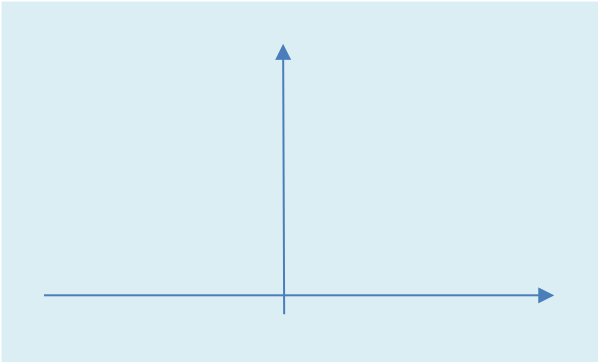
◎ One-sided PSD (Using symmetry of PSD)

$G_{XX}(\omega) = 2\Phi_{XX}(\omega)$, $\omega \geq 0$



Note:

$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-i\omega\tau)d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)(\cos\omega\tau - \sin\omega\tau)d\tau$$

$$= \frac{1}{\pi}\int_{0}^{\infty} R_{XX}(\tau)\cos\omega\tau\, d\tau$$

Therefore,

$$G_{XX}(\omega) = \frac{2}{\pi}\int_{0}^{\infty} R_{XX}(\tau)\cos\omega\tau\, d\tau$$

Inversely,

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\exp(i\omega\tau)\, d\omega$$

$$= \int_{0}^{\infty} \qquad \cos(\omega\tau)\, d\omega$$

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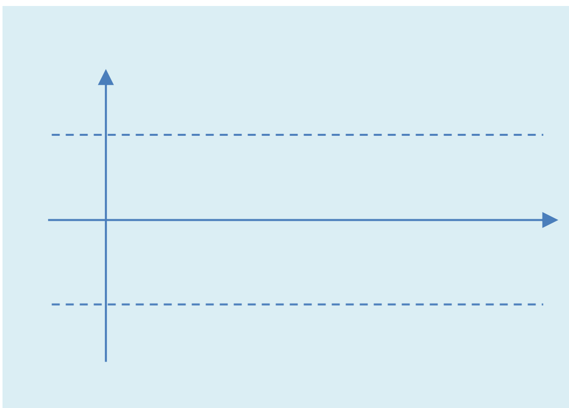
457.643 Structural Random Vibrations
In-Class Material: Class 11

II-2. Stochastic Calculus (contd.)

◎ Example: "Random Telegraph Process"

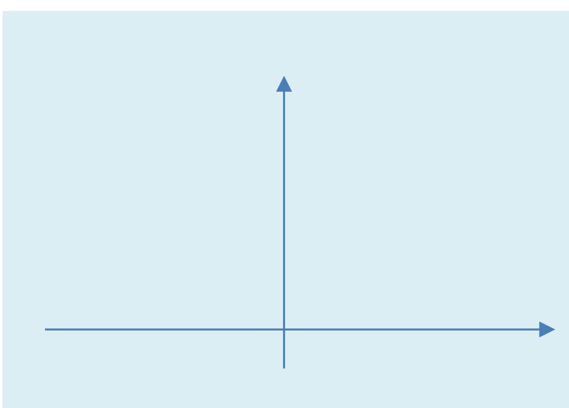
$X(t) = X_0 \cdot (-1)^{N(t)}$

where $X_0 \sim N(0, \sigma^2)$ and $N(t)$ is a homogeneous Poisson process with the mean occurrence rate ν



One can show the auto-correlation function of a random telegraph process is

$R_{XX}(\tau) = \sigma^2 \cdot e^{-2\nu|\tau|}$



The PSD of $X(t)$ is derived as follows:

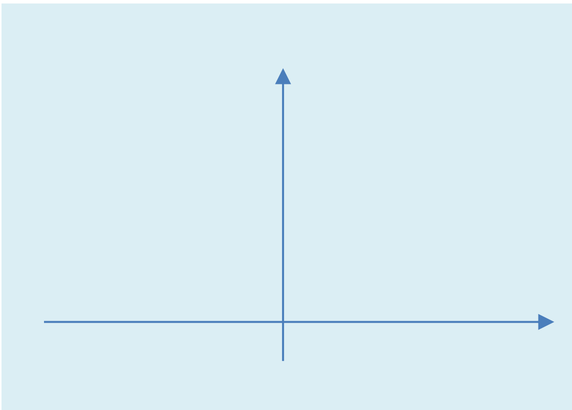
$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-i\omega\tau)\,d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma^2 \cdot e^{-2\nu|\tau|} \cdot \exp(-\mathrm{i}\omega\tau)\, d\tau$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \qquad\qquad\qquad\qquad d\tau$$

$$= \frac{\sigma^2}{\pi}\left[\frac{e^{-2\nu\tau}}{4\nu^2+\omega^2}(-2\nu\cos\omega\tau + \omega\sin\omega\tau)\right]_{\tau=0}^{\infty}$$

$$= \frac{\qquad\qquad}{}, -\infty < \omega < \infty$$



Note:

$$\int e^{cx}\cos bx\, dx = \frac{e^{cx}}{(c^2+b^2)}(c \cdot \cos bx + b \cdot \sin bx)$$

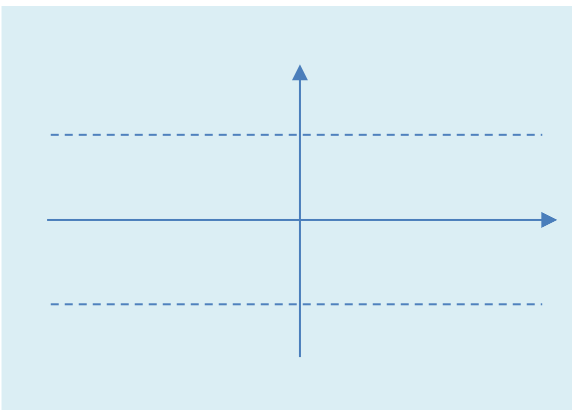
One-sided PSD:

$$G_{XX}(\omega) = \frac{\qquad\qquad}{} \qquad\qquad\qquad , \omega \geq 0$$

◎ Example

$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ where $\mathrm{E}[A] = \mathrm{E}[B] = 0$, $\mathrm{E}[A^2] = \mathrm{E}[B^2] = \sigma^2$, and $\rho_{AB} = 0$

It was shown that $\mu_X(t) = 0$ and $\phi_{XX}(t_1, t_2) = R_{XX}(\tau) = \sigma^2 \cos\omega_0\tau$ ("   " process)



$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)d\tau$$

$$= \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} \cos\omega_0\tau \cdot \cos\omega\tau\, d\tau$$

$$= \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}[\qquad\qquad\qquad\qquad\qquad\qquad]\, d\tau$$

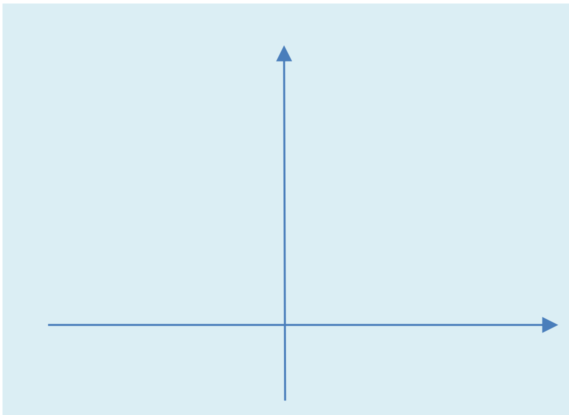
$$= \frac{\sigma^2}{2\pi}\int_{0}^{\infty} [\qquad\qquad\qquad\qquad\qquad\qquad]\, d\tau$$

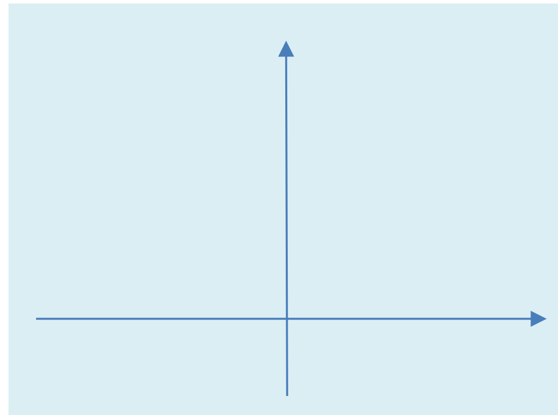
$$= \frac{\sigma^2}{2\pi}\left[\frac{\qquad\qquad}{} + \frac{\qquad\qquad}{}\right]_{\tau=0}^{\infty}$$

Note: $\left.\frac{\sin\omega t}{\omega}\right|_{t=0}^{\infty} = \pi \cdot \delta(\omega)$

Therefore,

$$\Phi_{XX}(\omega) = \frac{\sigma^2}{2\pi}[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] = \frac{\sigma^2}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



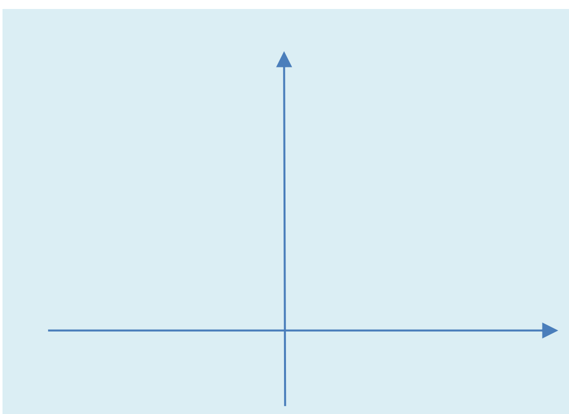


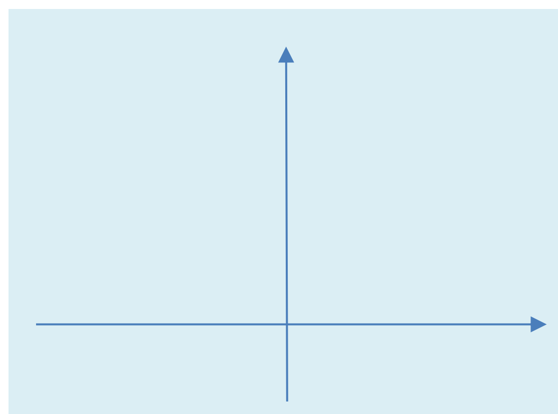
$\int_{-\infty}^{\infty} \Phi_{XX}(\omega)d\omega = \mathrm{E}[\quad] =$

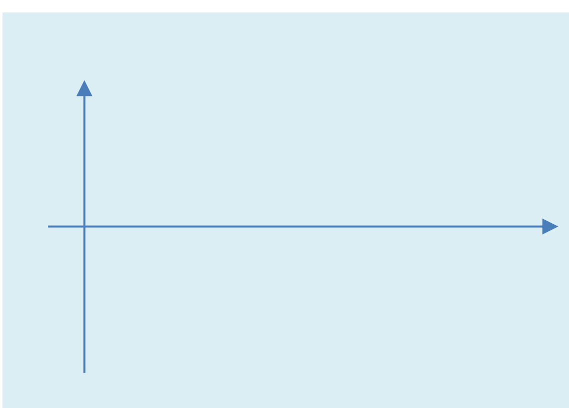
$\int_{0}^{\infty} G_{XX}(\omega)d\omega = \mathrm{E}[\quad] =$

◎ Special processes

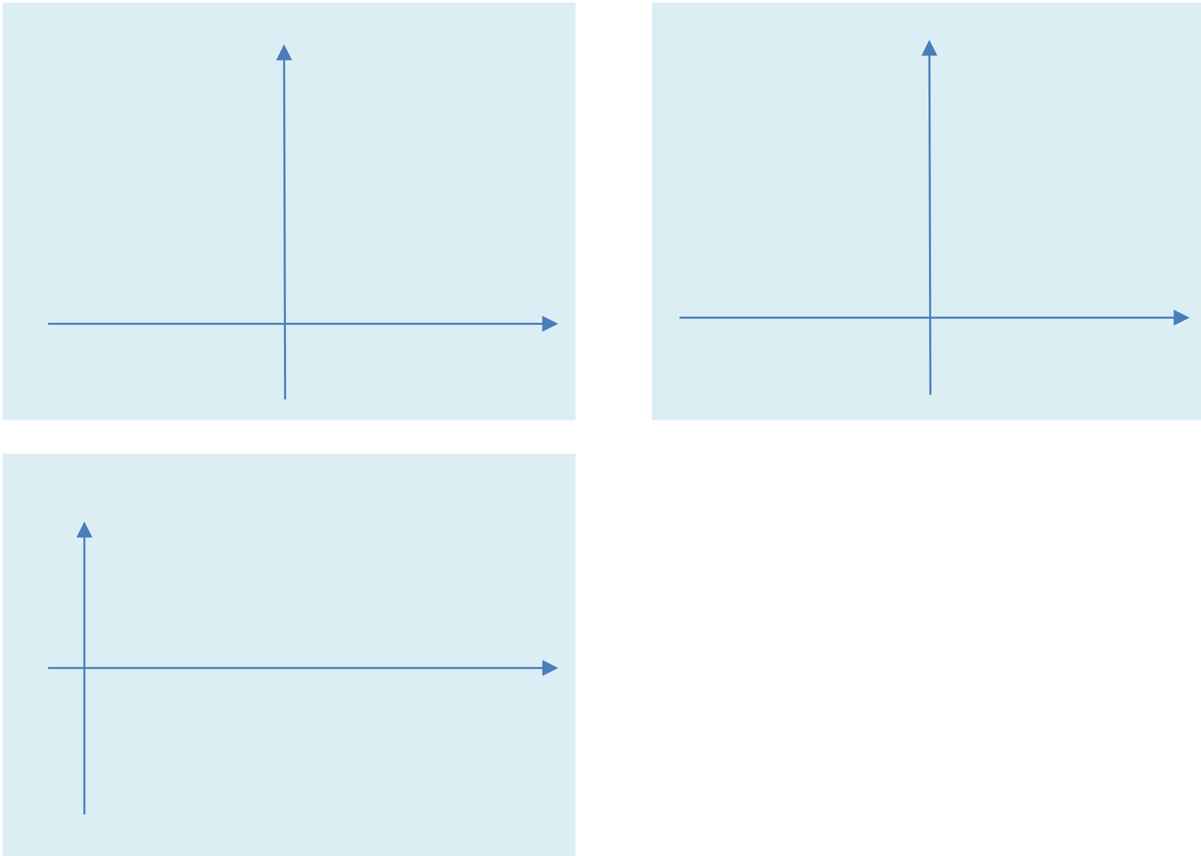
1) Narrow-band process (Example above is the ideal narrow-band process)



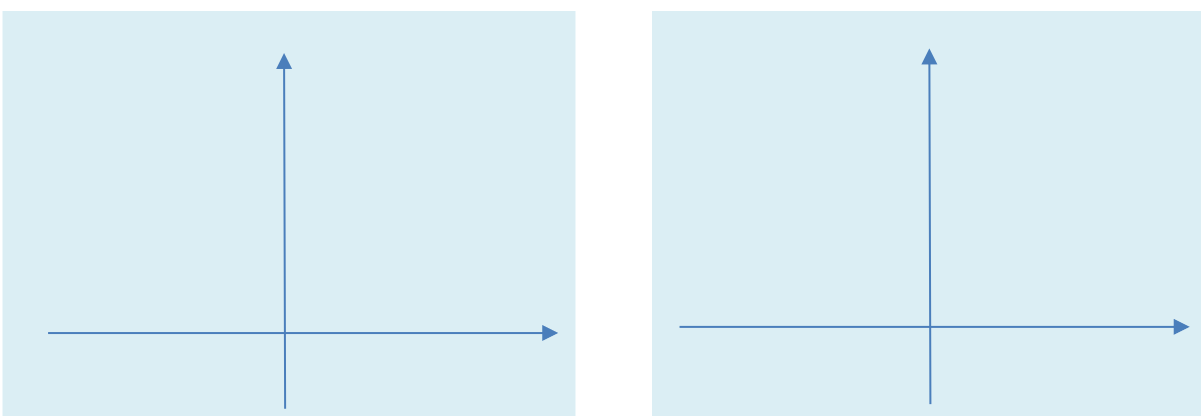




2) Wide-band process



3) White noise (ideal wide-band process)



$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\exp(i\omega\tau)d\omega = \int_{-\infty}^{\infty} \Phi_0 \exp(i\omega\tau)\, d\omega = 2\pi\Phi_0\delta(\tau)$$

Note: $1 = \int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}\, d\tau$ and thus $\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{i\omega\tau} d\omega$

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※ "Shot Noise"

$\mu_X(t) = 0$ and

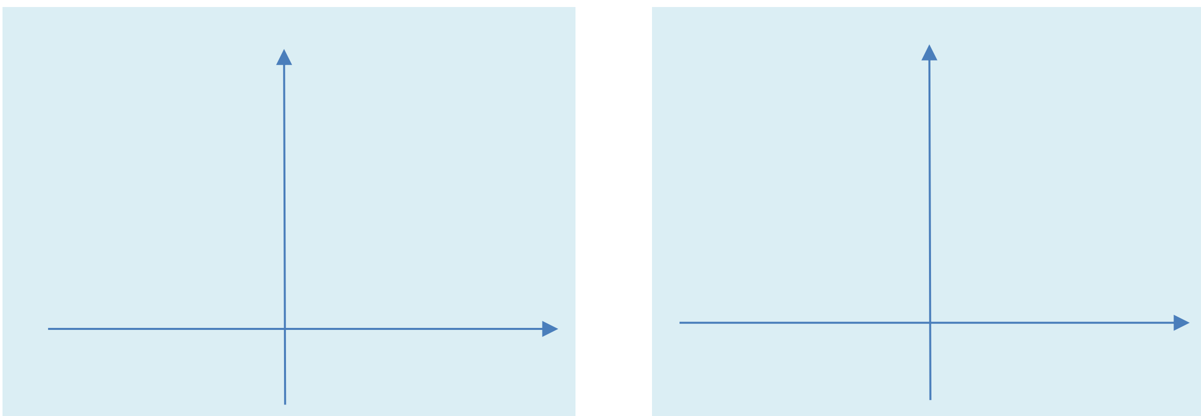
$\kappa_{XX}(t_1, t_2) = \phi_{XX}(t_1, t_2) = I(t_1) \cdot \delta(t_1 - t_2) = I(t_1) \cdot \delta(\tau)$

Here $I(t)$ is time-varying "intensity function."

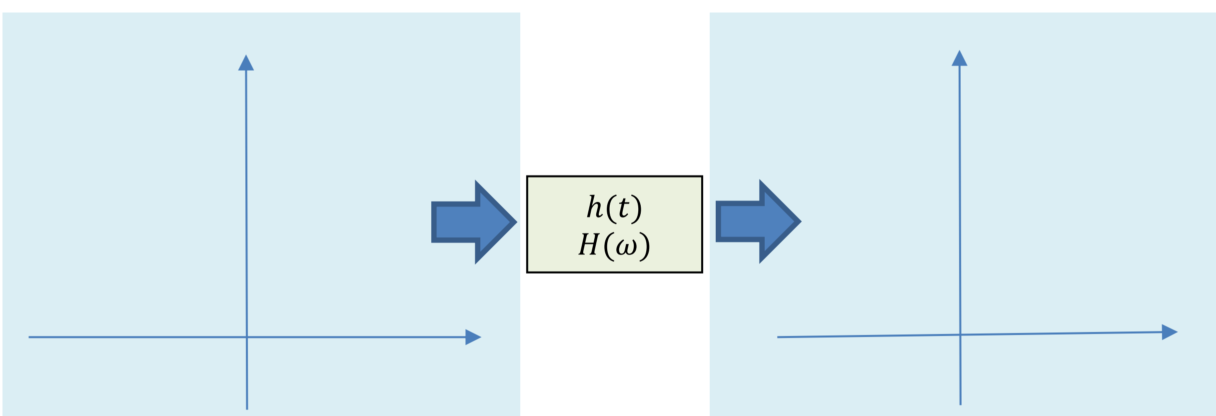
Therefore, a ___________ ____________ shot noise is a ___________ ___________

That is, $I(t_1) = I = 2\pi\Phi_0$ for WN

4) "Banded" white noise (more realistic WN)



5) "Filtered" white noise



e.g. SDOF oscillator (Kanai-Tajimi filter)

457.643 Structural Random Vibrations
In-Class Material: Class 12

II-2. Stochastic Calculus (contd.)

◎ Cross PSD

Consider jointly processes $X(t)$ and $Y(t)$, i.e.

$$\phi_{XY}(t_1, t_2) = R_{XY}(\tau)$$

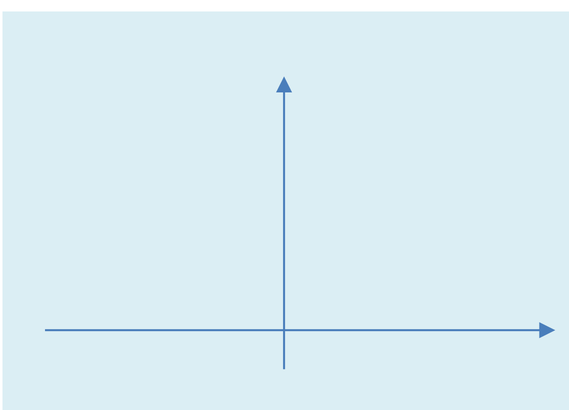
Cross PSD of $X(t)$ and $Y(t)$ is defined as

$$\Phi_{XY}(\omega) \equiv \lim_{T\to\infty} \frac{\pi}{T} E[\bar{X}(\omega, T)\bar{Y}^*(\omega, T)]$$

One can show

$$\begin{aligned}\Phi_{XY}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)\exp(-\mathrm{i}\omega\tau)d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau) \qquad\qquad d\tau - i\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau) \qquad\qquad d\tau\end{aligned}$$

"co-spectrum" "quad-spectrum"



Properties of Cross PSD $\Phi_{XY}(\omega)$

1) Hermitian

   $$\Phi_{XY}(\omega) = \Phi^*_{YX}(\omega)$$

   Note: $\mathrm{Re}\,\Phi_{\mathrm{XY}}(\omega) = \mathrm{Re}\Phi_{XY}(-\omega)$, $\mathrm{Im}\,\Phi_{\mathrm{XY}}(\omega) = -\mathrm{Im}\,\Phi_{XY}(-\omega)$

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Note: $\mathrm{E}[X(t)\cdot Y(t)] = R_{XY}(\ ) = \int_{-\infty}^{\infty} \qquad d\omega$

cf. $\mathrm{E}[X^2(t)] =$

2) If $\lim_{\omega\to 0} \omega \cdot \mathrm{Re}\,\Phi_{XY}(\omega) = 0$, $\mathrm{E}[X(t)\cdot Y(t)]$ is ____________.

3) $\mathrm{Im}\,\Phi_{XY}(0) =$

$$\mathrm{Re}\,\Phi_{XY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)\exp(-i\cdot 0\cdot\tau)\,d\tau =$$

※ Application example of cross PSD

Der Kiureghian, A. (1996). A coherency model for spatially varying ground motions. *Earthquake Engineering and Structural Dynamics*, 25:99-111.

Coherency function of ground acceleration processes $a_k(t)$ and $a_l(t)$ at stations k and l:

$$\gamma_{kl}(\omega) = \frac{G_{a_k a_l}(\omega)}{\sqrt{G_{a_k a_k}(\omega)\cdot G_{a_l a_l}(\omega)}}$$

Using the coherency function, one can characterize

(1) Incoherence effect: scattering of waves in the heterogeneous medium and differential superpositioning of waves

(2) Wave passage effect: delay in the arrival of the wave

(3) Attenuation effect: amplitude decreases due to geometric spreading, material damping and wave scattering

◎ PSD of derivative process

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$R_{\dot{X}X}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

$$= \int_{-\infty}^{\infty} (\qquad)\Phi_{XX}(\omega)\, e^{i\omega\tau} d\tau$$

Comparing the two equations above, we note

$$\boxed{\Phi_{\dot{X}X}(\omega) = \qquad\qquad}$$

It is also seen that

$$\boxed{\Phi_{X\dot{X}}(\omega) = \qquad\qquad}$$

We also know that

$$\begin{aligned} R_{\dot{X}\dot{X}}(\tau) &= -\frac{d^2 R_{XX}(\tau)}{d\tau^2} \\ &= -\int_{-\infty}^{\infty} (\qquad)^2 \Phi_{XX}(\omega) e^{i\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} \qquad\qquad\qquad e^{i\omega\tau} d\tau \end{aligned}$$

Therefore,

$$\boxed{\Phi_{\dot{X}\dot{X}}(\omega) = \qquad\qquad}$$

In general,

$$\Phi_{X^{(m)}Y^{(n)}}(\omega) = (i\omega)^m (-i\omega)^n \Phi_{XY}(\omega)$$

◎ Generation of artificial time histories by PSD

e.g. “Spectral representation” method

Shinozuka & Deodatis 1991: Stationary & Gaussian

Deodatis & Micaletti 2001: Non-Gaussian

$$\boxed{X(t) = \sum_{i=1}^{n} a_i \cos(\omega_i t + \theta_i)}$$

- a_i: contribution from the frequency ω_i ~ determined by ______
- ω_i: closely-spaced frequency values (>0)
- θ_i: random phase angle ~ $\mathrm{U}(0, 2\pi]$
- θ_i and θ_j are statistically independent $(i \neq j)$

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Check

1) $\mathrm{E}[X(t)]$

$$\mathrm{E}[\cos(\omega_i t + \theta_i)] = \int_0^{2\pi} \cos(\omega_i t + \theta_i) \qquad d\theta$$

$$=$$

Therefore, $X(t)$ is a _____-______ process

2) $\phi_{XX}(t_1, t_2)$

$$\phi_{XX}(t_1, t_2) = \sum_{i=1}^{n}\sum_{i=1}^{n} a_i a_j \mathrm{E}\left[\cos(\omega_i t_1 + \theta_i) \cdot \cos\left(\omega_j t_2 + \theta_j\right)\right]$$

$(i \neq j)$

$$\mathrm{E}\left[\cos(\omega_i t_1 + \theta_i) \cdot \cos\left(\omega_j t_2 + \theta_j\right)\right] = \mathrm{E}[\cos(\omega_i t_1 + \theta_i)] \cdot \mathrm{E}\left[\cos\left(\omega_j t_2 + \theta_j\right)\right]$$

$$=$$

$(i = j)$

$$\mathrm{E}[\cos(\omega_i t_1 + \theta_i) \cdot \cos(\omega_i t_2 + \theta_i)] = \frac{1}{2}\left\{\mathrm{E}[\cos(\omega_i(t_1 + t_2) + 2\theta_i)] + \mathrm{E}\left[\cos\left(\omega_i(t_1 - t_2)\right)\right]\right\}$$

$$= \frac{1}{2}\cos\left(\omega_i(t_1 - t_2)\right)$$

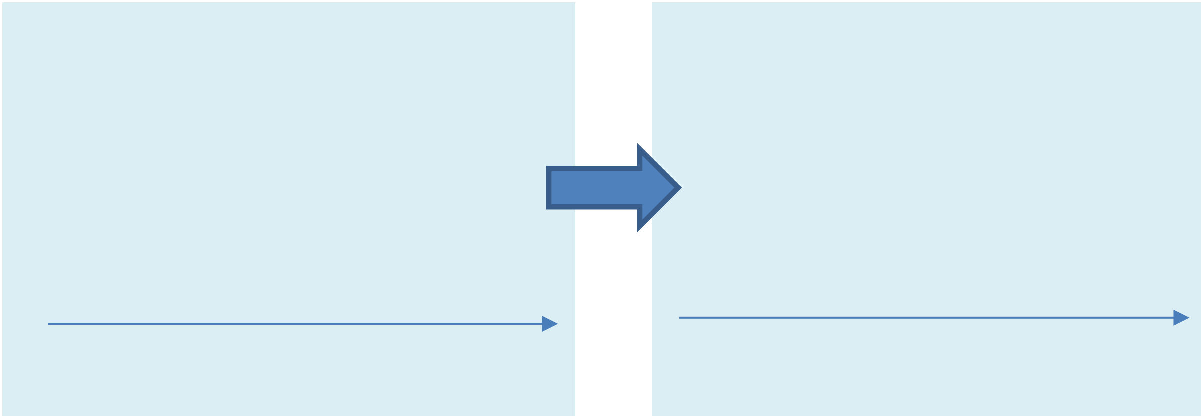
$= \frac{1}{2}\cos(\omega_i \quad) \rightarrow X(t)$ is a __________ process

Therefore,

$$R_{XX}(\tau) = \frac{1}{2}\sum_{i=1}^{n} a_i^2 \cos(\omega_i \tau)$$

$$\Phi_{XX}(\omega) = \frac{1}{4}\sum_{i=1}^{n} a_i^2 [\delta(\omega + \omega_i) + \delta(\omega - \omega_i)]$$

Recall $R_{XX}(\tau) = \sigma^2 cos\omega_o \tau \rightarrow \Phi_{XX}(\omega) = \frac{\sigma^2}{2}[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$



Given PSD

Spectral Representation

How to determine a_i?

… such that the powers $\mathrm{E}[X^2]$ in the corresponding intervals are equivalent

$$\int_{\frac{\omega_{i-1}+\omega_i}{2}}^{\frac{\omega_i+\omega_{i+1}}{2}} \Phi_{XX}(\omega)d\omega =$$

When $\Delta\omega$ is small,

$$\Phi_{\mathrm{XX}}(\omega_i) \cdot \frac{\omega_{i+1} - \omega_{i-1}}{2} =$$

Therefore,

$$\therefore \;\; a_i = 2\sqrt{\Phi_{XX}(\omega_i) \cdot \frac{\omega_{i+1} - \omega_{i-1}}{2}}$$
$$= 2\sqrt{\Phi_{XX}(\omega_i) \cdot \quad}$$

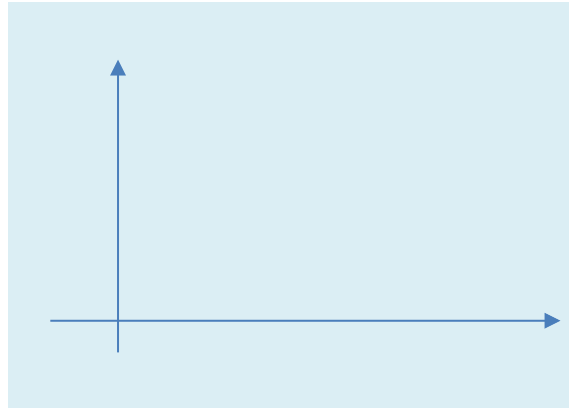
The generated process has Gaussianity and Ergodicity (proof: Deodatis 2001)

◎ Spectral moments

- VanMarcke (1972, ASME JEM): first introduced
- Michaelov et al. (1999, Structural Safety): a good summary and extension to nonstationary case

The m-th order spectral moment is defined as

$$\boxed{\lambda_m = \int_0^{\infty} \omega^m G_{XX}(\omega) d\omega}$$



1) Help compute variances easily

$$\lambda_0 = \int_0^{\infty} G_{XX}(\omega) d\omega =$$

$$\lambda_2 = \int_0^{\infty} \omega^2 G_{XX}(\omega) d\omega =$$

$$\lambda_4 = \int_0^{\infty} \omega^4 G_{XX}(\omega) d\omega =$$

$$\lambda_{2n} = \int_0^{\infty} \omega^{2n} G_{XX}(\omega) d\omega =$$

2) Capture frequency-related characteristics of a random process

(in analogy to spectral moments $\mathrm{E}[X^m]$ capturing characteristics of a random process)

- Central frequency

$$\boxed{\omega_c = \frac{\lambda_1}{\lambda_0} = \frac{\int_0^{\infty} \omega G_{XX}(\omega) d\omega}{\int_0^{\infty} G_{XX}(\omega) d\omega}}$$

In analogy to the mean

$$\mathrm{E}[X] = \frac{\int_{-\infty}^{\infty} x \cdot f_X(x) dx}{\int_{-\infty}^{\infty} f_X(x) dx} =$$

Geometric “center” of probability density function

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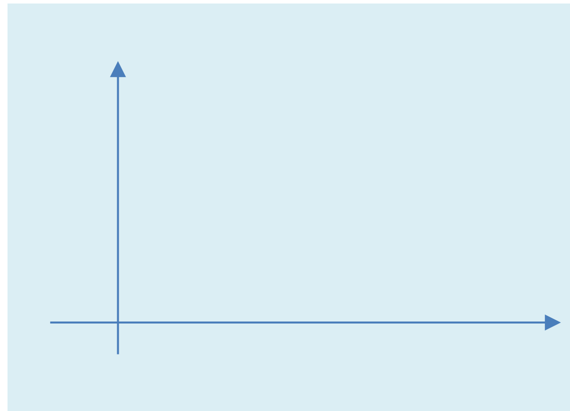
- Normalized radius of gyration of PSD

$$s = \frac{\sqrt{\frac{\lambda_2}{\lambda_0} - \left(\frac{\lambda_1}{\lambda_0}\right)^2}}{\omega_c}$$

$$= \frac{\sqrt{\frac{\lambda_2}{\lambda_0} - \omega_c^2}}{\omega_c}$$

$$= \boxed{\sqrt{\frac{\lambda_0 \lambda_2}{\lambda_1^2} - 1}}, \quad 0 \le s < \infty$$



- Bandwidth factor

$$\delta = \frac{s}{\sqrt{s^2 + 1}}$$

If $s \to 0$, $\delta =$

$s \to \infty$, $\delta =$

Therefore, $0 \le \delta \le 1$

If $s = \sqrt{\frac{\lambda_0 \lambda_2}{\lambda_1^2} - 1}$ is substituted,

$$\boxed{\delta = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}}}$$

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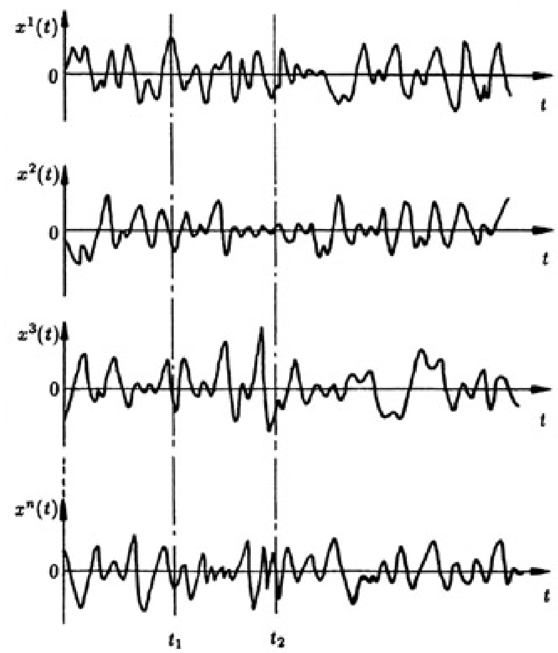
In-Class Material: Class 13

II-2. Stochastic Calculus (contd.)

◎ Ergodicity



Ensemble Average: $\mathrm{E}[X(t)] =$

Temporal Average: $m_T =$

Average over the time domain is actually a _________ variable. Consider

$$M_T = \frac{1}{T}\int_0^T X(t)dt = \langle X(t)\rangle$$

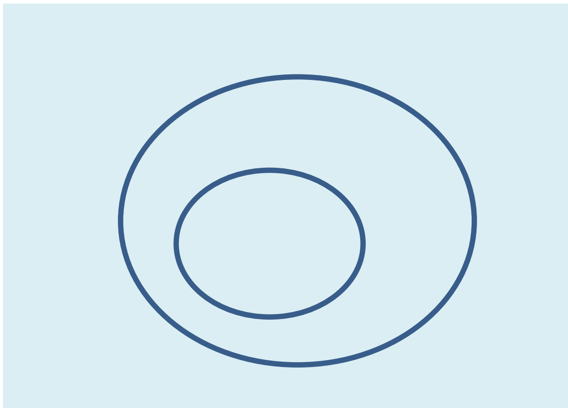
$$\phi_T(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)X(t)dt$$

These are random because the results depend on random outcome (selection) of a time history.

1) "Ergodic" process: If $X(t)$ is an ergodic process, one can use the temporal average from a time history $x(t)$ as an substitute for an ___________ expectation.

2) Basic __________ condition for ergodicity: stationarity

Stationary Ergodic



What if $X(t)$ is NOT stationary? $\mu(t_1) \neq \mu(t_2)$

3) Condition for ergodicity in the mean M_T

- $\lim_{T\to\infty} \mathrm{E}[M_T] = \mu_X(t) = \mathrm{E}[X(t)]$ ~ automatically satisfied for _________ process

- $\lim_{T\to\infty} Var[M_T] =$

$$\mathrm{Var}[M_T] = \mathrm{E}\left[\left(\frac{1}{T}\int_0^T X(t)\,dt - \mu_X\right)^2\right]$$

$$= \mathrm{E}\left[\left\{\frac{1}{T}\int_0^T (X(t) - \mu_X)\,dt\right\}^2\right]$$

$$= \frac{1}{T^2}\int_0^T \int_0^T \Gamma_{XX}(t_1 - t_2)dt_1 dt_2$$

$$= \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)\Gamma_{XX}(\tau)d\tau$$

"Flip and Rotate" trick used

$\therefore \lim_{T\to\infty} \mathrm{Var}[M_T] = 0 \leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_0^T \Gamma_{XX}(\tau)d\tau = 0$ (See Lin 1967, p. 64)

This is the condition for "ergodicity in the mean"

4) Condition for ergodicity in the correlation function $\phi_T(\tau)$

- $\lim_{T\to\infty} \mathrm{E}[\phi_T(\tau)] = R_{XX}(\tau)$ ~ automatically satisfied for _________ process

- $\lim_{T\to\infty} \mathrm{Var}[\phi_T(\tau)] = 0$

The latter is equivalent to (Lin 1967, p. 65)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{E}\{[X(t+\tau)X(t) - R_{XX}(\tau)][X(t+\tau+u)X(t+u) - R_{XX}(\tau)]\}du = 0$$

Example: Telegraph Random Process

$\Gamma_{XX}(\tau) = \sigma^2 \exp(-2\nu|\tau|)$

Is the process ergodic in the mean?

Example: $Y(t) = A + X(t)$ where $X(t)$ is the random telegraph process, and $\mathrm{E}[A] = 0$ and $\mathrm{Var}[A] = \sigma^2$. Is $Y(t)$ ergodic in the mean?

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III. Random Vibration of Linear Structures

◎ Stochastic response of "linear" structural system

Recall the system equation introduced in **0. Introduction**

$\mathcal{D}[\mathcal{X}(x,t)] = \mathcal{Y}(x,t), \qquad t \geq 0, \qquad x \in D \subset \mathcal{R}^d$

1. Deterministic systems and input (457.516 Dynamics of Structures)
2. Deterministic systems and stochastic input **(457.643 Structural Random Vibrations)**
3. Stochastic systems and deterministic input (457.646 Topics in Structural Reliability)
4. Stochastic systems and input

We consider the second case in this course. The system is "linear" when the differential equation is linear, i.e. s__________ principle works.

e.g. if $x_1(t)$ is the response to $y_1(t)$, and $x_2(t)$ is the response to the input $y_2(t)$, the response to $y_1(t) + y_2(t)$ is ______________

III-1. Response Functions of Structural Systems

◎ Characterization of linear systems

1) Time-domain: **"Impulse Response Function"**

2) Frequency-domain: **"Frequency Response Function"**

◎ Impulse response function of a linear system

Consider the differential equation (DE) of a general linear system

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 = p(t)$$

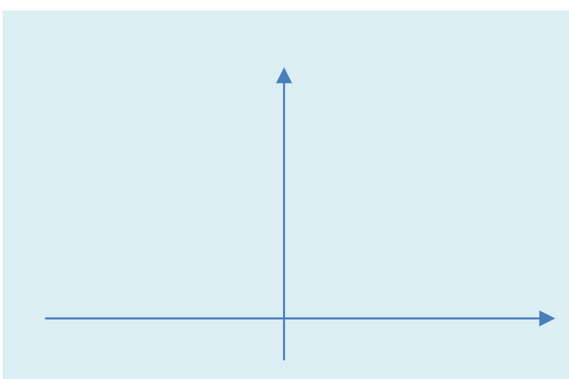
Impulse response function of a linear system $h(t)$ is the solution of the DE when $p(t) = \delta(t)$, i.e.

$$a_n \frac{d^n h}{dt^n} + a_{n-1} \frac{d^{n-1} h}{dt^{n-1}} + \cdots + a_1 \frac{dh}{dt} + a_0 = \delta(t) \ \cdots \ (*)$$

$$h(t) = h_h(t) + h_p(t)$$

where $h_h(t)$: homogeneous solution and $h_p(t)$: particular solution

Strategy: Model the dirac delta input function by a triangular function for $t \in (-\epsilon, \epsilon)$. Then, $h(t) = h_h(t)$ with the initial conditions caused by the impulse.



From (*), it is clear that only the ________est term can be dirac delta function because if an non-highest-order term is dirac delta, the higher-order-terms will blow up.

$$\therefore a_n \frac{d^n h}{dt^n} = \delta(t),\ 0^- < t < 0^+$$

$$a_n \left. \frac{d^{n-1} h}{dt^{n-1}} \right|_{0^-}^{0^+} = \int_{0^-}^{0^+} \delta(t) dt = \qquad \rightarrow a_n \left. \frac{d^{n-1} h}{dt^{n-1}} \right|_{t=0^+} =$$

Therefore, the initial conditions at $t = 0^+$ are

$$\frac{d^{n-1} h}{dt^{n-1}} = \qquad , \qquad \frac{d^{n-2} h}{dt^{n-2}} = \cdots = h =$$

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457.643 Structural Random Vibrations
In-Class Material: Class 14

III-1. Response Functions of Structural Systems (contd.)

◎ Characterization of linear systems (contd.)

Example: IRF of an SDOF oscillator

Equations of Motion:

1) External force: $p(t)$

   $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = p(t)$

2) Ground acceleration: $\ddot{x}_g(t)$

   $m\ddot{x} + c\dot{u} + ku = 0$

   - x: total displacement, u: relative displacement
   - Thus, $x =$

   $m\ddot{u} + c\dot{u} + ku =$

Divide the equation by m,

$\ddot{s}(\mathrm{t}) + 2\xi\omega_0\dot{s}(t) + \omega_0^2 s(t) = f(t)$

where $\omega_0 = \sqrt{\frac{k}{m}}$, $\xi = \frac{c}{2m\omega_0} = \frac{c}{2\sqrt{km}}$, and

$f(t) = \frac{p(t)}{m}$ or $-\ddot{x}_g(t)$

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IRF of the system is obtained from

$$\ddot{h}(t) + 2\xi\omega_0\dot{h}(t) + \omega_0^2 h(t) = \delta(t)$$

Initial conditions at $t = 0^+$:

- $\left.\frac{d^{n-1}h}{dt^{n-1}}\right|_{0^+} = \frac{1}{a_n}$ → $\left.\frac{dh}{dt}\right|_{0^+} = h'(0^+) =$
- Lower-order derivatives are zero → $h(0^+) =$

Find the homogeneous solution $h_h(t)$ from the equation of motion

$$\ddot{h}(t) + 2\xi\omega_0\dot{h}(t) + \omega_0^2 h(t) = 0$$

Setting $h(t) = e^{rt}$ and substituting it into the equation,

$$r^2 \cdot e^{rt} + 2\xi\omega_0 r \cdot e^{rt} + \omega_0^2 \cdot e^{rt} = 0$$

$$r^2 + 2\xi\omega_0 r + \omega_0^2 = 0$$

$$r = -\xi\omega_0 \pm \sqrt{\xi^2\omega_0^2 - \omega_0^2}$$

For $0 \le \xi \le 1$ (most practical situation),

$$r = -\xi\omega_0 \pm \sqrt{\xi^2\omega_0^2 - \omega_0^2} = -\xi\omega_0 \pm i\omega_D$$

where $\omega_D = \omega_0\sqrt{1-\xi^2}$ (damped natural frequency)

$$\begin{aligned} h(t) &= A_1 e^{(-\xi\omega_0 + i\omega_D)t} + A_2 e^{(-\xi\omega_0 - i\omega_D)t} \\ &= e^{-\xi\omega_0 t}\left(A_1 e^{i\omega_D t} + A_2 e^{-i\omega_D t}\right) \\ &= e^{-\xi\omega_0 t}(B_1 \cos\omega_D t + B_2 \sin\omega_D t) \end{aligned}$$

Determine B_1 and B_2 by the IC's, i.e. $h'(0^+) = 1$ and $h(0^+) = 0$

$$h(0) = B_1 =$$

$$h'(t) = e^{-\xi\omega_0 t}(-B_1\omega_D \sin\omega_D t + B_2\omega_D \cos\omega_D t - \xi\omega_0 B_1 \cos\omega_D t - \xi\omega_0 B_2 \sin\omega_D t)$$

$$\therefore h'(0) = B_2\omega_D =$$

Finally,

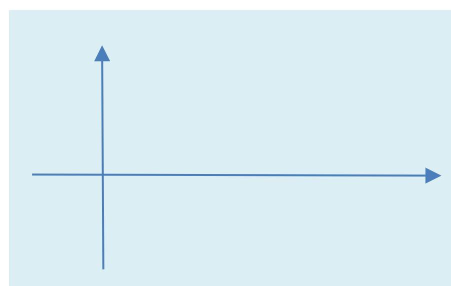
$$\boxed{h(t) = \frac{1}{\omega_D} e^{-\xi\omega_0 t} \sin \omega_D t\,, \quad t > 0 \qquad (0 \text{ otherwise})}$$

Note: This is the IRF when the mass-normalized force $f(t)$ is given as $\delta(\mathrm{t})$. Therefore, if the force $p(t)$ is given as $\delta(\mathrm{t})$, the IRF is

$$\boxed{h(t) = \frac{1}{m\omega_D} e^{-\xi\omega_0 t} \sin \omega_D t\,, \quad t > 0 \qquad (0 \text{ otherwise})}$$

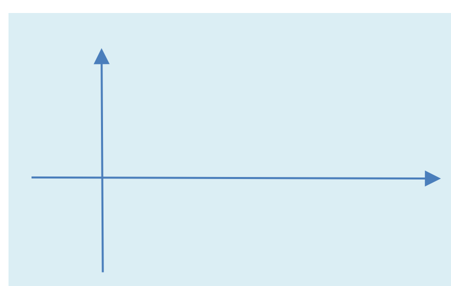
For $\xi = 0$ (no damping; undamped system),

$$h(t) = \frac{1}{\omega_0} \sin \omega_0 t\,, \quad t > 0$$



For $\xi = 1$ ("critical damping"),

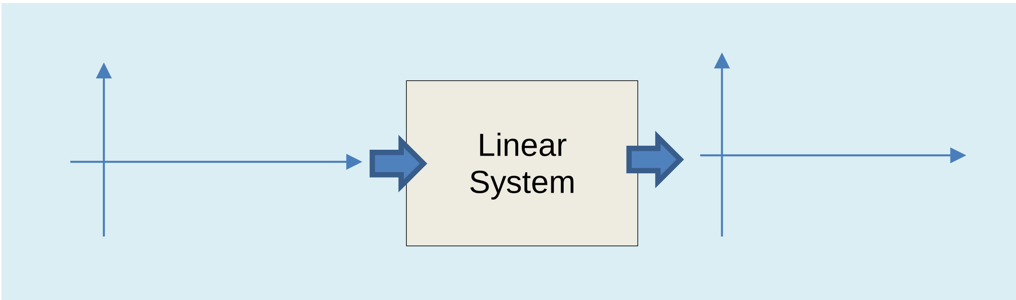
- Try e^{rt} and $r \cdot e^{rt}$ in solving the DE; or
- $h(t) = \lim_{\xi \to 1} \frac{1}{\omega_0\sqrt{1-\xi^2}} e^{-\xi\omega_0 t} \sin \omega_D t$



Either way, the IRF is derived as

$$h(t) = t \cdot e^{-\omega_0 t}, \quad t > 0$$

◎ Response of a linear system to general loading (by IRF)



Loading at $t = \tau$: $f(\tau)\delta(t-\tau)d\tau$

➔ Response at time t caused by $f(\tau)$: $f(\tau)h(t-\tau)d\tau$

$$x(t) = x_h(t) + x_p(t)$$

$$= x_h(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

Note:

- Homogeneous solution $x_h(t)$ is determined by ______
- Particular solution $x_p(t)$ is alternatively obtained by $\int_0^t f(t-\tau)h(\tau)d\tau$ (convolution integral or Duhamel's integral) → works because of the __________ rule (linear system)
- The force and the IRF in the integral should be consistent in terms of _________ by mass

e.g. Standard SDOF oscillator

$$x(t) = x(0)g(t) + \dot{x}(0)h(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

where

$h(t) = \frac{1}{\omega_D} e^{-\xi\omega_0 t} \sin\omega_D t, \quad t > 0$ and

$$g(t) = \mathrm{e}^{-\xi\omega_0 t} \cdot \left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right), \quad t > 0$$

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◎ Frequency response function of a linear system

For a "stable" system subjected to a harmonic input $f(t) = e^{i\omega t}$, its "steady-state" response is $x(t) = H(\omega)e^{i\omega t}$

$$x(t) = x_h(t) + \int_0^t f(t-\tau)h(\tau)d\tau = x_h(t) + \int_0^t e^{i\omega(t-\tau)}h(\tau)d\tau$$

Then, for a stable system, i.e. $\lim_{t\to\infty} x_h(t) = 0$,

$$\lim_{t\to\infty} x(t) = e^{i\omega t} \cdot \int_0^\infty e^{-i\omega\tau}h(\tau)d\tau = e^{i\omega t}H(\omega)$$

Therefore, the relationship between IRF and FRF is

$$\boxed{\begin{aligned} H(\omega) &= \int_0^\infty h(t)e^{-i\omega\tau}d\tau = \int_{-\infty}^\infty h(t)e^{-i\omega\tau}d\tau \\ h(t) &= \frac{1}{2\pi}\int_{-\infty}^\infty H(\omega)e^{i\omega\tau}d\omega \end{aligned}}$$

FRF and IRF form a Fourier pair and describe the linear system in the _________ and _______ domain respectively.

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457.643 Structural Random Vibrations
In-Class Material: Class 15

III-1. Response Functions of Structural Systems (contd.)

◎ Direct derivation of $\mathbf{H}(\boldsymbol{\omega})$

Assume $f(t) = e^{i\omega t}$ and find the particular solution in the form $x_p(t) = H(\omega)e^{i\omega t}$

Example 1

$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = 2\xi\omega_0\dot{f}$

$H(\omega)$ for the response $x(t)$ to the input $f(t)$?

- $f(t) = e^{i\omega t}$ and $\dot{f}(t) =$
- $x(t) = H(\omega)e^{i\omega t}$, $\dot{x}(t) =$ and $\ddot{x}(t) =$

Substituting these into the equation, one finds

$H(\omega)e^{i\omega t}[\qquad\qquad] = \qquad e^{i\omega t}$

Therefore,

$$H(\omega) = \frac{2i\xi\omega_0\omega}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega}$$

Example 2: Standard SDOF oscillator

$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = f$

$H(\omega)$ for the response $x(t)$ to the input $f(t)$?

- $f(t) = e^{i\omega t}$
- $x(t) = H(\omega)e^{i\omega t}$, $\dot{x}(t) =$ and $\ddot{x}(t) =$

$H(\omega)e^{i\omega t}[\qquad\qquad] = e^{i\omega t}$

$$H(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega}$$

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Note: Derivation of $H(\omega)$ for “state-space” formulation

Chen, C.T. (1999). *Linear System Theory & Design*, Oxford University Press.

$$\dot{\mathbf{z}}(t) = \mathbf{A} \cdot \mathbf{z}(t) + \mathbf{B} \cdot \mathbf{w}(t)$$

$$\mathbf{y}(t) = \mathbf{C_y} \cdot \mathbf{z}(t) + \mathbf{D_y} \cdot \mathbf{w}(t)$$

For example, consider the standard SDOF oscillator $\ddot{x}(t) + 2\xi\omega_0\dot{x}(t) + \omega_0^2 x(t) = f(t)$. If the system is described by a state-space formulation,

$\mathbf{z}(t) = \{\quad\}$, $\mathbf{A} = [\quad\quad]$, $\mathbf{B} = [\quad]$ and $\mathbf{w}(t) = [\quad]$

If one is interested in the responses $x(t)$ and $\dot{x}(t)$, $\mathbf{y}(t) =$, thus $\mathbf{C_y}$ and $\mathbf{D_y}$ are and respectively.

As shown in Chen (1999), the frequency response function vector is in general obtained by

$$\mathbf{h}(\omega) = \mathbf{C_y}(i\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D_y}$$

(Derive the transfer function by Laplace transform and replace "s" by "$i\omega$")

◎ **Response to a general loading by $\mathbf{H}(\boldsymbol{\omega})$**

$$x(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} f(t-\tau)\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega\tau}d\omega\, d\tau$$

$$= \int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t-\tau)e^{-i\omega(t-\tau)}\, d\tau d\omega$$

For the green-colored integral, we change the variable, i.e. $\tilde{t} = t - \tau$, then it becomes

$$\bar{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tilde{t})e^{-i\omega\tilde{t}}d\tilde{t}$$

i.e. Fourier transform of the input time history.

Finally, the response time history is obtained by

$$\boxed{x(t) = \int_{-\infty}^{\infty} \bar{f}(\omega)H(\omega)e^{i\omega t}d\omega}$$

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◎ Derivation of $\mathbf{h(t)}$ from $\mathbf{H(\omega)}$

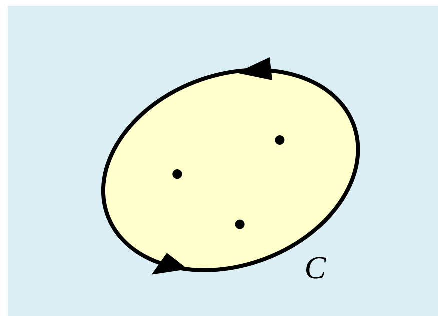
$$\boxed{h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega}$$

➔ Need to use "Residue Theorem" – appears in textbooks on complex analysis, e.g. *Advanced Calculus for Applications* (Hildebrand, 1976)

Residue Theorem

Consider $z = a + ib$ is a complex value. Let $f(z)$ be "analytic" i.e. single-valued and finite derivative on contour C, and its inside except at a finite number of points, $z_1, z_2, \cdots, z_m$ ("poles") inside C. Then,

$$\boxed{\oint f(z)dz = 2\pi i \sum_{j=1}^{m} \text{Res}\, f(z_j)}$$



(counter-clockwise)

where $\text{Res}\, f(z_j)$ is the "residue" of $f(z)$ at $z = z_j$

For example, $f(z) = 1/(z - z_1)(z - z_2)$ has two poles,

For a single pole, i.e. $(z - z_j)$ appears in the denominator of $f(z)$,

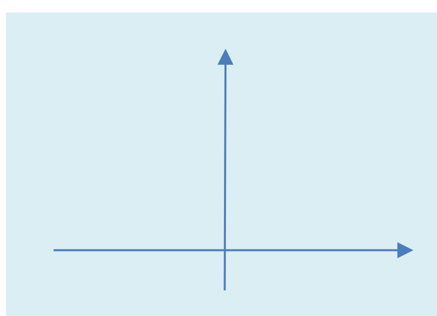
$$\text{Res}\, f(z_j) = \lim_{z \to z_j} (z - z_j) \cdot f(z)$$

For a n-th order pole, i.e. $(z - z_j)^n$ appears in the denominator of $f(z)$

$$\text{Res}\, f(z_j) = \lim_{z \to z_j} (z - z_j)^n \cdot f(z)$$

Application of the residue theorem to a line integral, e.g. $\int_{-\infty}^{\infty} f(\omega)d\omega$

$$\oint_C f(z)dz = \int_{-r}^{r} f(z)dz + \int_S f(z)dz$$

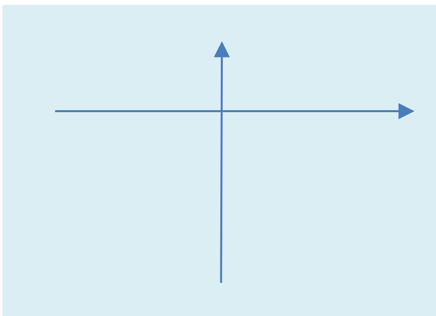


Provided $\int_S f(z)dz$ vanishes as $r \to \infty$

$$\boxed{\int_{-\infty}^{\infty} f(z)dz = \oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \text{Res}\, f(z_j)}$$

(upper half-plane)

What if we need to use the lower half-plane to make $\int_S f(z)dz$ vanish?



$$\oint_C f(z)dz = \int_r^{-r} f(z)dz + \int_S f(z)dz$$

Therefore, for the lower half-plane,

$$\boxed{\int_{-\infty}^{\infty} f(z)dz = -\oint_C f(z)dz = -2\pi i \sum_{j=1}^{m} \text{Res}\, f(z_j)}$$

Example: Derive IRF of an SDOF oscillator from its FRF

Note its FRF is

$$H(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega}$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega} d\omega$$
$$= \frac{1}{2\pi}\oint_C \frac{e^{izt}}{\omega_0^2 - z^2 + 2i\xi\omega_0 z} dz - \frac{1}{2\pi}\int_S f(z)dz$$

Which half-plane should be used to make the second integral vanish?

Consider poles of the analytic function $f(z)$, i.e. roots of the equation $\omega_0^2 - z^2 + 2i\xi\omega_0 z = 0$

- $z_1 = \omega_0\left(-\sqrt{1-\xi^2} + i\xi\right)$
- $z_2 = \omega_0\left(\sqrt{1-\xi^2} + i\xi\right)$

When $z = a + ib$,

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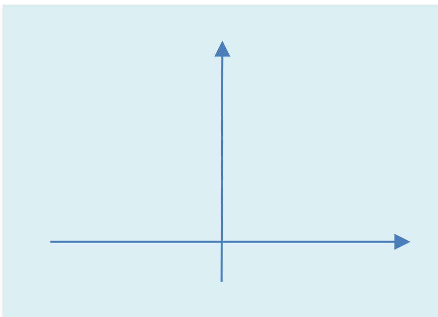
$e^{izt} = e^{i(a+ib)t} = e^{iat} \cdot e^{-bt}$

When b 0, the $\frac{1}{2\pi}\int_S f(z)dz$ vanishes as $r \to \infty$. Therefore, we should use the ( ) half-plane.



$$h(t) = \frac{1}{2\pi}\oint_C f(z)dz = \frac{1}{2\pi} \cdot 2\pi i \sum_{j=1}^{2} \text{Res}\, f(z_j) = i \cdot \left(\text{Res}\, f(z_1) + \text{Res}\, f(z_2)\right)$$

$$f(z) = \frac{e^{izt}}{-(z-z_1)(z-z_2)}$$

- $\text{Res}\, f(z_1) = \lim_{z \to z_1} (z - z_1) \cdot f(z) = \frac{e^{iz_1 t}}{-(z_1 - z_2)} = \frac{e^{-\omega_0\left(\xi + i\sqrt{1-\xi^2}\right)t}}{2\omega_D}$
- $\text{Res}\, f(z_2) = \lim_{z \to z_2} (z - z_2) \cdot f(z) = \frac{e^{iz_2 t}}{-(z_2 - z_1)} = \frac{e^{-\omega_0\left(\xi - i\sqrt{1-\xi^2}\right)t}}{-2\omega_D}$

Finally,

$$\begin{aligned} h(t) &= i \cdot \left(\text{Res}\, f(z_1) + \text{Res}\, f(z_2)\right) \\ &= \frac{1}{\omega_D} e^{-\xi\omega_0 t} \sin \omega_D t \quad (t > 0) \end{aligned}$$

III-2. Random Vibration Analysis of Linear Structures

…

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457.643 Structural Random Vibrations
In-Class Material: Class 16

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Response of a linear system to a stochastic input process

Deterministic input

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 = p(t)$$

$$x(t) = \sum_{i=1}^{n-1} x^{(i)}(0) g_i(t) + \int_0^t p(\tau) h_p(t-\tau) d\tau$$

$$\left(x(t) = \sum_{i=1}^{n-1} x^{(i)}(0) g_i(t) + \int_0^t f(\tau) h_f(t-\tau) d\tau \right)$$

Stochastic input

$$a_n \frac{d^n X}{dt^n} + \cdots + a_1 \frac{dX}{dt} + a_0 = P(t)$$

$$X(t) = \sum_{i=1}^{n-1} X^{(i)}(0) g_i(t) + \int_0^t P(\tau) h_p(t-\tau) d\tau$$

◎ Example: stochastic response of standard SDOF oscillator

$$\ddot{x}(t) + 2\xi\omega_0 \dot{x}(t) + \omega_0^2 x = f(t)$$

$$x(t) = x(0) g(t) + \dot{x}(0) h(t) + \int_0^t f(\tau) h(t-\tau) d\tau$$

- $g(t) = e^{-\xi\omega_0 t} \cdot \left(\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}} \sin\omega_D t\right) \cdot U(t) \rightarrow g_1(t)$ above
- $h(t) = \frac{1}{\omega_D} e^{-\xi\omega_0 t} \cdot \sin\omega_D t \cdot U(t) \rightarrow g_2(t)$ above

When there exists randomness in the initial conditions and the external force, the response is a stochastic response

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$$X(t) = S_1 \cdot g(t) + S_2 \cdot h(t) + \int_0^t F(\tau) h(t-\tau) d\tau$$

Question: $\mu_X(t)$, $\phi_{XX}(t_1, t_2)$, …?

Assuming the integral (↗) exists in the mean-square sense, we can derive the moment functions as follows.

1) $\mu_X(t) = \mathrm{E}[X(t)]$

$$= \qquad \cdot g(t) + \qquad \cdot h(t) + \int_0^t \qquad h(t-\tau) d\tau$$

2) $\phi_{XX}(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)]$

$$= \qquad \cdot g(t_1)g(t_2) + \qquad \cdot h(t_1)h(t_2) + \qquad \cdot \{g(t_1)h(t_2) + g(t_2)h(t_1)\}$$

$$+ \int_0^{t_1} \int_0^{t_2} \phi_{FF}(\tau_1, \tau_2) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_2 d\tau_1$$

$$+\mathrm{E}\left\{[S_1 g(t_1) + S_2 h(t_1)] \int_0^{t_2} F(\tau_2) h(t_2 - \tau_2) d\tau_2\right\}$$

$$+\mathrm{E}\left\{[S_1 g(t_2) + S_2 h(t_2)] \int_0^{t_1} F(\tau_1) h(t_1 - \tau_1) d\tau_1\right\}$$

3) $\kappa_{XX}(t_1, t_2) = COV[X(t_1), X(t_2)] = \mathrm{E}\left[\left(X(t_1) - \mu_X(t_1)\right) \cdot \left(X(t_2) - \mu_X(t_2)\right)\right]$

$$= \int_0^{t_1} \int_0^{t_2} \kappa_{FF}(\tau_1, \tau_2) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_2 d\tau_1$$

$$+\sigma_{S_1}^2 g(t_1) g(t_2) + \sigma_{S_2}^2 h(t_1) h(t_2) + COV[S_1, S_2] \cdot [g(t_1) \cdot h(t_2) + g(t_2) \cdot h(t_1)]$$

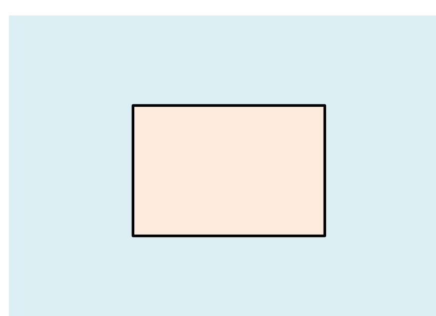
$+$ terms involving covariances between S_1 and F, and those between S_2 and F (usually zero)

◎ Response of a linear system under multiple stochastic inputs

Assuming zero IC's for simplicity, suppose a linear system is subjected to multiple stochastic loads

$F_1(t), F_2(t), \cdots$

Then, the stochastic response and its moment functions are



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- $X(t) = \sum_{i=1}^{n} \int_0^t F_i(\tau)h(t-\tau)d\tau$
- $\mu_X(t) = \sum_{i=1}^{n} \int_0^t \mu_{F_i}(\tau)h(t-\tau)d\tau$
- $\phi_{XX}(t_1,t_2) = \sum_{i=1}^{n} \int_0^{t_1}\int_0^{t_2} \phi_{F_iF_j}(\tau_1,\tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_2 d\tau_1$
- $\kappa_{XX}(t_1,t_2) = \sum_{i=1}^{n} \int_0^{t_1}\int_0^{t_2} \kappa_{F_iF_j}(\tau_1,\tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_2 d\tau_1$

If $F_i(t)$ and $F_j(t)$ $(i \neq j)$ are statistically independent of each other, the double integral becomes

◎ Cross covariance between response and excitation

$$\kappa_{XF}(t_1,t_2) = \mathrm{E}[(X(t_1) - \qquad)(F(t_2) - \qquad)]$$

$$X(t_1) - \mu_X(t_1) = \int_0^{t_1} F(\tau)h(t_1-\tau)d\tau - \int_0^{t_1} \mu_F(\tau)h(t_1-\tau)d\tau =$$

$$\therefore \kappa_{X,F}(t_1,t_2) = \int_0^{t_1} \mathrm{E}\{[F(\tau)-\mu_F(\tau)][F(t_2) - \qquad]\}h(t_1-\tau)d\tau$$

Therefore,

$$\kappa_{X,F}(t_1,t_2) = \int_0^{t_1} \kappa_{FF}(\quad,\quad)h(t_1-\tau)d\tau$$

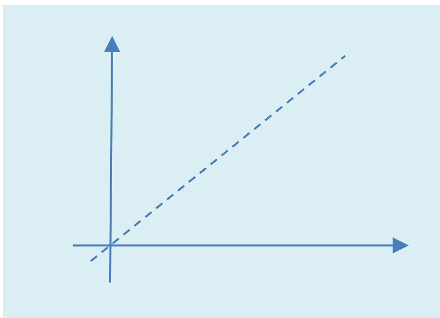
for $t_1 \geq t_2$

When $t_1 < t_2$, $\kappa_{XF}(t_1,t_2) =$

◎ Example: Response to shot noise (Delta-correlated process)

Recall: Shot noise is white noise with time-varying intensity

- $\mu_F(t) =$
- $\kappa_{FF}(t_1,t_2) = I(t_1)\delta(t_1-t_2)$



When $I(t) = I$, i.e. constant, the shot noise becomes __________ noise

Assuming zero IC's

$$\kappa_{XX}(t_1, t_2) = \int_0^{t_2} \int_0^{t_1} \kappa_{FF}(\tau_1, \tau_2) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_1 d\tau_2$$

$$= \int_0^{t_2} \int_0^{t_1} \qquad h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_1 d\tau_2$$

$$= \int_0^{\min(t_1, t_2)} I(\tau) h(t_1 - \tau) h(t_2 - \tau) d\tau$$

Example: Massless SDOF oscillator under shot noise

Equation of motion:

$c\dot{x} + kx = p(t)$

$\dot{x} + \alpha x = f(t)$

where $\alpha = k/c$ and $f(t) = p(t)/c$

Chracterization of the system: impulse response function?

$\dot{h}(t) + \alpha h(t) = \delta(\mathrm{t})$

Initial condition: $h(0^+) = \frac{1}{1} =$

Homogeneous solution:

Set $h(t) = e^{rt}$

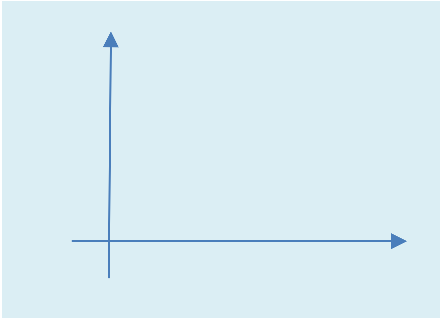
$r =$

Therefore,

$h(t) = A \cdot e^{-\alpha t}$

Applying IC, the impulse response function is $h(t) =$ for t 0

Suppose the intensity function of the shot noise $F(t)$ is given as



$I(t) = I$ for $0 < t \le t_0$ and 0 otherwise

Then,

$$\kappa_{XX}(t_1, t_2) = \int_0^{\min(t_1,t_2)} I(\tau)h(t_1-\tau)h(t_2-\tau)d\tau$$

$$= I\int_0^{\min(t_0,t_1,t_2)} e^{-\alpha(t_1-\tau)} \cdot e^{-\alpha(t_2-\tau)}d\tau$$

$$= I \cdot e^{-\alpha(t_1+t_2)}\int_0^{t^*} e^{2\alpha\tau}d\tau$$

$$= \frac{I}{2\alpha} \cdot e^{-\alpha(t_1+t_2)}[\exp(2\alpha t^*) - 1]$$

$\kappa_{XX}(t,t) = \sigma_X^2 = ?$

For $t \le t_0$, i.e. $t^* = t$

$\sigma_X^2 =$

For $t > t_0$, i.e. $t^* = t_0$

$\sigma_X^2 =$

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457.643 Structural Random Vibrations
In-Class Material: Class 17

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Response of a linear system to weakly stationary input

$\kappa_{FF}(t_1, t_2) = \Gamma_{FF}(\tau)$ where $\tau = t_1 - t_2$

Assuming zero initial conditions,

$$\kappa_{XX}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} \kappa_{FF}(\tau_1, \tau_2) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_2 d\tau_1$$

$$= \int_0^{t_1}\int_0^{t_2} \Gamma_{FF}(\tau) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_2 d\tau_1$$

where $\tau = \tau_1 - \tau_2$

Note that $\Gamma_{FF}(\tau) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega) e^{i\omega\tau} d\omega$

Thus,

$$\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} \Phi_{FF}(\omega) h(t_1 - \tau_1) h(t_2 - \tau_2) e^{i\omega\tau} d\tau_2 d\tau_1 \, d\omega$$

$$= \int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} \Phi_{FF}(\omega) h(t_1 - \tau_1) h(t_2 - \tau_2) e^{-i\omega(t_1-\tau_1)} e^{i\omega(t_2-\tau_2)} e^{i\omega(t_1-t_2)} d\tau_2 d\tau_1 \, d\omega$$

By changing variable $u = t_1 - \tau_1$, one can show

$$\int_0^{t_1} h(t_1 - \tau_1) e^{-i\omega(t_1-\tau_1)} d\tau_1 = \int_0^{t_1} h(u) e^{-i\omega u} du$$

$$\boxed{= \int_{-\infty}^{t_1} h(u) e^{-i\omega u} du = \mathcal{H}(\omega, t_1)}$$

This is so-called "incomplete" Fourier transform of the impulse response function.

cf. "complete" FT of IRF gives the FRF

$$\boxed{H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt}$$

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Therefore, $\kappa_{XX}(t_1, t_2)$ for a weakly stationary input $F(t)$ is expressed as

$$\boxed{\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\, \mathcal{H}(\omega, t_1)\mathcal{H}^*(\omega, t_2) e^{i\omega\tau} d\omega}$$

Note:

- The response of a linear system to a stationary input is ___________ stationary necessarily.
- However, as $t_1, t_2 \to \infty$, the incomplete FTs becomes independent of t_1 and t_2, Therefore, $\kappa_{XX}(t_1, t_2)$ depends only on $\tau = t_1 - t_2$

Observations:

1. $\lim_{t\to\infty} \mathcal{H}(\omega, t) = H(\omega)$ for a "stable" system

   Therefore, the response of a linear system to a stationary input becomes ____________ e__________

2. $\kappa_{XX}(0,0)$ must be ________ and it means $\sigma_X^2(0) =$. This makes sense because we assumed _________ IC's

3. For the stationary response, i.e. $t_1, t_2 \to \infty$

$$\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\, \mathcal{H}(\omega, t_1)\mathcal{H}^*(\omega, t_2) e^{i\omega\tau} d\omega$$

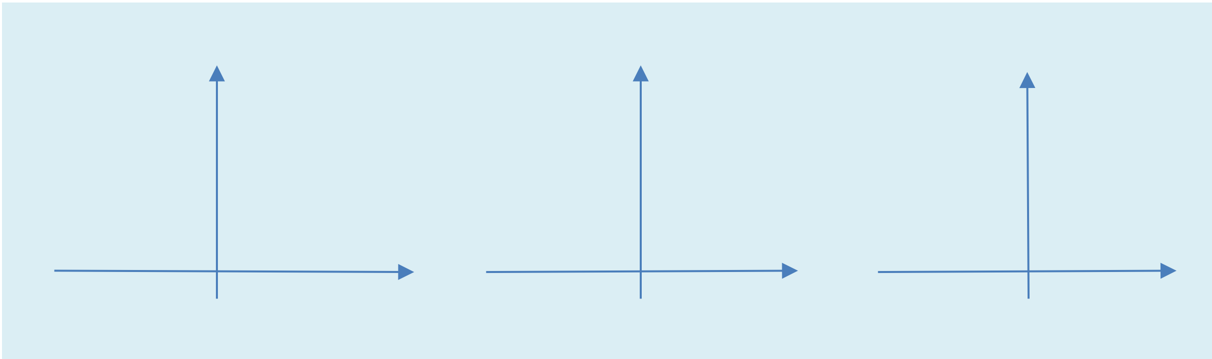
$$= \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\, |H(\omega)|^2 e^{i\omega\tau} d\omega$$

   That is,

$$\boxed{\Gamma_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\, |H(\omega)|^2 e^{i\omega\tau} d\omega}$$

4. From this result, for a <u>stationary</u> response, it is found that

$$\boxed{\Phi_{XX}(\omega) = \qquad\qquad}$$



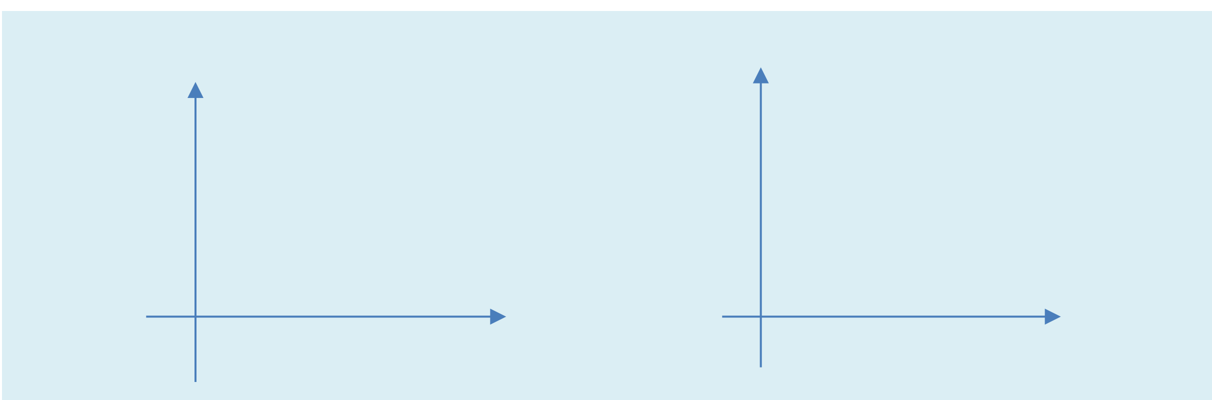
For example, let us consider…

◎ $\boldsymbol{\mathcal{H}}(\omega, t)$ and $\mathrm{H}(\omega)$ of standard SDOF oscillator

Recall

- $h(t) = \frac{1}{\omega_D} e^{-\xi\omega_0 t} \sin\omega_D t$
- $H(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega} = \frac{\omega_0^2 - \omega^2 - 2i\xi\omega_0\omega}{\left(\omega_0^2 - \omega^2\right)^2 + (2\xi\omega_0\omega)\hat{}2}$

$$|H(\omega)|^2 = \frac{1}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$



$$\boldsymbol{\mathcal{H}}(\omega, t_1) = \int_{-\infty}^{t} \frac{1}{\omega_D} e^{-\xi\omega_0 t} \sin\omega_D t \cdot U(t) \cdot e^{-i\omega t} dt$$

$$= H(\omega)\left[1 - \left(\cos\omega_D t + \frac{\xi\omega_0 + i\omega}{\omega_D}\sin\omega_D t\right) \cdot e^{-\xi\omega_0 t} \cdot e^{-i\omega t}\right]$$

From this result, the terms in ( ) has the same order as 1, and $e^{-i\omega t}$ oscillates. Therefore, the rate of the convergence of the terms in [ ] to _____ is determined by _______

In other words, "sufficient" time to achieve stationarity depends on $\xi\omega_0 t = \xi \frac{2\pi}{T_0} t$

Suppose we set $\xi\omega_0 t = 2\pi\xi \frac{t}{T_0} = \pi$ (note $e^{-\pi} = 4\%$) and solve it for t, i.e. time to make the exponentially decaying term as 4%, $t_{4\%} = \frac{T_0}{2\xi}$

e.g. $\xi = 0.1$ → $t_{4\%} \cong 5T_0$, $\xi = 0.05$ → $t_{4\%} \cong 10T_0$

※ Alternative (empirical) method:

Wang, Z., and Song, J. (2016) Equivalent linearization method using Gaussian mixture (GM-ELM) for nonlinear random vibration analysis, *Structural Safety*, http://dx.doi.org/10.1016/j.strusafe.2016.08.005

3.1.1. Remark 1: Selecting sample points

One issue in selecting sample points in the aforementioned algorithm is that the nonlinear response takes a certain amount of time to achieve stationarity, thus using the whole time series including a nonstationary part will introduce errors to the estimated PDF. To reduce this error, for each of the M response histories obtained from the first step of the algorithm, we need to select $\bar{N}$ stationary response values as the sample points.

Here we provide a method to crudely estimate the time that the system would take to achieve stationarity. To begin with, the standard deviation of the response at a sequence of time points, denoted as $\mathrm{std}[Z(j\Delta t)]$, in which $j = 1, 2, \ldots$ and Δt is the time step of the nonlinear analysis, is estimated using the recorded M response histories, and then a sigmoid function expressed as

$$f_{fit}(j) = \frac{1}{1 + e^{-aj\Delta t + b}} \tag{12}$$

is employed to fit the $\mathrm{std}[Z(j\Delta t)]$ curve. Note that $f_{fit}(\cdot) \in (0, 1)$, thus the $\mathrm{std}[Z(j\Delta t)]$ curve should be scaled by a factor $J / \sum_{j=1}^{J} \mathrm{std}[Z(j\Delta t)]$ ($J\Delta t$ is the duration of the excitation) so that it approximately ranges from 0 to 1. The parameters a and b in Eq. (12) can be determined from a least-square regression analysis. A typical scaled $\mathrm{std}[Z(j\Delta t)]$ curve and its corresponding fitting function $f_{fit}(\cdot)$ is illustrated in Fig. 3. With $f_{fit}(t)$ available, the time the system takes to achieve stationarity, denoted by $j_{ns}\Delta t$, can be estimated via

$$j_{ns} = argmin\{j | 1 - f(j\Delta t) \leqslant Tol, j = 1, 2, \ldots\} \tag{13}$$

where Tol denotes a specified tolerance. With j_{ns} determined, for each of the M response histories, $\bar{N} = J - j_{ns}$ time points corresponding to the stationary responses are selected to be the sample points, and the total number of sample points is $N = M \cdot \bar{N} = M \cdot (J - j_{ns})$.

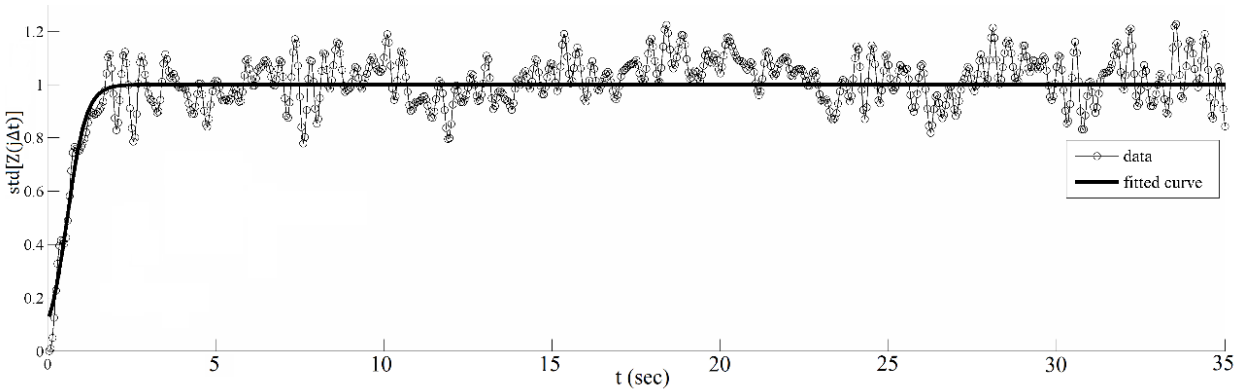
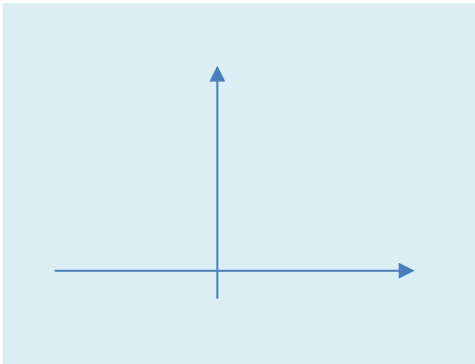


Figure 3. A typical scaled $\mathrm{std}[Z(j\Delta t)]$ curve and the fitting function

◎ Stationary response of standard SDOF oscillator to "white noise"

Useful for linear random vibration analysis of MDOF systems using modal combination, i.e. each mode is represented by a standard SDOF oscillator (will be shown later)

$$\Phi_{\mathrm{FF}}(\omega) = \Phi_0$$



PSD of the stationary response

$$\Phi_{XX}(\omega) = \Phi_0 |H(\omega)|^2$$
$$= \frac{\Phi_0}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$

Thus,

$$\Gamma_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$
$$= \Phi_0 \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2} d\omega$$

How? We can use ____________ theorem

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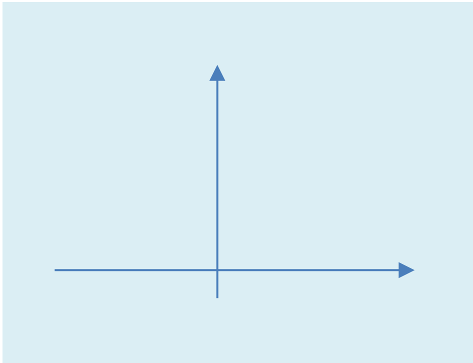
457.643 Structural Random Vibrations
In-Class Material: Class 18

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Stationary response of standard SDOF oscillator to "white noise"

Useful for linear random vibration analysis of MDOF systems using modal combination, i.e. each mode is represented by a standard SDOF oscillator (will be shown later)

$\Phi_{\mathrm{FF}}(\omega) = \Phi_0$



PSD of the stationary response

$$\Phi_{XX}(\omega) = \Phi_0 |H(\omega)|^2$$
$$= \frac{\Phi_0}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$

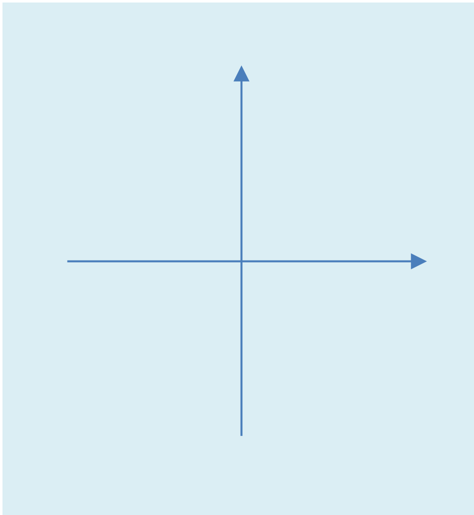
Thus,

$$\Gamma_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$
$$= \Phi_0 \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2} d\omega$$

How? We can use ____________ theorem

Poles? $f(z) = \frac{\Phi_0 e^{iz\tau}}{\left(\omega_0^2 - z^2\right)^2 + 4\xi^2\omega_0^2 z^2} = \frac{\Phi_0 e^{iz\tau}}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

Solve $(\omega_0^2 - z^2)^2 + 4\xi^2\omega_0^2 z^2 = 0$ for z



$\omega_0^2 - z^2 = \pm 2i\xi\omega_0 z$

$z^2 \pm 2i\xi\omega_0 z - \omega_0^2 = 0$

- $z_1 = \omega_D - i\xi\omega_0$
- $z_2 = -\omega_D - i\xi\omega_0$
- $z_3 = \omega_D + i\xi\omega_0$
- $z_4 = -\omega_D + i\xi\omega_0$

First consider $\tau > 0$, note $z = a + ib$

$$e^{iz\tau} = e^{i(a+ib)\tau} = e^{ia\tau} \cdot e^{-b\tau}$$

Therefore, the function insider the integral vanishes as $r \to \infty$ if $b \quad 0$.

That is, we should use upper/lower half-plane for the residue theorem.

$$\begin{aligned}\int_{-\infty}^{\infty} f(\omega)d\omega &= \oint_C f(z)dz \\ &= 2\pi i\left(\text{Res}\, f(\quad) + \text{Res}\, f(\quad)\right) \\ &= 2\pi i\left(\frac{\Phi_0 e^{i \quad \tau}}{(\quad -z \quad)(\quad -z \quad)(\quad -z \quad)} + \frac{\Phi_0 e^{i \quad \tau}}{(\quad -z \quad)(\quad -z \quad)(\quad -z \quad)}\right)\end{aligned}$$

As a result,

$$\Gamma_{XX}(\tau) = \frac{\pi\Phi_0}{2\xi\omega_0^3} e^{-\xi\omega\tau}\left(\cos\omega_D\tau + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D\tau\right), \tau > 0$$

From ___________, for $\forall\tau$

$$\boxed{\Gamma_{XX}(\tau) = \frac{\pi\Phi_0}{2\xi\omega_0^3} e^{-\xi\omega|\tau|}\left(\cos\omega_D\tau + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D|\tau|\right)}$$

1) Variance of the stationary response of standard SDOF oscillator to “white noise”

$$\boxed{\sigma_X^2 = \Gamma_{XX}(\quad) = \frac{\pi\Phi_0}{2\xi\omega_0^3}}$$

2) Mean-square continuous?

3) Mean-square differentiable?

4) Cross-covariance of the response and its time derivative

$$\Gamma_{\dot{X}X}(\tau) = -\Gamma_{X\dot{X}}(\tau) = \frac{d\Gamma_{XX}(\tau)}{d\tau} = -\frac{\pi\Phi_0}{2\xi\omega_0\omega_D} e^{-\xi\omega_0\tau}\sin\omega_D\tau \quad \text{for } \tau > 0$$

Therefore, for $\forall\tau$,

$$\boxed{\Gamma_{\dot{X}X}(\tau) = -\frac{\pi\Phi_0}{2\xi\omega_0\omega_D} e^{-\xi\omega_0|\tau|}\sin\omega_D\tau}$$

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5) Auto-covariance of the time derivative

$$\boxed{\Gamma_{\dot{X}\dot{X}}(\tau) = -\frac{d^2\Gamma_{XX}(\tau)}{d\tau^2} = \frac{\pi\Phi_0}{2\xi\omega_0}e^{-\xi\omega_0|\tau|}\left(\cos\omega_D\tau - \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D|\tau|\right)}$$

6) Note that the time derivative of the SDOF response to white noise is _______ differentiable (in the mean-square sense).

Setting $Y(t) = \dot{X}(t)$, $\Gamma_{YY}(\tau) = \Gamma_{\dot{X}\dot{X}}(\tau)$

$\frac{d\Gamma_{YY}(\tau)}{d\tau}$ does not exist at $\tau = 0$

7) PSD of the time derivative (velocity)

$$\boxed{\begin{aligned}\Phi_{\dot{X}\dot{X}}(\omega) &= \omega^2\Phi_{XX}(\omega)\\ &= \frac{\Phi_0\omega^2}{(\omega_0^2-\omega^2)^2+4\xi^2\omega_0^2\omega^2}\end{aligned}}$$

8) Note

$$\Gamma_{\dot{X}\dot{X}}(\tau) = \Phi_0\int_{-\infty}^{\infty}\frac{\omega^2 e^{i\omega\tau}}{(\omega_0^2-\omega^2)^2+4\xi^2\omega_0^2\omega^2}d\omega = \text{Result in (5)}$$

The term inside the integral is $o(\omega^2)/o(\omega^4)$: decays faster than $1/\omega$

How about $\ddot{X}(t)$? $\Phi_{\ddot{X}\ddot{X}}(\omega) = \omega^4\Phi_{XX}(\omega)$

Therefore, $\Gamma_{\ddot{X}\ddot{X}}(\tau)$ is the integral of the term proportional to $o(\omega^4)/o(\omega^4)$: does NOT decay faster than $1/\omega$. Thus $\Gamma_{\ddot{X}\ddot{X}}(\tau) \to \infty$

9) Variance of the time derivative $\dot{X}(t)$

$$\boxed{\sigma_{\dot{X}}^2 = \Gamma_{\dot{X}\dot{X}}(\quad) = \frac{\pi\Phi_0}{2\xi\omega_0}}$$

◎ Non-stationary response of standard SDOF oscillator to "white noise"

$$\kappa_{XX}(t_1,t_2) = \int_{-\infty}^{\infty}\Phi_0\mathcal{H}(\omega,t_1)\mathcal{H}^*(\omega,t_2)e^{i\omega(t_1-t_2)}d\omega$$

◎ Supplementary Materials: SDOF responses to WN

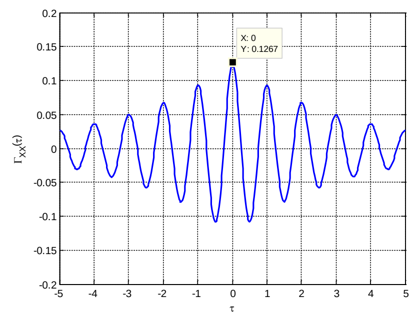
I. "Stationary" responses of the standard SDOF oscillator to white noise: $X(t)$ and its derivative $Y(t) = dX(t)/dt$

(Plots generated for $\omega_0 = 2\pi$, $\zeta = 0.05$ and $\Phi_0 = 1.0$)

(1) Autocovariance function of $X(t)$:

$$\Gamma_{XX}(\tau) = \frac{\pi\Phi_0}{2\zeta\omega_0^3} e^{-\zeta\omega_0|\tau|}\left(\cos\omega_D\tau + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|\tau|\right)$$

$$\Gamma_{XX}(0) = \sigma_X^2 = \frac{\pi\Phi_0}{2\zeta\omega_0^3}$$

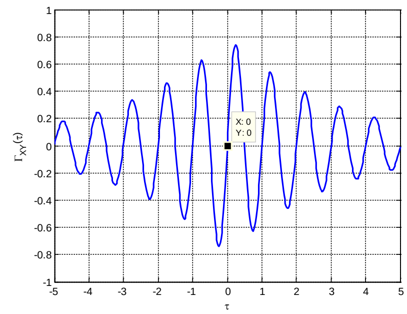


(2) Crosscovariance function of $X(t)$ and $Y(t)$:

$$\Gamma_{XY}(\tau) = -\frac{d\Gamma_{XX}(\tau)}{d\tau} = \frac{\pi\Phi_0}{2\zeta\omega_0\omega_D} e^{-\zeta\omega_0|\tau|}\sin\omega_D\tau$$

$$\Gamma_{XY}(0) = 0$$

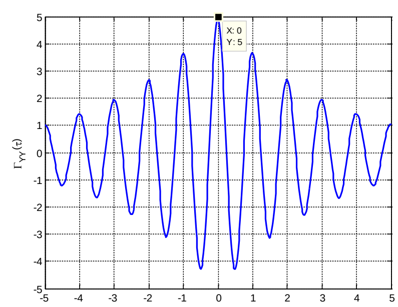
(Note: A stationary r.p. and its derivative are always orthogonal, i.e. $R_{XY}(0) = 0$.)



(3) Autocovariance function of $Y(t)$:

$$\Gamma_{YY}(\tau) = -\frac{d^2\Gamma_{XX}(\tau)}{d\tau^2} = \frac{\pi\Phi_0}{2\zeta\omega_0} e^{-\zeta\omega_0|\tau|}\left(\cos\omega_D\tau - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|\tau|\right)$$

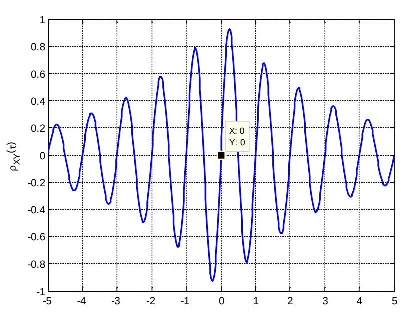
$$\Gamma_{YY}(0) = \sigma_Y^2 = \frac{\pi\Phi_0}{2\zeta\omega_0}$$



(4) Crosscorrelation coefficient function of $X(t)$ and $Y(t)$:

$$\rho_{XY}(\tau) = \frac{\Gamma_{XY}(\tau)}{\sqrt{\Gamma_{XX}(0)\Gamma_{YY}(0)}} = e^{-\zeta\omega_0|\tau|}\frac{1}{\sqrt{1-\zeta^2}}\sin\omega_D\tau$$

$$\rho_{XY}(0) = 0$$



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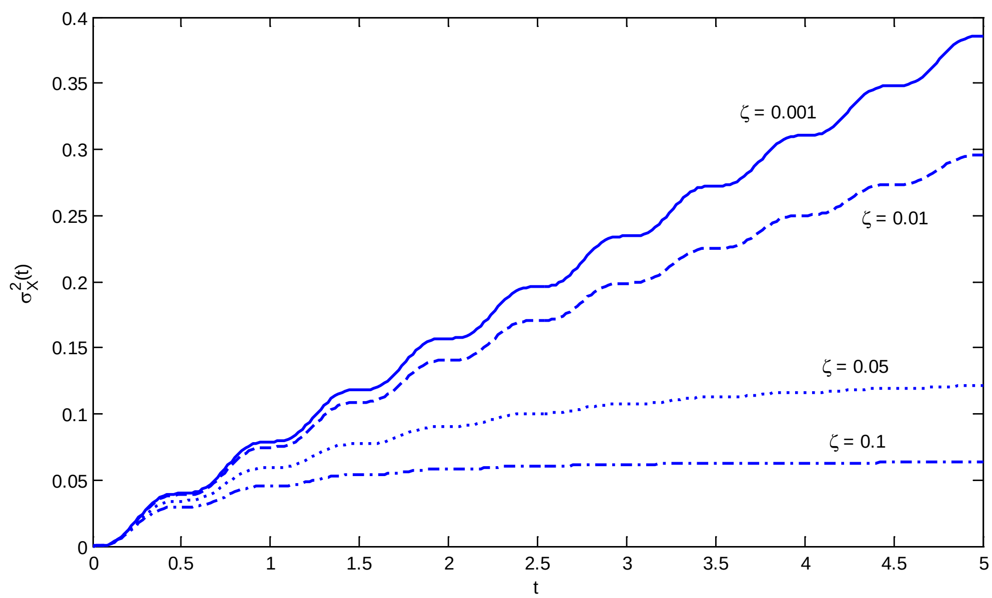
II. "Nonstationary" responses of the standard SDOF oscillator to white noise: $X(t)$ and its derivative $Y(t)=dX(t)/dt$

(1) Autocovariance function of $X(t)$:

$$\kappa_{XX}(t_1,t_2)=\int_{-\infty}^{\infty}\Phi_0 H(\omega,t_1)H^*(\omega,t_2)e^{i\omega(t_1-t_2)}d\omega=2\pi\Phi_0\int_0^{\min(t_1,t_2)}h(t_1-\tau)h(t_2-\tau)d\tau$$

$$=\frac{\pi\Phi_0}{2\zeta\omega_0^3}\left\{\begin{aligned}&e^{-\zeta\omega_0|t_1-t_2|}\left[\cos\omega_D(t_1-t_2)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|t_1-t_2|\right]-\\&e^{-\zeta\omega_0(t_1+t_2)}\left[\frac{1}{1-\zeta^2}\cos\omega_D(t_1-t_2)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D(t_1+t_2)-\frac{\zeta^2}{1-\zeta^2}\cos\omega_D(t_1+t_2)\right]\end{aligned}\right\}$$

$$\kappa_{XX}(t,t)=\sigma_X^2(t)=\frac{\pi\Phi_0}{2\zeta\omega_0^3}\left\{1-e^{-2\zeta\omega_0 t}\left[\frac{1}{1-\zeta^2}+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin 2\omega_D t-\frac{\zeta^2}{1-\zeta^2}\cos 2\omega_D t\right]\right\}$$



Note:

1) Eventually approaches to the variance of the stationary response, $\dfrac{\pi\Phi_0}{2\zeta\omega_0^3}$

2) The "sufficient" time to achieve stationarity depends on $\zeta\omega_0$.

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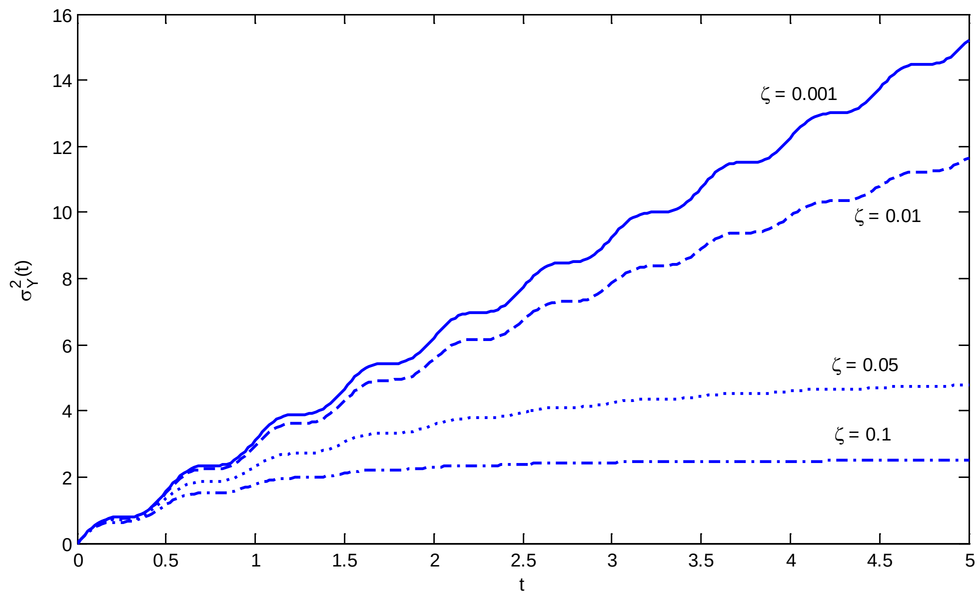
(2) Autocovariance function of $Y(t)$:

$$\kappa_{YY}(t_1,t_2) = \frac{\partial^2 \kappa_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}$$

$$= \frac{\pi\Phi_0}{2\zeta\omega_0}\left\{ \begin{array}{l} e^{-\zeta\omega_0|t_1-t_2|}\left[\cos\omega_D(t_1-t_2) - \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|t_1-t_2|\right] - \\ e^{-\zeta\omega_0(t_1+t_2)}\left[\dfrac{1}{1-\zeta^2}\cos\omega_D(t_1-t_2) - \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D(t_1+t_2) - \dfrac{\zeta^2}{1-\zeta^2}\cos\omega_D(t_1+t_2)\right] \end{array} \right\}$$

$$\kappa_{YY}(t,t) = \sigma_Y^2(t)$$

$$= \frac{\pi\Phi_0}{2\zeta\omega_0}\left\{1 - e^{-2\zeta\omega_0 t}\left[\frac{1}{1-\zeta^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin 2\omega_D t - \frac{\zeta^2}{1-\zeta^2}\cos 2\omega_D t\right]\right\}$$



Note:

1) Eventually approaches to the variance of the stationary response, $\dfrac{\pi\Phi_0}{2\zeta\omega_0}$

2) The "sufficient" time to achieve stationarity depends on $\zeta\omega_0$.

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(3) Crosscovariance of $X(t)$ and $Y(t)$:

$$\kappa_{XY}(t_1,t_2)=\frac{\partial\kappa_{XX}(t_1,t_2)}{\partial t_2}$$

$$=\frac{\pi\Phi_0}{2\zeta\omega_0\omega_D}\left\{\begin{matrix} e^{-\zeta\omega_0|t_1-t_2|}\sin\omega_D|t_1-t_2|- \\ e^{-\zeta\omega_0(t_1+t_2)}\left[\sin\omega_D|t_1-t_2|-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_D(t_1-t_2)+\dfrac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_D(t_1+t_2)\right]\end{matrix}\right\}$$

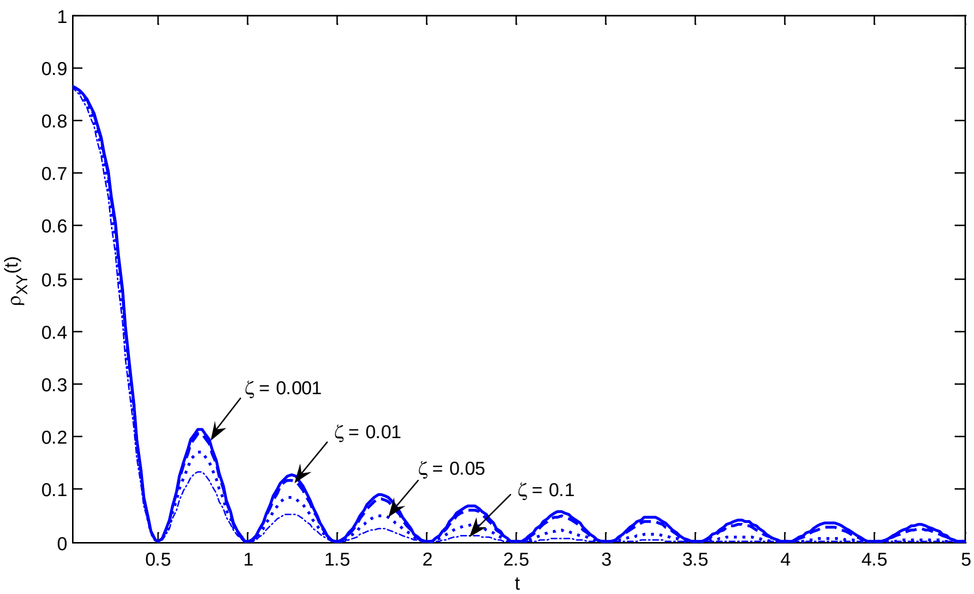
(4) Crosscorreltion coefficient function of $X(t)$ and $Y(t)$:

$$\rho_{XY}(t_1,t_2)=\frac{\kappa_{XY}(t_1,t_2)}{\sqrt{\kappa_{XX}(t_1,t_1)\kappa_{YY}(t_2,t_2)}}$$

$$\rho_{XY}(t,t)=\frac{\kappa_{XY}(t,t)}{\sqrt{\kappa_{XX}(t,t)\kappa_{YY}(t,t)}}$$

in which

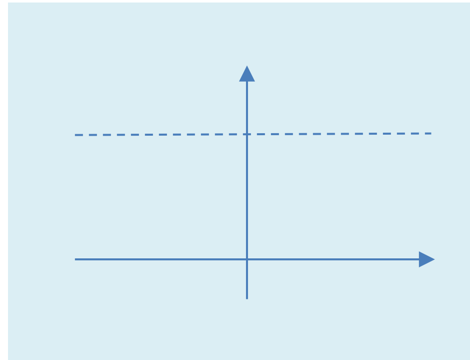
$$\kappa_{XY}(t,t)=\frac{\pi\Phi_0}{2\omega_D^2}e^{-2\zeta\omega_0 t}\left(1-\cos 2\omega_D t\right)$$



Note:

1) $\rho_{XY}(t)$ is not zero due to the nonstationarity.
2) Therefore, $\rho_{XY}(t)$ can be used as a criterion for checking stationarity.

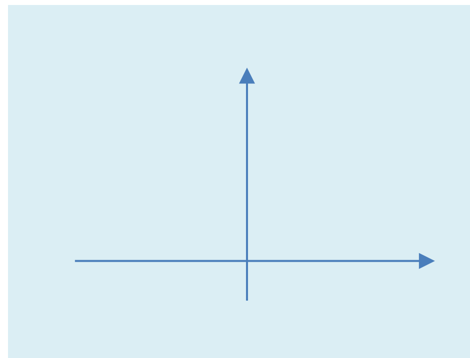
◎ Stationary response of standard SDOF oscillator to wide-band inputs: approximation by "white-noise" response



$$\Phi_{XX}(\omega) = \Phi_{FF}(\omega)|H(\omega)|^2$$
$$\cong \Phi_0 |H(\omega)|^2$$

where $\Phi_0 = \Phi_{FF}(\omega_0)$

The accuracy of the WN approximation depends on



- ξ: Bandwidth of $|H(\omega)|^2$ (accurate if it is narrow band, i.e. $\xi \cong 0$)
- Bandwidth parameter of $F(t)$ (e.g. δ, s, ξ_g) accurate if it is wideband, e.g. $\xi_g \gg 0$
- $\frac{\omega_0}{\omega_g} \cong 1$

◎ Spectral moments of stationary response of SDOF to white noise input

$$\lambda_m = \int_0^\infty \omega^m G_{XX}(\omega) d\omega$$
$$= 2\Phi_0 \int_0^\infty \omega^m |H(\omega)|^2 d\omega$$

- $\lambda_0 = \mathrm{E}\left[X^2\right] = \sigma_X^2 = \frac{\pi\Phi_0}{2\xi\omega_0^3}$
- $\lambda_1 = \frac{\pi\Phi_0}{2\xi\omega^2} \times \frac{2}{\pi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \cong \frac{\pi\Phi_0}{2\xi\omega^2} \times \left(1 - \frac{2\xi}{\pi}\right)$ for $\xi \cong 0$

  ➔ Useful for identifying the bandwidth of the process, e.g. $\delta = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}}$

- $\lambda_2 = \mathrm{E}\left[\dot{X}^2(t)\right] = \sigma_{\dot{X}}^2 = \frac{\pi\Phi_0}{2\xi\omega_0}$
- $\lambda_m \to \infty$ if $m > 3$

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457.643 Structural Random Vibrations
In-Class Material: Class 19

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Spectral representation of nonstationary process

Used PSD $\Phi_{XX}(\omega)$ for spectral representation of a stationary process $X(t)$. What to use if $X(t)$ is non-stationary?

Main purpose: describe the change of the frequency content over time

1) **Generalized PSD** $\hat{\Phi}_{XX}(\omega_1, \omega_2)$: Fourier transform of $\phi_{XX}(t_1, t_2)$

Assuming $\tilde{X}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)e^{-i\omega t}dt$ exists in the mean-square sense,

$$\hat{\Phi}_{XX}(\omega_1, \omega_2) \equiv \mathrm{E}[\tilde{X}(\omega_1)\tilde{X}^*(\omega_2)]$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{E}[X(t_1)X^*(t_2)]e^{-i(\omega_1 t_1 - \omega_2 t_2)}dt_1 dt_2$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \qquad e^{-i(\omega_1 t_1 - \omega_2 t_2)}dt_1 dt_2$$

Can show (from the formulation for $\phi_{XX}(t_1, t_2)$ of a linear system)

$$\boxed{\hat{\Phi}_{XX}(\omega_1, \omega_2) = \hat{\Phi}_{FF}(\omega_1, \omega_2)H(\omega_1)H^*(\omega_2)}$$

It is also noted that

$$\phi_{XX}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \qquad e^{i(\omega_1 t_1 - \omega_2 t_2)}d\omega_1 d\omega_2$$

Question: Practical? No, because

- It is difficult to assign physical meaning (two ω's)
- The time term does not appear in the PSD although it is important for nonstationary process

2) **Instantaneous PSD** $\Phi^i(\omega, t)$ (Page 1952)

$$\phi^i(\tau, t) = \mathrm{E}\left[X\left(t - \frac{\tau}{2}\right)X\left(t + \frac{\tau}{2}\right)\right]$$

$$\Phi^i(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi^i(\tau, t)e^{-i\omega\tau}d\tau$$

3) **Physical PSD** (Mark 1970)

$$\Phi^p(\omega, t)_{\mathrm{w}} = \mathrm{E}\left[\left|\int_{-\infty}^{\infty} w(t-\tau)X(\tau)e^{-i\omega\tau}d\tau\right|^2\right]$$

where $w(t-\tau)$ is the "window" function that captures PSD around the time t

4) **Evolutionary PSD** (Pristley 1965, 1967)

Consider two different versions of inverse FT

- $X(t) = \int_{-\infty}^{\infty} \tilde{X}(\omega)e^{i\omega t}d\omega$ Riemann integral
- $X(t) = \int_{-\infty}^{\infty} e^{i\omega t}dS(\omega)$ Stieltjes integral

  ~ generalization of Riemann integral by use of "increment process" $dS(\omega)$

※ Increment Process $dS(\omega)$

(1) Can use Fourier integral even when $\tilde{X}(\omega) = \frac{dS(\omega)}{d\omega}$ does not exist, i.e. $dS(\omega)$ is smoother

$$X(t) = \int_{-\infty}^{\infty} e^{i\omega t}dS(\omega)$$

-- Fourier-Stieltjes integral

(2) "Orthogonal" increment process $dS(\omega)$

$$\boxed{\mathrm{E}[dS(\omega_1)dS^*(\omega_2)] = \Phi(\omega_1)\delta(\omega_1 - \omega_2)d\omega_1 d\omega_2}$$

It has been proved that (Lin & Cai 1995), for an orthogonal increment process,

$\boxed{X(t) = \int_{-\infty}^{\infty} e^{i\omega t}dS(\omega) \text{ exists} \leftrightarrow X(t) \text{ is weakly stationary}}$

Note that $\tilde{X}(\omega)$ does not exist for stationary process

Proof for (→):

$$\phi_{XX}(t_1,t_2) = \mathrm{E}[X(t_1)X^*(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega_1 t_1 - i\omega_2 t_2}\mathrm{E}[dS(\omega_1)dS^*(\omega_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega_1 t_1 - i\omega_2 t_2}\Phi(\omega_1)\delta(\omega_1-\omega_2)d\omega_1 d\omega_2$$
$$= \int_{-\infty}^{\infty} e^{i\omega(t_1-t_2)}\Phi(\omega_1)d\omega_1$$
$$=$$

a) Priestley's idea (toward "evolutionary" PSD)

$$\boxed{X(t) = \int_{-\infty}^{\infty} A(\omega,t)e^{i\omega t}dS(\omega)}$$

where

- $A(\omega,t)$: frequency-time modulating function
- $dS(\omega)$: orthogonal increment process representing a stationary "base" process $X_s(t) = \int_{-\infty}^{\infty} e^{i\omega t}dS(\omega)$

b) In this case, the auto-correlation function is derived as

$$\phi_{XX}(t_1,t_2) = \mathrm{E}[X(t_1)X^*(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega_1,t_1)A^*(\omega_2,t_2)e^{i(\omega_1 t_1-\omega_2 t_2)}\mathrm{E}[dS(\omega_1)dS(\omega_2)]$$
$$= \int_{-\infty}^{\infty} A(\omega,t_1)A^*(\omega,t_2)e^{i\omega(t_1-t_2)}\Phi_{SS}(\omega)d\omega$$

Note, for a stationary process:

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\Phi_{XX}(\omega)d\omega$$

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c) For $t_1 = t_2$,

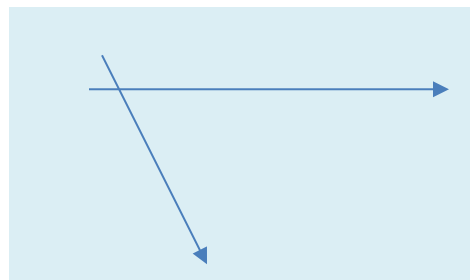
$$\mathrm{E}[X^2(t)] = \int_{-\infty}^{\infty} |A(\omega,t)|^2\, \Phi_{SS}(\omega) d\omega$$

Note, for a stationary process:

$$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\, d\omega$$

Comparing the two equations, $|A(\omega,t)|^2 \Phi_{SS}(\omega)$ seems to describe the evolution of the spectral representation of the non-stationary process at time t, so we can…

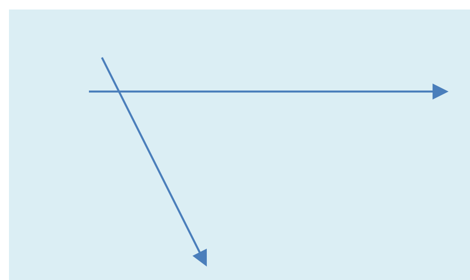
d) Define "Evolutionary" PSD (EPSD) as



$$\boxed{\Phi(\omega,t) = \Phi_{SS}(\omega)|A(\omega,t)|^2}$$

to describe the evolution of the frequency content over time using the frequency-time modulating function $A(\omega,t)$

e) Special case: uniformly modulated evolutionary process



$$A(\omega,t) = A(t)$$

In this case, it is noted

$$X(t) = \int_{-\infty}^{\infty} A(t) e^{i\omega t} dS(\omega) = A(t) \int_{-\infty}^{\infty} e^{i\omega t} dS(\omega) = A(t) X_s(t)$$

$$\Phi_{XX}(\omega,t) = |A(t)|^2 \Phi_{SS}(\omega)$$

$$\mathrm{E}[X^2(t)] = |A(t)|^2 E[X_S^2(t)] = |A(t)|^2 E[X_S^2]$$

How to determine $A(\omega,t)$? Examples:

- Kubo & Penzien (1976): Identified $A(t)$ by statistical analysis of San Fernando earthquake records (Clough & Penzien 1993)

  $$\ddot{X}_g(t) = A(t) \cdot X_s(t)$$

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The "envelope function" $A(t)$ was identified in the form $a_1 t \cdot \exp(-a_2 t)$

- Jangid (2004, EESD): provided an extensive survey of envelope functions and investigated SDOF nonstationary responses

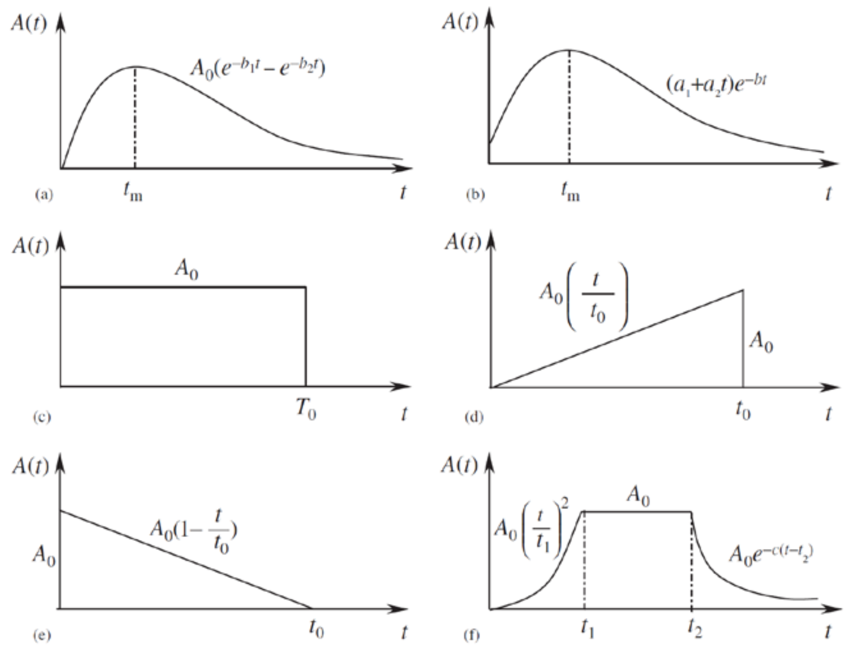

Figure 1. Different modulating functions: (a) exponential—I; (b) exponential—II; (c) box-car; (d) triangular—I; (e) triangular—II; and (f) Amin and Ang type.

- Other ways for spectral representation of nonstationary processes: Wavelet transform (Kareem, Spanos, …), Hilbert-Huang transform (with empirical model decomposition; Wen & Gu, 2004, 2009 in JEM)

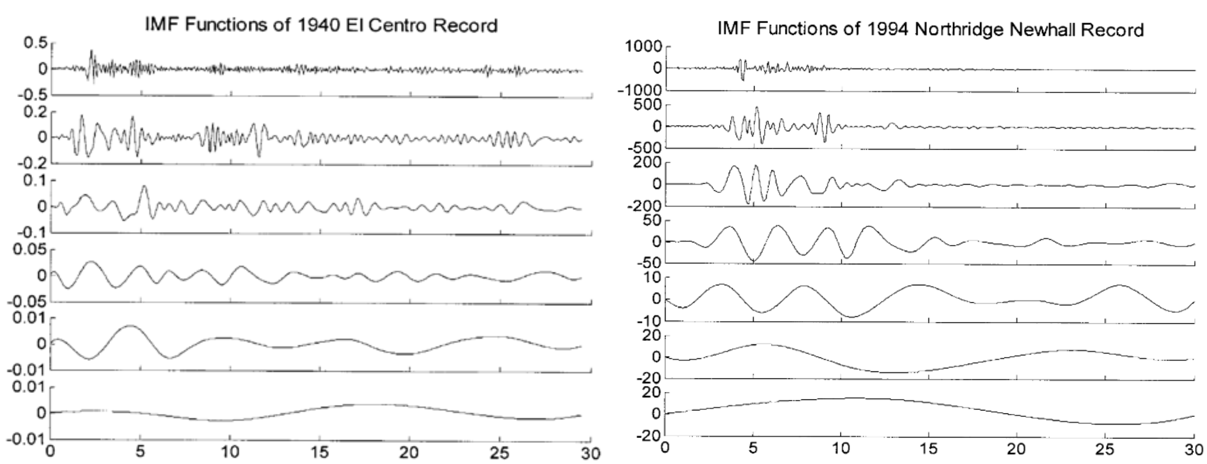

Fig. 2. Intrinsic mode functions (IMFs) of El Centro record and Newhall record

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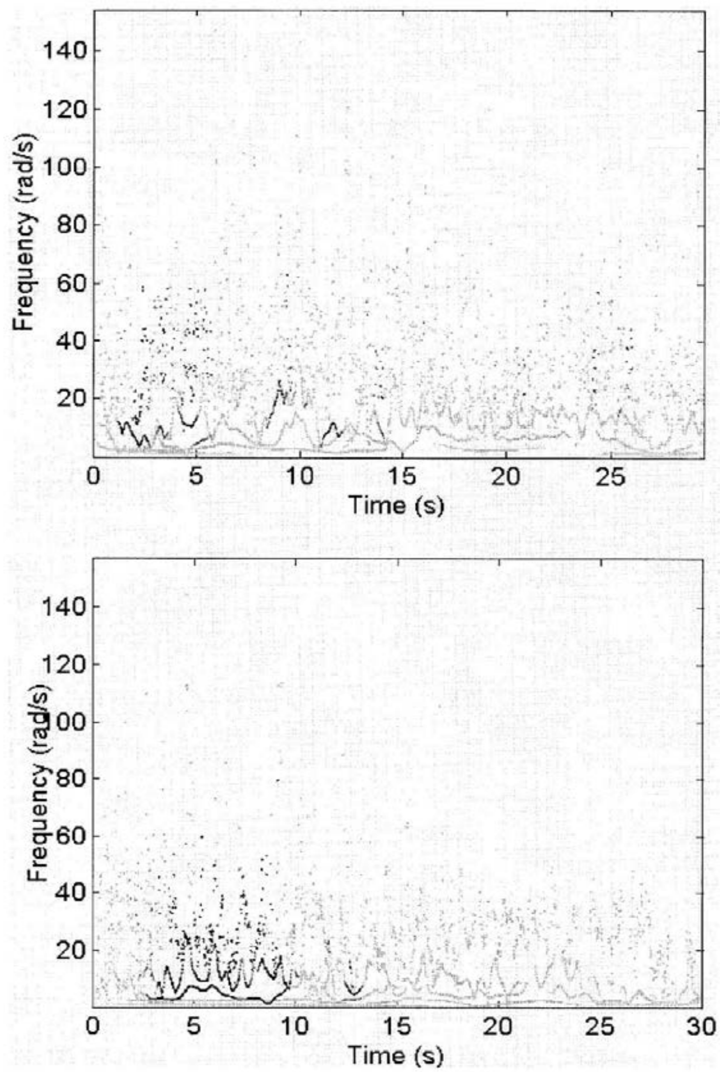

Fig. 3. Hilbert spectra of El Centro record (top) and Newhall record (bottom)

f) Input-output relationship when evolutionary PSD is used (to be continued)

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457.643 Structural Random Vibrations
In-Class Material: Class 20

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Spectral representation of <u>nonstationary</u> process (contd.)

4) **Evolutionary PSD** (Pristley 1965, 1967; contd.)

   f) Input-output relationship when evolutionary PSD is used?

   In general, it was shown (see Class 16)

$$\kappa_{XX}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\kappa_{FF}(\tau_1,\tau_2)h(t_1-\tau_1)h(t_2-\tau_2)d\tau_1 d\tau_2$$

   If the input process $F(t)$ is modeled as an evolutionary process, i.e.

$$\kappa_{FF}(\tau_1,\tau_2)=\int_{-\infty}^{\infty}\Phi_{SS}(\omega)A_F(\omega,\tau_1)A_F^*(\omega,\tau_2)e^{i\omega(\tau_1-\tau_2)}d\omega$$

   Substituting this into the equation above and exchanging the order of the integrals,

$$\kappa_{XX}(t_1,t_2)=\int_{-\infty}^{\infty}\Phi_{SS}(\omega)\times\left[\int_{-\infty}^{\infty}A_F(\omega,\tau_1)h(t_1-\tau_1)e^{-i\omega(t_1-\tau_1)}d\tau_1\right]$$
$$\times\left[\int_{-\infty}^{\infty}A_F^*(\omega,\tau_2)h(t_2-\tau_2)e^{i\omega(t_2-\tau_2)}d\tau_2\right]e^{i\omega(t_1-t_2)}d\omega$$

   Here we define

$$\mathrm{m}(\omega,t)=\int_{-\infty}^{\infty}A_F(\omega,\tau)h(t-\tau)e^{-i\omega(t-\tau)}d\tau$$

   It is noted that the lower boundary value can be replaced by 0 because $A_{\mathrm{F}}(\omega,\tau)=$ for $\forall\tau<0$ and the upper boundary value can be replaced by t because $h(t-\tau)=$ for $t-\tau\quad 0$.

   Using the function $\mathrm{m}(\omega,t)$, the auto-covariance function is determined as

$$\boxed{\kappa_{XX}(t_1,t_2)=\int_{-\infty}^{\infty}\Phi_{SS}(\omega)\mathrm{m}(\omega,t_1)\mathrm{m}^*(\omega,t_2)\,e^{i\omega(t_1-t_2)}d\omega}$$

   Note that, for a stationary input, the auto-covariance function is

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$$\boxed{\kappa_{XX}(t_1,t_2) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\mathcal{H}(\omega,t_1)\mathcal{H}^*(\omega,t_2)\, e^{i\omega(t_1-t_2)}d\omega}$$

That is, we use

- $\mathcal{H}(\omega,t) = \int_{-\infty}^{t} h(u)e^{-i\omega u}du$ for stationary input
- $\mathrm{m}(\omega,t) = \int_{-\infty}^{\infty} A_F(\omega,\tau)h(t-\tau)e^{-i\omega(t-\tau)}d\tau$ for nonstationary input
  (→ information regarding the nonstationary added)

If $A_F(\omega,\mathrm{t}) = 1$, $\mathrm{m}(\omega,t)$ $\quad \mathcal{H}(\omega,t)$

Note: The mean-square of the nonstationary output is derived as

$$\mathrm{E}[X^2(t)] = \int_{-\infty}^{\infty} \Phi_{SS}(\omega)|\mathrm{m}(\omega,t)|^2 d\omega$$

Evolutionary PSD of the nonstationary response?

$$\boxed{\Phi_{XX}(\omega,t) = |\mathrm{m}(\omega,t)|^2\Phi_{SS}(\omega)}$$

It is found that the response is evolutionary if the input is evolutionary (because $\Phi_{XX}(\omega,t) = |A_X(\omega,t)|^2\Phi_{SS}(\omega)$). The frequency-time modulating function of the response is

$$A_X(\omega,t) = \qquad = \int_{-\infty}^{\infty} A_F(\omega,\tau)h(t-\tau)e^{-i\omega(t-\tau)}d\tau$$

For the stationary input and output, the PSD is

$$\Phi_{XX}(\omega) = |H(\omega)|^2\Phi_{FF}(\omega)$$

Example: (Consider again) nonstationary response of standard oscillator to WN

$$h(t) = \frac{1}{\omega_D}e^{-\xi\omega_0 t}\sin\omega_D t$$

WN input: $\Phi_{FF}(\omega) = \Phi_0$

If WN input is modeled by use of the evolutionary model,

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$$A_F(\omega, t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$\Phi_{ss}(\omega) = \Phi_0$$

$$\mathrm{m}(\omega, t) = \int_0^{\mathrm{t}} A_F(\omega, t) h(t-\tau) e^{-i\omega(t-\tau)} d\tau$$

$$= \mathcal{H}(\omega, \mathrm{t}) \quad \because \text{stationary input}$$

Therefore,

$$\Phi_{XX}(\omega, t) = |\mathrm{m}(\omega, t)|^2 \Phi_{ss}(\omega)$$

$$= |\mathcal{H}(\omega, t)|^2 \Phi_0$$

As $t \to \infty$, $\Phi_{XX}(\omega, t) \to |H(\omega)|^2 \Phi_0$

Describe $X(t)$ as an evolutionary process? What is the frequency-time modulating function in

$\Phi_{XX}(\omega, t) = |A_X(\omega, t)|^2 \Phi_{ss}(\omega)$?

Comparing this with the equations above, it is clear in this example that

$$A_X(\omega, t) = m(\omega, t) = \mathcal{H}(\omega, t) = \int_0^t h(u) e^{-i\omega u} du$$

$$= \frac{1}{(\omega_0^2 - \omega^2) + 2i\xi\omega_0\omega} \left[1 - e^{-(\xi\omega_0 + i\omega)t} \left(\cos\omega_D t + \frac{\xi\omega_0 + i\omega}{\omega_D} \sin\omega_D t\right)\right]$$

IV. Random Vibration Analysis of MDOF Systems

- State-space approach (Section 8.6, 9.8 & 10.6 in L&S)
- Modal approach (discussed in this course)

◎ Modal analysis of MDOF system (Review)

1) Equation of Motion

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{PF}$$

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where $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are $(n \times n)$ mass, damping and stiffness matrices; $\ddot{\mathbf{x}}, \dot{\mathbf{x}}$ and $\mathbf{x}$ are $(n \times 1)$ vectors of acceleration, velocity and displacement; $\mathbf{P}$ is $(n \times m)$ matrix that determines the contributions of the external forces to the DOFs; and $\mathbf{F}$ is $(m \times 1)$ vector of the forces

2) Let $\mathbf{x} = \mathbf{\Phi q}(t)$

where $\mathbf{\Phi} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n]$ is the matrix containing n modal shape vectors, and $\mathbf{q}(\mathrm{t}) = [q_1(t) \;\; q_2(t) \;\; \cdots \;\; q_n(t)]^{\mathrm{T}}$ is the vector of scales of the modes at time t

➔ Superposition of multiple modes, each of which is scaled by $q_i(t)$ at time t

Thus, $\dot{\mathbf{x}} = \mathbf{\Phi}\dot{\mathbf{q}}(t)$ and $\ddot{\mathbf{x}} = \mathbf{\Phi}\ddot{\mathbf{q}}(t)$

3) We select $\mathbf{\Phi}$ to be the solution to an eigenvalue problem

$$\mathbf{K\Phi} = \boldsymbol{\lambda}\mathbf{M\Phi}$$

where $\boldsymbol{\lambda} = \mathrm{diag}[\lambda_i] = \mathrm{diag}[\omega_i^2]$, i.e. the diagonal matrix of the eigenvalues (real & positive because $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive-definite), and $\mathbf{\Phi}$ is the eigenmatrix

(Pre-)multiply E.O.M. by $\mathbf{\Phi}^{\mathrm{T}}$

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi}\ddot{\mathbf{q}} + \mathbf{\Phi}^{\mathrm{T}}\mathbf{C\Phi}\dot{\mathbf{q}} + \mathbf{\Phi}^{\mathrm{T}}\mathbf{K\Phi q} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{PF}$$

where

- $\mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi} = \mathrm{diag}[M_i]$: Modal masses
- $\mathbf{\Phi}^{\mathrm{T}}\mathbf{K\Phi} = \mathrm{diag}\left[\omega_i^2 M_i\right]$: Modal stiffnesses ($\omega_i = \sqrt{\frac{K_i}{M_i}}$, modal frequency)
- $\mathbf{\Phi}^{\mathrm{T}}\mathbf{C\Phi} = \mathrm{diag}[C_i]$: Modal damping coefficients (= $\mathrm{diag}[2\xi_i\omega_i M_i]$ for classical damping; ξ_i: modal damping ratio)

When $\mathbf{F}(t) = [F(t)]$, i.e. single input process,

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{P} = \begin{Bmatrix} \vdots \\ \gamma_i M_i \\ \vdots \end{Bmatrix}$$ and $\gamma_i = \frac{\boldsymbol{\phi}_i^{\mathrm{T}}\mathbf{P}}{M_i}$, so-called "modal participation factor"

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4) The *i*-th de-coupled (thanks to the orthogonality) equation is

$$\ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = \gamma_i F(t),\ i = 1, \dots, n$$

Recall

$$q(t) = g_i(t)q_i(0) + h_i(t)\dot{q}_i(0) + \gamma_i \int_0^t F(\tau)h_i(t-\tau)d\tau$$
$$= g_i(t)q_i(0) + h_i(t)\dot{q}_i(0) + \gamma_i s_i(t)$$

where $h_i(t)$ is the unit impulse response function of the *i*-th mode (per-unit-mass force), i.e.

$$h_i(t) = \frac{1}{\omega_{D_i}} e^{-\xi_i \omega_i t} \sin \omega_{D_i} t \text{ and } \omega_{D_i} = \sqrt{1-\xi_i^2} \cdot \omega_i$$

Recall $\mathbf{x} = \mathbf{\Phi q}(t)$, for zero IC's,

$$\boxed{\mathbf{x} = \mathbf{\Phi \Gamma s}(t)}$$

where $\mathbf{\Gamma} = \text{diag}[\gamma_i]$ and $\mathbf{s}(t) = \{s_1(t) \cdots s_n(t)\}^{\mathrm{T}}$

5) Generic response

$$\mathbf{y} = \mathbf{Qx} = \mathbf{Q\Phi\Gamma s}(t) = \mathbf{As}(t)$$

(scalar version) $y_k = \sum_{i=1}^n a_{k,i} s_i(t)$

where

- $\mathbf{A}$: "effective" participation matrix (participation of each mode to each generic response)
- $\mathbf{s}(t)$: vector of normalized modal responses

Examples:

(1) The relative displacement of $x_1(t)$ and $x_2(t)$

$$\mathbf{y} = \mathbf{Qx} = [1 \quad -1] \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

(2) Resisting force: $\mathbf{Q} = [k_s \quad -k_s]$

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457.643 Structural Random Vibrations
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IV. Random Vibration Analysis of MDOF Systems (contd.)

◎ Modal analysis of MDOF system (Review; contd.)

6) Derivation of $h(t)$ and $H(\omega)$

$h_{pq}(t)$: u.i.r.f of the *p*-th response to the loading applied at the *q*-th DOF

To derive $h_{pq}(t)$, set

- $\mathbf{Q} = [0\ \cdots 0\ \ 1\ \ 0\ \cdots 0]$ ("1" at the *p*-th element only): $\mathbf{y} = y = x_p$
- $\mathbf{P} = \{0 \cdots 0\ \ 1\ \ 0 \cdots 0\}^{\mathrm{T}}$("1" at the *q*-th element only): (impulse) load applied at the *q*-th DOF
- $\mathbf{F} = F(t) = \delta(t) \rightarrow s_i(t) = \int_0^t F(\tau) h_i(t-\tau) d\tau = h_i(t)$
- $\gamma_i = \frac{\boldsymbol{\phi}_i^{\mathrm{T}} \mathbf{P}}{M_i} = \frac{\phi_{qi}}{M_i}$

Then, obtain the generic response $\mathbf{y} = \mathbf{Q\Phi\Gamma s}(t)$ to obtain $h_{pq}(t)$

$$h_{pq}(t) = \mathbf{y}$$
$$= \mathbf{Q\Phi\Gamma s}(t)$$
$$= [0\ \cdots 0\ \ 1\ \ 0\ \cdots 0]\boldsymbol{\Phi}\mathrm{diag}\left[\frac{\phi_{qi}}{M_i}\right]\begin{Bmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{Bmatrix}$$
$$= [\phi_{p1}\ \ \phi_{p2}\ \cdots\ \phi_{pn}]\ \mathrm{diag}\left[\frac{\phi_{qi}}{M_i}\right]\begin{Bmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{Bmatrix}$$

Thus,

$$\boxed{\begin{aligned} h_{pq}(t) &= \sum_{i=1}^{n} \left(\frac{\phi_{pi}\phi_{qi}}{M_i}\right) h_i(t) \\ H_{pq}(\omega) &= \sum_{i=1}^{n} \left(\frac{\phi_{pi}\phi_{qi}}{M_i}\right) H_i(\omega) \end{aligned}}$$

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7) Multiple inputs ($m \neq 1$): By superposition,

$$\boxed{\mathbf{y} = \sum_{k=1}^{n} \mathbf{A}_k \mathbf{s}_k(t)}$$

where $\mathbf{A}_k = \mathbf{Q\Phi\Gamma}_k$, $\mathbf{\Gamma}_k = \text{diag}\left[\frac{\boldsymbol{\phi}_i^{\mathrm{T}}\boldsymbol{P}_k}{M_i}\right]$, $\mathbf{s}_k(t) = \int_0^t F_k(\tau)\boldsymbol{h}(t-\tau)d\tau$

Scalar version:

$$y_p(t) = \sum_{\mathrm{k}=1}^{\mathrm{m}} \sum_{i=1}^{n} \left(a_{pi}\right)_k s_{ik}(t)$$

where $s_{ik}(t) = \int_0^t F_k(\tau)h_i(t-\tau)d\tau$

◎ MDOF response to stochastic input: moment functions

When the inputs are modeled by a vector of "random" processes $\mathbf{F}(t) = \{F_1(t)\ F_2(t) \cdots F_m(t)\}^{\mathrm{T}}$

$$\mathbf{Y}(t) = \sum_{k=1}^{m} \mathbf{A}_k \boldsymbol{S}_k(t)$$

1) Mean response:

$$\boxed{\mathrm{E}[\mathbf{Y}(t)] = \sum_{k=1}^{m} \mathbf{A}_k \mathrm{E}[\boldsymbol{S}_k(t)]}$$

where

$$\mathrm{E}[\mathbf{S}_k(t)] = \int_0^t \mathrm{E}[F_k(\tau)]\boldsymbol{h}(t-\tau)d\tau = \int_0^t \mu_k(\tau)\boldsymbol{h}(t-\tau)d\tau$$

2) Auto- and cross-covariances of the responses:

$$\mathbf{Y}(t_1)\mathbf{Y}(t_2)^{\mathrm{T}} = \sum_{k=1}^{m}\sum_{l=1}^{m} \mathbf{A}_k \mathbf{S}_k(t_1)\mathbf{S}_l^{\mathrm{T}}(t_2)\mathbf{A}_l^{\mathrm{T}}$$

$$\boxed{\mathbf{\Sigma}_{\mathbf{YY}}(t_1,t_2) = \mathrm{E}[\mathbf{Y}(t_1)\mathbf{Y}(t_2)^{\mathrm{T}}] = \sum_{k=1}^{m}\sum_{l=1}^{m} \mathbf{A}_k \mathbf{\Sigma}_{\mathbf{S}_k\mathbf{S}_l}(t_1,t_2)\,\mathbf{A}_l^{\mathrm{T}}}$$

(single input case: $\mathbf{\Sigma}_{\mathbf{YY}}(t_1,t_2) = \mathbf{A}\mathbf{\Sigma}_{\mathbf{SS}}(t_1,t_2)\mathbf{A}^{\mathrm{T}}$)

Typical element (i,j) in the matrix $\boldsymbol{\Sigma}_{\mathbf{S}_k\mathbf{S}_l}(t_1,t_2)$:

$$\kappa_{S_kS_l}^{(i,j)}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2}\kappa_{F_kF_l}(\tau_1,\tau_2)h_i(t_1-\tau_1)h_j(t_2-\tau_2)d\tau_2 d\tau_1$$

3) PSD of <u>stationary</u> response to stationary input:

$$\boxed{\boldsymbol{\Phi}_{\mathbf{YY}}(\omega) = \sum_{k=1}^{m}\sum_{l=1}^{m}\mathbf{A}_k\boldsymbol{\Phi}_{\mathbf{S}_k\mathbf{S}_l}(\omega)\mathbf{A}_l^T}$$

Typical element (i,j) in the matrix $\boldsymbol{\Phi}_{\mathbf{S}_k\mathbf{S}_l}(\omega)$:

$$\Phi_{\mathbf{S}_k\mathbf{S}_l}^{(i,j)}(\omega) = \Phi_{F_kF_l}(\omega)H_i(\omega)H_j^*(\omega)$$

(single input case: $\boldsymbol{\Phi}_{\mathbf{YY}}(\omega) = \mathbf{A}\boldsymbol{\Phi}_{\mathbf{SS}}(\omega)\mathbf{A}^{\mathrm{T}}$)

4) Response to evolutionary excitation, i.e.

$$\Phi_{F_kF_l}(\omega,t) = A_k(\omega,t)A_l^*(\omega,t)\Phi_{F_kF_l}^S(\omega)$$

Note $A_k(\omega,t)$ is not the effective participation matrix, but the frequency-time modulation function; and $\Phi_{F_kF_l}^S(\omega)$ is the cross-PSD of the base stationary process $F_k^S(t)$ and $F_l^S(t)$ that appear in the evolutionary process of $F_k(t) = A_k(\omega,t)F_k^S(t)$ and $F_l(t) = A_l(\omega,t)F_l^S(t)$

$$\Phi_{\mathbf{S}_k\mathbf{S}_l}^{(i,j)} = \mathrm{m}_{ik}(\omega,t)\mathrm{m}_{jl}^*(\omega,t)\Phi_{F_kF_l}^S(\omega)$$

where

$$\mathrm{m}_{ik}(\omega,t) = \int_0^t A_k(\omega,t-\tau)h_i(\tau)e^{-i\omega\tau}d\tau$$

◎ "Stationary" response to of MDOF system to (single) WN – important for CQC

Generic response (displacement, stress, internal forces, etc.)

$$y_p(t) = \sum_{i=1}^{n}a_{pi}s_i(t)$$

- the *p*-th element of $\mathbf{y} = \mathbf{As}(t) = \mathbf{Q\Phi\Gamma s}(t)$

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1) PSD of the stationary response:

$$\boxed{\Phi_{y_p y_q}(\omega) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{pi}a_{qj}\Phi_{s_i s_j}(\omega)}$$

where

$$\boxed{\Phi_{s_i s_j}(\omega) = \Phi_{FF}(\omega)H_i(\omega)H_j^*(\omega)}$$

For white noise $F(t)$, i.e. $\Phi_{FF}(\omega) = \Phi_0$

$\therefore \Phi_{s_i s_j}(\omega) = \Phi_0 H_i(\omega)H_j^*(\omega)$ and $H_i(\omega) = \frac{1}{\omega_i^2 - \omega^2 + 2i\xi_i\omega_i\omega}$

2) Cross-correlation functions of modal responses (needed to derive spectral moments $\lambda_{m,ij}$):

$$\begin{aligned} R_{ij}(\tau) &= \mathrm{E}\left[S_i(t+\tau)S_j(t)\right] \\ &= \int_{-\infty}^{\infty} \Phi_{S_i S_j}(\omega)e^{i\omega\tau}d\omega \\ &= \Phi_0 \int_{-\infty}^{\infty} H_i(\omega)H_j^*(\omega)e^{i\omega\tau}d\omega \\ &= \pi\Phi_0\left[\alpha_{ij}g_i(\tau) + \beta_{ij}h_i(\tau)\right], \qquad \tau > 0 \text{ (use } |\tau| \; for \; \forall\tau) \end{aligned}$$

where

- $h_i(t) = \frac{1}{\omega_{D_i}} e^{-\xi_i\omega_i t} \sin \omega_{D_i} t$ (for $t > 0$)
- $g_i(t) = e^{-\xi_i\omega_i t}\left[\cos \omega_{D_i} t + \frac{\xi_i}{\sqrt{1-\xi_i^2}} \sin \omega_{D_i} t\right]$
- $\alpha_{ij} = \frac{4(\omega_i\xi_i + \omega_j\xi_j)}{K_{ij}}$, $\beta_{ij} = \frac{2(\omega_j^2 - \omega_i^2)}{K_{ij}}$
- $K_{ij} = \left(\omega_i^2 - \omega_j^2\right)^2 + 4\omega_i\omega_j\left(\omega_i\xi_i + \omega_j\xi_j\right)\cdot\left(\omega_i\xi_j + \omega_j\xi_i\right)$

Using $R_{ij}(\tau)$, we can derive

$$\boxed{\lambda_{m,ij} = 2\int_0^{\infty} \omega^m \Phi_{S_i S_j}(\omega)d\omega}$$

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3) Zeroth order spectral moment:

$$\lambda_{0,ij} = \mathrm{E}\left[S_i(t)S_j(t)\right] = R_{ij}(0) = \pi\Phi_0\alpha_{ij} = \frac{4\pi\Phi_0\left(\omega_i\xi_i + \omega_j\xi_j\right)}{K_{ij}}$$

4) $\mathrm{E}\left[\dot{S}(t)S_j(t)\right] = \frac{d}{d\tau}R_{ij}(\tau)\Big|_{\tau=0} = -\mathrm{E}\left[S(t)\dot{S}_j(t)\right]$

5) 2nd order spectral moment:

$$-\frac{d^2}{d\tau^2}R_{ij}(\tau)\Bigg|_{\tau=0} = \mathrm{E}\left[\dot{S}(t)\dot{S}_j(t)\right] = \lambda_{2,ij} = \frac{4\pi\Phi_0\omega_i\omega_j\left(\omega_i\xi_j + \omega_j\xi_i\right)}{K_{ij}}$$

6) 1st order spectral moments

$$\lambda_{1,ij} = 2\int_0^\infty \omega^1\Phi_{S_iS_j}(\omega)\,d\omega$$

7) $\lambda_{0,ii}$, $\lambda_{2,ii}$, $\lambda_{1,ij}$, $\lambda_{1,ii}$: see the summary

8) Cross-modal correlation coefficient:

$$\rho_{m,ij} = \lambda_{m,ij}/\sqrt{\lambda_{m,ii}\lambda_{m,jj}}$$

See the summary for $\rho_{0,ij}$ and $\rho_{2,ij}$ ~ correlation between $S_i(t)$ and $S_j(t)$, and between $\dot{S}(t)$ and $\dot{S}_j(t)$, respectively

See the summary for $\rho_{1,ij}$

Summary

Spectral moments of stationary modal responses $s_i(t)$ and $s_j(t)$ to a white noise input whose power spectral density function is Φ_0.

$$\boxed{\begin{aligned}\lambda_{m,ij} &= 2\int_0^\infty \omega^m \Phi_{s_i s_j}(\omega)d\omega = 2\Phi_0\int_0^\infty \omega^m H_i(\omega)H_j^*(\omega)d\omega \\ \rho_{m,ij} &= \lambda_{m,ij}/\sqrt{\lambda_{m,ii}\lambda_{m,jj}}\end{aligned}}$$

Note: $K_{ij} = (\omega_i^2 - \omega_j^2)^2 + 4\omega_i\omega_j(\omega_i\zeta_i + \omega_j\zeta_j)(\omega_i\zeta_j + \omega_j\zeta_i)$

(1) $\lambda_{0,ij} = \mathrm{E}[s_i(t)\cdot s_j(t)] = \dfrac{4\pi\Phi_0(\omega_i\zeta_i + \omega_j\zeta_j)}{K_{ij}}$

(2) $\lambda_{2,ij} = \mathrm{E}[\dot{s}_i(t)\cdot \dot{s}_j(t)] = \dfrac{4\pi\Phi_0\omega_i\omega_j(\omega_i\zeta_j + \omega_j\zeta_i)}{K_{ij}}$

(3) $\lambda_{0,ii} = \mathrm{E}[s_i^2(t)] = \dfrac{\pi\Phi_0}{2\zeta_i\omega_i^3}$

(4) $\lambda_{2,ii} = \mathrm{E}[\dot{s}_i^2(t)] = \dfrac{\pi\Phi_0}{2\zeta_i\omega_i}$

(5) $$\lambda_{1,ij} = \frac{2\Phi_0}{K_{ij}}\left\{\begin{aligned}&\left[(\omega_i^2+\omega_j^2)\zeta_i + 2\omega_i\omega_j\zeta_j\right]\frac{1}{\sqrt{1-\zeta_i^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta_i^2}}{\zeta_i}\right) - (\omega_i^2-\omega_j^2)\ln\left(\frac{\omega_i}{\omega_j}\right)\\ &+\left[(\omega_i^2+\omega_j^2)\zeta_j + 2\omega_i\omega_j\zeta_i\right]\frac{1}{\sqrt{1-\zeta_j^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta_j^2}}{\zeta_j}\right)\end{aligned}\right\}$$

(6) $\lambda_{1,ii} = \dfrac{\pi\Phi_0}{2\zeta_i\omega_i^2}\dfrac{2}{\pi\sqrt{1-\zeta_i^2}}\tan^{-1}\left(\dfrac{\sqrt{1-\zeta_i^2}}{\zeta_i}\right)$

(7) $\rho_{0,ij} = [8\sqrt{\zeta_i\zeta_j\omega_i\omega_j}(\omega_i\zeta_i + \omega_j\zeta_j)\omega_i\omega_j]/K_{ij}$

(8) $\rho_{2,ij} = [8\sqrt{\zeta_i\zeta_j\omega_i\omega_j}(\omega_i\zeta_j + \omega_j\zeta_i)\omega_i\omega_j]/K_{ij}$

(9) In case $\zeta_i = \zeta_j = \zeta$,

$$\rho_{0,ij} = \rho_{2,ij} = \frac{8\zeta^2 r^{3/2}}{(1+r)[(1-r)^2 + 4\zeta^2 r]} \quad \text{in which } r = \omega_j/\omega_i$$

(10) $$\rho_{1,ij} \cong \frac{2\sqrt{\zeta_i\zeta_j}\cdot\left[(\omega_i+\omega_j)^2(\zeta_i+\zeta_j) - \frac{4}{\pi}(\omega_i-\omega_j)^2\right]}{4(\omega_i-\omega_j)^2 + (\zeta_i+\zeta_j)^2(\omega_i+\omega_j)^2}$$

457.643 Structural Random Vibrations
In-Class Material: Class 22

IV. Random Vibration Analysis of MDOF Systems (contd.)

◎ Spectral moments of MDOF generic response $y_p = \sum_{i=1}^{n} a_{pi} s_i(t)$

$$\lambda_m = \int_0^\infty \omega^m 2\Phi_{y_p y_p}(\omega) d\omega$$

Note

$$\Phi_{y_p y_p}(\omega) = \sum_i \sum_j a_{pi} a_{pj} \Phi_{s_i s_j}(\omega)$$

Thus,

$$\lambda_m = \sum_i \sum_j a_{pi} a_{pj} \int_0^\infty \omega^m 2\Phi_{s_i s_j}(\omega) d\omega$$

$$\boxed{\lambda_m = \sum_i \sum_j a_{pi} a_{pj} \qquad\qquad}$$

In words, the *m*-th order spectral moment of the generic response y_p can be obtained by the weighted sum of the *m*-th order (cross) spectral moments of the modal responses $s_i(t), i = 1, \ldots, n$

If WN approximation is made, the spectral moment is approximated as

$$\boxed{\lambda_m \cong \sum_i \sum_j a_{pi} a_{pj} \qquad\qquad}$$

Can use the closed-form formulas provided in the previous classnotes for $\lambda_{m,ij}^{WN}$

$$\lambda_{0,ij} = \mathrm{E}[s_i(t) \cdot s_j(t)] = \frac{4\pi\Phi_0(\omega_i\zeta_i + \omega_j\zeta_j)}{K_{ij}}, \quad \lambda_{2,ij} = \mathrm{E}[\dot{s}_i(t) \cdot \dot{s}_j(t)] = \frac{4\pi\Phi_0\omega_i\omega_j(\omega_i\zeta_j + \omega_j\zeta_i)}{K_{ij}}$$

$$\lambda_{1,ij} = \frac{2\Phi_0}{K_{ij}}\left\{\left[(\omega_i^2+\omega_j^2)\zeta_i + 2\omega_i\omega_j\zeta_j\right]\frac{1}{\sqrt{1-\zeta_i^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta_i^2}}{\zeta_i}\right) - (\omega_i^2-\omega_j^2)\ln\left(\frac{\omega_i}{\omega_j}\right) + \left[(\omega_i^2+\omega_j^2)\zeta_j + 2\omega_i\omega_j\zeta_i\right]\frac{1}{\sqrt{1-\zeta_j^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta_j^2}}{\zeta_j}\right)\right\}$$

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Example:

A frame structure with a light equipment attached (α 1) and small damping (ξ_1, ξ_2 1)

The ground acceleration process is assumed to be a white noise with the intensity Φ_0

Question: the mean square of the displacements $\mathrm{E}[x_1^2]$ and $\mathrm{E}[x_2^2]$

$$\mathbf{M} = \begin{bmatrix} & 0 \\ 0 & \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \alpha k & \\ & (1+\alpha)k \end{bmatrix}$$

E.O.M.

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{M}\mathbf{R}(-\ddot{x}_g)$$

Here $\mathbf{P} =$ and $\mathbf{F}(t) = F(t) =$

Eigenvalue analysis

$$|\mathbf{K} - \lambda\mathbf{M}| = 0$$

$$\begin{vmatrix} \alpha k - \lambda\alpha m & \\ & (1+\alpha)k - \lambda m \end{vmatrix} = 0$$

$$\lambda^2 - (2+\alpha)\frac{k}{m}\lambda + \frac{k^2}{m^2} = 0$$

$$\lambda = \left(1 + \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 + 4\alpha}\right)\frac{k}{m}$$

For small $\alpha \ll 1$,

$$\omega_1 \cong \left(1 - \frac{\sqrt{\alpha}}{2}\right)\sqrt{\frac{k}{m}}$$

$$\omega_2 \cong \left(1 + \frac{\sqrt{\alpha}}{2}\right)\sqrt{\frac{k}{m}}$$

The corresponding modal vectors are

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} \frac{1}{\sqrt{\alpha}} \\ 1 \end{Bmatrix}$$

$$\boldsymbol{\phi}_2 = \begin{Bmatrix} -\frac{1}{\sqrt{\alpha}} \\ 1 \end{Bmatrix}$$

Modal masses:

$$M_1 = \boldsymbol{\phi}_1^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_1 = 2m$$

$$M_2 = \boldsymbol{\phi}_2^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_2 = 2m$$

Modal participation factors:

$$\gamma_i = \frac{\boldsymbol{\phi}_i^{\mathrm{T}}\mathbf{P}}{M_i} = \frac{\boldsymbol{\phi}_i^{\mathrm{T}} \qquad}{M_i}$$

$$\gamma_1 = \frac{1+\sqrt{\alpha}}{2} \cong$$

$$\gamma_2 = \frac{1-\sqrt{\alpha}}{2} \cong$$

Effective modal participation factor:

$$\mathbf{A} = \mathbf{Q\Phi\Gamma} = \begin{bmatrix} \qquad & \qquad \\ & \end{bmatrix}\begin{bmatrix} \qquad & \qquad \\ & \end{bmatrix}\begin{bmatrix} \qquad & \qquad \\ & \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{\alpha}} & -\frac{1}{2\sqrt{\alpha}} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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Recall

$$\mathrm{E}[x_p^2] = \lambda_0^{(p)} = \sum_i \sum_j a_{pi} a_{pj} \lambda_{0,ij} = \sum_i \sum_j a_{pi} a_{pj} \rho_{0,ij} \sqrt{\lambda_{0,ii} \lambda_{0,jj}}$$

WN modal responses:

$$\lambda_{0,ii} = \frac{\pi \Phi_0}{2\xi_i \omega_i^3} \cong \frac{\pi \Phi_0}{2\xi \omega_0^3}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$

$$\rho_{0,12} = \frac{8\xi^2 r^{\frac{3}{2}}}{[(1-r)^2 + 4\xi^2 r](1+r)}$$

where $r = \omega_1/\omega_2$, and

$$1 - r = 1 - \frac{\frac{1-\sqrt{\alpha}}{2}}{1+\frac{\sqrt{\alpha}}{2}} = \frac{\sqrt{\alpha}}{1+\frac{\sqrt{\alpha}}{2}} \cong \sqrt{\alpha},\;\; 1 + r = 1 + \frac{\frac{1-\sqrt{\alpha}}{2}}{1+\frac{\sqrt{\alpha}}{2}} = \frac{2}{1+\frac{\sqrt{\alpha}}{2}} \cong 2$$

Therefore,

$$\rho_{0,12} \cong \frac{4\xi^2}{4\xi^2 + \alpha}$$

Finally, from $\mathbf{A} = \begin{bmatrix} \frac{1}{2\sqrt{\alpha}} & -\frac{1}{2\sqrt{\alpha}} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $\mathrm{E}[x_p^2] = \sum_i \sum_j a_{pi} a_{pj} \rho_{0,ij} \sqrt{\lambda_{0,ii} \lambda_{0,jj}}$, the mean square responses are finally derived as

$$\mathrm{E}[x_1^2] = \left(\frac{1}{2\sqrt{\alpha}}\right)^2 \lambda_{0,11} + 2\left(\frac{1}{2\sqrt{\alpha}}\right)\left(-\frac{1}{2\sqrt{\alpha}}\right) \rho_{0,12} \sqrt{\lambda_{0,11}\lambda_{0,22}} + \left(-\frac{1}{2\sqrt{\alpha}}\right)^2 \lambda_{0,22}$$
$$= \frac{\pi \Phi_0}{2\xi \omega_0^3} \cdot \frac{1}{2\alpha} \cdot \left(1 - \frac{4\xi^2}{4\xi^2 + \alpha}\right)$$

$$\mathrm{E}[x_2^2] = \frac{\pi \Phi_0}{2\xi \omega_0^3} \cdot \frac{1}{2} \cdot \left(1 + \frac{4\xi^2}{4\xi^2 + \alpha}\right)$$

For $\xi = 0.05$, the standard deviations of the displacements normalized by $\pi \Phi_0 / 2\xi \omega_0^3$ are

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	$\sigma_{x_1}/\sqrt{\frac{\pi\Phi_0}{2\xi\omega_0^3}}$			$\sigma_{x_2}/\sqrt{\frac{\pi\Phi_0}{2\xi\omega_0^3}}$		
	Exact	$\rho_{0,12}$ neglected	Error (%)	Exact	$\rho_{0,12}$ neglected	Error (%)
$\alpha = 0.01$	5	7.07	41	0.866	0.707	-18
$\alpha = 0.001$	6.71	22.4	233	0.975	0.707	-27

◎ Random vibration theory behind modal combination rules

Recall $y_r = \sum_{i=1}^{n} a_{ri} s_i$ and $(y_r^{max})^2 \cong (\sum a_{ri}\, s_i^{max})^2$ where

$y_r^{max} = \max\limits_{0<t\leq\tau} |y_r(t)|$ and $s_i^{max} = \max\limits_{0<t\leq\tau} |s_i(t)|$

1) Modal combination rules

SRSS (Rosenblueth 1951):

$$\boxed{y_r^{max} \cong \left(\sum_{i=1}^{n} a_{ri}^2 (s_i^{max})^2\right)^{1/2}}$$

CQC (Der Kiureghian 1981, EESD)

$$y_r^{max} \cong \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN}\, s_i^{max} s_j^{max}\right)^{\frac{1}{2}}$$

$$= \left[\sum_{i=1}^{n} (a_{ri})^2 (s_i^{max})^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN}\, s_i^{max} s_j^{max}\right]^{1/2}$$

2) Random vibration theory

Recall

$$\lambda_m = \sum_i \sum_j a_{ri} a_{rj} \rho_{m,ij} \sqrt{\lambda_{m,ii}\lambda_{m,jj}}$$

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For example, consider $m = 0$

$$\lambda_0 = \mathrm{E}[Y_r^2] = \sigma_{Y_r}^2 = \sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \sqrt{\lambda_{0,ii}\lambda_{0,jj}}$$

$$= \sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \sigma_{s_i} \sigma_{s_j}$$

That is,

$$\sigma_{Y_r} = \left(\sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \sigma_{s_i} \sigma_{s_j} \right)^{\frac{1}{2}}$$

Assume $\mathrm{E}[y_r^{max}] = p\sigma_{Y_r}$ and $\mathrm{E}[s_i^{max}] = p\sigma_{s_i}$, $\mathrm{E}\left[s_j^{max}\right] = p\sigma_{s_j}$

$$\frac{\mathrm{E}[y_r^{max}]}{p} = \left(\sum_i \sum_j a_{ri} a_{rj} \rho_{0,ij} \frac{\mathrm{E}[s_i^{max}]}{p} \frac{\mathrm{E}\left[s_j^{max}\right]}{p} \right)^{\frac{1}{2}}$$

Inspired by this, CQC rule is proposed as

$$y_r^{max} \cong \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN} s_i^{max} s_j^{max} \right)^{\frac{1}{2}}$$

This actually means

$$\mathrm{E}[y_r^{max}] \cong \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ri} a_{rj} \rho_{0,ij}^{WN} \mathrm{E}[s_i^{max}] \mathrm{E}\left[s_j^{max}\right] \right)^{\frac{1}{2}}$$

Herein $\mathrm{E}[s_i^{max}]$ and $\mathrm{E}\left[s_j^{max}\right]$ are obtained from ___________ spectrum.

3) SRSS works well when modal frequencies are well-separated, say $r = \frac{\omega_i}{\omega_j} < \frac{0.2}{0.2+\xi_i+\xi\_j}$ $(\omega_j > \omega_i)$ because $\rho_{0,ij} \cong 0.10$

4) Approximation introduced in CQC for practicality

$$\rho_{0,ij} \cong \rho_{0,ij}^{WN}$$

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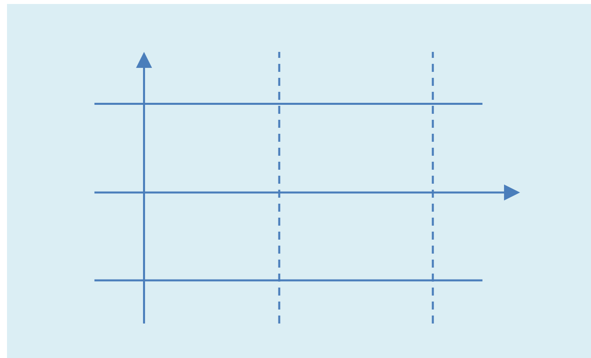
457.643 Structural Random Vibrations
In-Class Material: Class 23

V. Crossings & Failure Analysis

◎ Failure probabilities

1) Instantaneous failure probability

$P(|X(t)| > a)$ or $P(X(t) > a)$



e.g. Gaussian with $\mu_X(t)$ and $\sigma_X(t)$

$X(t) \sim N(\mu_X(t), \sigma_X^2(t))$

$$P(X(t) > a) = 1 - F_{X(t)}(a)$$
$$= 1 - \Phi\left(\frac{\quad}{\quad}\right)$$

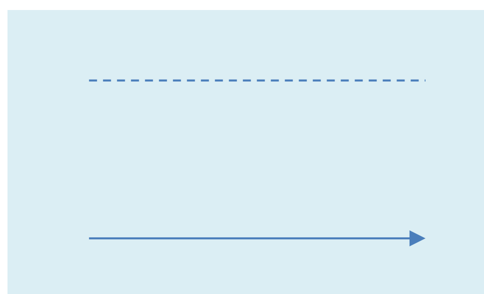
2) First-passage failure probability

$$P\left(\max_{0<t\leq\tau} |X(t)| > a\right) = P(at\ least \qquad crossing\ in\ (0, \tau])$$

can be estimated by checking the probability distribution of _______ values, or

by deriving from __________ rates and other characteristics

3) Accumulated damage



e.g. Fatigue damage index
(L&S 11.8~11.11, 12.9)

$D(t)$: damage measure (counts)

◎ Crossing statistics

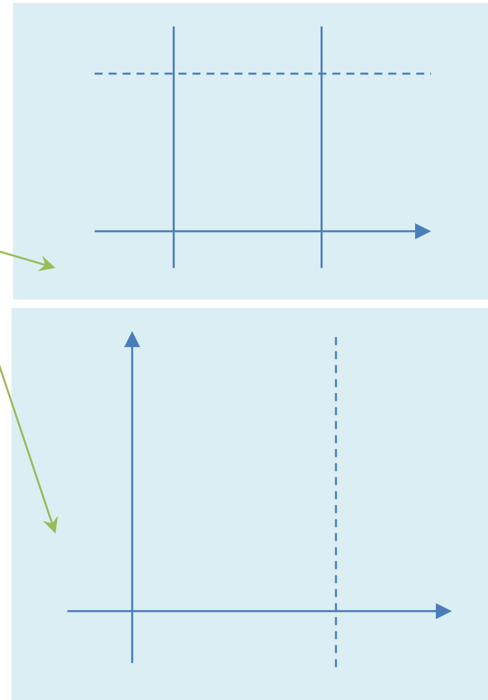
1) $N^+(a; t)$: Number of upcrossings of level a in $(0, t)$

$p^+(a; t)$: Probability of an uncrossing of level a in $(t, t + dt]$

Upcrossing event at $(t, t+dt]$

Conditions:

- $X(t) \quad a$
- $\dot{X}(t) \quad 0$
- $X(t+dt) \cong X(t) + \dot{X}(t)dt \qquad a$



Therefore,

$$p^+(a;t) = P[\{ \qquad < X(t) < \quad \}$$
$$\cap \{\dot{X}(t) \quad 0\}]$$
$$= \int_0^\infty \int f_{X\dot{X}}(x, \dot{x}; t)\, dx d\dot{x}$$
$$= \int_0^\infty f_{X\dot{X}}(a, \dot{x}; t) \dot{x} dt d\dot{x}$$
$$= dt \int_0^\infty \dot{x} f_{X\dot{X}}(a, \dot{x}; t) d\dot{x}$$

For the bottom figure, the third condition is interpreted as $a - \dot{x}dt < x$ and thus $\dot{x} > -\frac{1}{dt}x + \frac{1}{dt}a$

2) $dN^+(a;t) (= \frac{\partial N^+(a;t)}{\partial t} dt)$: Number of crossings in $(t, t+dt]$

$$\mathrm{E}[dN^+(a;t)] = 0 \times P(0 \text{ crossings}) + 1 \times P(1 \text{ crossing}) + 2 \times P(2 \text{ crossings}) + \cdots$$
$$\cong P(1 \text{ crossing in } (t, t+dt])$$
$$= p^+(a;t)$$
$$= dt \int_0^\infty \dot{x} f_{X\dot{X}}(a, \dot{x}; t) d\dot{x}$$

3) Average number of upcrossings in $(t, t+dt]$, i.e. "mean upcrossing rate"

$$\boxed{\nu^+(a;t) = \mathrm{E}\left[\frac{dN^+(a;t)}{dt}\right] = \int_0^\infty \dot{x} f_{X\dot{X}}(a, \dot{x}; t) d\dot{x}}$$

S.O. Rice (1944; 1945) → "Rice formula"

Downcrossing rate?

$$\nu^-(a;t) = -\int_{-\infty}^{0} \dot{x} f_{X\dot{X}}(a,\dot{x};t)d\dot{x} = \int_{-\infty}^{0} |\dot{x}| f_{X\dot{X}}(a,\dot{x};t)d\dot{x}$$

All crossings?

$$\begin{aligned} \nu(a;t) &= \nu^+(a;t) + \nu^-(a;t) \\ &= \int_{-\infty}^{\infty} |\dot{x}| f_{X\dot{X}}(a,\dot{x};t)d\dot{x} \end{aligned}$$

- More rigorous derivation available in L&S (p. 265)

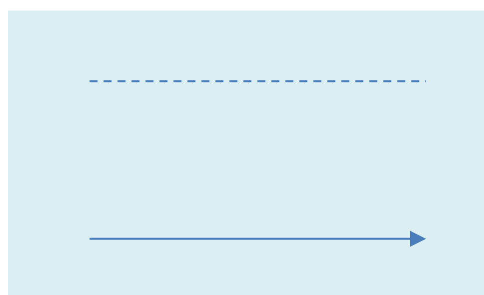
4) Mean number of crossing in $(t_1, t_2]$

$$\mathrm{E}[N(a;t_2) - N(a;t_1)] = \int_{t_1}^{t_2} \nu(a;t)dt$$

5) If $X(t)$ is stationary,

- $f_{X\dot{X}}(x,\dot{x};t) \rightarrow f_{X\dot{X}}(x,\dot{x})$ (if zero-mean Gaussian, $f_X(x) \cdot f_{\dot{X}}(\dot{x})$)
- $\nu(a;t) \rightarrow \nu(a)$
- $\mathrm{E}[N(a;t_2) - N(a;t_1)] \rightarrow \nu(a) \cdot (t_2 - t_1)$

6) Relationship between crossing rate and peak distribution (approximation for narrow-band processes)



If $X(t)$ is stationary narrow-band process, almost every upcrossings over μ is associated with one and only one peak, then…

$$\begin{aligned} P(\text{a randomly selected peak} > a) &\cong \frac{\nu^+(a)}{\nu^+(\mu)} \\ &\cong 1 - F_p(a) \end{aligned}$$

where $F_p(\cdot)$ is the CDF of a local peak

PDF $f_p(a) \cong -\frac{1}{\nu^+(\mu)} \cdot \frac{d\nu^+(a)}{da}$

Example: A stationary Gaussian process with zero-mean

$$
\begin{aligned}
f_{X\dot{X}}(x,\dot{x}) &= f_X(x)\cdot f_{\dot{X}}(\dot{x}) \\
&= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left\{-\frac{1}{2}\left[\left(\frac{x}{\sigma^2}\right)^2+\left(\frac{\dot{x}}{\sigma_{\dot{X}}^2}\right)^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\nu^+(a) &= \int_0^\infty \dot{x} f_{X\dot{X}}(a,\dot{x})d\dot{x} \\
&= \int_0^\infty \dot{x}\frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left\{-\frac{1}{2}\left[\left(\frac{a}{\sigma_X^2}\right)^2+\left(\frac{\dot{x}}{\sigma_{\dot{X}}^2}\right)^2\right]\right\}d\dot{x} \\
&= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}}\exp\left(-\frac{a^2}{2\sigma_X^2}\right)\int_0^\infty \dot{x}\exp\left(-\frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right)d\dot{x}
\end{aligned}
$$

One can show that $\int_0^\infty \dot{x}\exp\left(-\frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right)d\dot{x} = \sigma_{\dot{X}}^2$ (hint: change variable $\dot{x}^2 \to t$)

Therefore,

$$
\boxed{
\begin{aligned}
\nu^+(a) &= \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X}\exp\left(-\frac{a^2}{2\sigma_X^2}\right) \\
&= \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\sigma_X^2}\right)
\end{aligned}
}
$$

Some notable results:

- $\nu^-(a) =$
- $\nu(a) =$
- $\nu^+(0) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}$
- $\sqrt{\frac{\lambda_2}{\lambda_0}} = 2\pi\nu^+(0)$: circular apparent frequency

  e.g. WN response: $\lambda_2 = \frac{\pi\Phi_0}{2\xi\omega_0}$ and $\lambda_0 = \frac{\pi\Phi_0}{2\xi\omega_0^3}$ → $\sqrt{\frac{\lambda_2}{\lambda_0}} = \omega_0$
- NB approximation for local peak distribution: $f_p(a) \cong -\frac{1}{\nu^+(0)}\cdot\frac{d\nu^+(a)}{da} = \frac{a}{\lambda_0}\exp\left(-\frac{a^2}{2\lambda_0}\right)$

  → "Rayleigh" distribution

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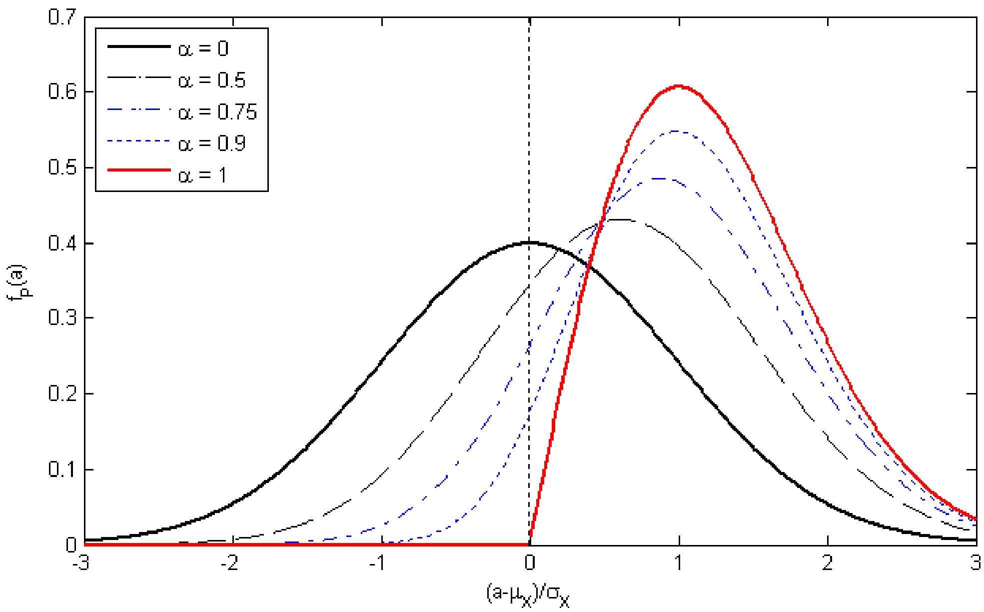
◎ **Distribution of local peaks (NOT NB approximation; L&S pp. 488-490)**

$$F_p(a;t) = \frac{\int_{-\infty}^{0}\int_{-\infty}^{a}|\ddot{x}| f_{X\dot{X}\ddot{X}}(x,0,\ddot{x};t)dx d\ddot{x}}{\int_{-\infty}^{0}|\ddot{x}| f_{\dot{X}\ddot{X}}(0,\ddot{x};t)d\ddot{x}}, \quad f_p(a;t) = \frac{dF_p(a;t)}{da} = \frac{\int_{-\infty}^{0}|\ddot{x}| f_{X\dot{X}\ddot{X}}(a,0,\ddot{x};t)d\ddot{x}}{\int_{-\infty}^{0}|\ddot{x}| f_{\dot{X}\ddot{X}}(0,\ddot{x};t)d\ddot{x}}$$

Example: The PDF and CDF of the local peaks of a stationary Gaussian process $X(t)$:
(Rice distribution; Ex 11.1 in L&S)

$$f_P(a) = \frac{\sqrt{1-\alpha^2}}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{(a-\mu_X)^2}{2(1-\alpha^2)\sigma_X^2}\right] + \frac{\alpha(a-\mu_X)}{\sigma_X^2}\exp\left[-\frac{(a-\mu_X)^2}{2\sigma_X^2}\right]\Phi\left[\frac{\alpha(a-\mu_X)}{\sqrt{1-\alpha^2}\sigma_X}\right]$$

$$F_P(a) = \Phi\left(\frac{a-\mu_X}{\sqrt{1-\alpha^2}\sigma_X}\right) - \alpha\exp\left[-\frac{(a-\mu_X)^2}{2\sigma_X^2}\right]\Phi\left[\frac{\alpha(a-\mu_X)}{\sqrt{1-\alpha^2}\sigma_X}\right]$$



Note

(1) $\alpha = 0$: wide-band $\quad f_P(a) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{(a-\mu_X)^2}{2\sigma_X^2}\right]$ (Gaussian)

(2) $\alpha = 1$: narrow-band $\quad f_P(a) = \frac{(a-\mu_X)}{\sigma_X^2}\exp\left[-\frac{(a-\mu_X)^2}{2\sigma_X^2}\right]$ (Rayleigh)

(3) The average fraction of local peaks below the mean value.

$$F_P(\mu_X) = \frac{1-\alpha}{2}$$

0.5 for $\alpha = 0$ (Gaussian) and 0 for $\alpha = 1$ (Rayleigh)

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※ How was it derived?

- $f_{X\dot{X}\ddot{X}}(x,\dot{x},\ddot{x}) = f_{X\ddot{X}}(x,\ddot{x}) \cdot f_{\dot{X}}(\dot{x})$ ($\because$ stationary and Gaussian)

- $\rho_{X\ddot{X}}$? Note $\mathrm{COV}[X,\ddot{X}] = -\lambda_2$

  $\because \Phi_{X\ddot{X}}(\omega) = (-i\omega)^2\Phi_{XX}(\omega) = -\omega^2\Phi_{XX}(\omega) = -\Phi_{\dot{X}\dot{X}}(\omega)$

  $$\therefore \rho_{X\ddot{X}} = -\frac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} = -\alpha$$

- $\alpha = \frac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} = \frac{\nu_X^+(0)}{\nu_{\dot{X}}^+(0)} = \frac{\frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}}{\frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}}}$

Note: α is another measure of the bandwidth (cf. $0 < s < \infty$ and $0 < \delta < 1$)

- $0 < \alpha < 1$

- $\alpha \cong 0$: $\nu_{\dot{X}}^+(0) \gg \nu_X^+(0)$ wide-band process

- $\alpha \cong 1$: $\nu_{\dot{X}}^+(0) \cong \nu_X^+(0)$ narrow-band process

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457.643 Structural Random Vibrations
In-Class Material: Class 24

V. Crossings & Failure Analysis (contd.)

◎ (Upper) bound on first-passage probability using crossing rate

$$P(\text{at least one failure in } (0,t]) = \sum_{i=1}^{\infty} P(i \text{ crossing(s) in } (0,t])$$

Note

$$\int_0^t \nu(a;t)dt = \mathrm{E}[N(a;t)]$$
$$= \text{mean no. of crossings in } (0,t]$$
$$= \sum_{i=1}^{\infty} i \cdot P(i \text{ crossing(s) in } (0,t])$$

$$\therefore P(\text{at least one failure}) \leq \int_0^t \nu(a;t)dt$$

This approximation works well when crossing events are rare, but may not work if it is a narrow-band process (because if there is crossing, multiple crossings can occur).

◎ Probability distribution of "global" peak and first-passage probability

$X_\tau = \max_{0\leq t\leq\tau} X(t)$ (cf. $|X(t)|$ ~ two-sided)

Relationship between first-passage probability and CDF of the global peak:

$$p_X(a;\tau) = \quad - F_{X_\tau}(a)$$

$$F_{X_\tau}(a) = P(X(0) \leq a \cap \qquad \text{upcrossings above level } a \text{ in } (0,t])$$
$$\cong F_X(a;0) \cdot P(\qquad \text{upcrossings above level } a \text{ in } (0,\mathrm{t}])$$

Two methods to obtain the probability of __________ upcrossings:

- Poisson assumption
- Vanmarcke's formula (Prof. Erik Vanmarcke)

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◎ First-passage probability by Poisson assumption

In this approach, it is assumed that upcrossing events form a Poisson process.

This approach works relatively well if the threshold value a is ________ or the process is a _____-band process (because correlation between crossing events is _____ in these cases).

$$P(x \text{ crossing(s) in } (0,\tau]) = \frac{m(t)^x}{x!}\exp[-m(\tau)]$$

$$\therefore P(0 \text{ crossings in } (0,\tau]) = \exp[-m(\tau)]$$

$$= \exp\left[-\int_0^{\tau} \qquad dt\right]$$

Therefore, the first-passage probability by Poisson assumption is

$$\boxed{p_X(a;\tau) = 1 - F_X(a;0)\cdot\exp\left[-\int_0^{\tau} \nu^+(a;t)dt\right]}$$

Note: the first-passage probability takes the form $1 - A\cdot L_X(a;\tau) = 1 - A\cdot\exp\left(-\int_0^{\tau}\alpha(a;t)dt\right)$. The approach by Vanmarcke aims to improve the accuracy of A and $\alpha(a;t)$.

Example: Stationary Gaussian process with zero-mean

$$\nu^+(a) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\lambda_0}\right)$$

$$F_X(a;0) = P(X \le a) = \Phi\left(\frac{a}{\sigma_X}\right) = \Phi(r)$$

Thus,

$$\boxed{p_X(a;\tau) = 1 - \Phi\left(\frac{a}{\sigma_X}\right)\cdot\exp\left[-\frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\lambda_0}\right)\cdot\tau\right]}$$

Note: For two-sided crossing, $F_{|X|}(a;0) = 1 - 2\Phi(-r)$ and $2\nu_X^+(a)$ are used instead.

Furthermore, from the CDF of the global peak, $F_{X_\tau}(a) = \exp\left[-\frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\lambda_0}\right)\cdot\tau\right]$,

Davenport (1964) derived the relationship between the statistics of the global peak (μ_{X_τ} and σ_{X_τ}) and the standard deviation of the process $X(t)$ as follows:

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$\mu_{X_\tau} = p\sigma_X$ and $\sigma_{X_\tau} = q\sigma_X$

The so-called "peak factors" were derived as

$$p = \sqrt{2\ln[\nu_X^+(0)\tau]} + \frac{0.5772}{\sqrt{2\ln[\nu_X^+(0)\tau]}}$$

$$q = \frac{\pi}{\sqrt{6}}\frac{1}{\sqrt{2\ln[\nu_X^+(0)\tau]}}$$

Note:

- For the two-sided peak, replace $\nu_X^+(0)$ by $\nu_X(0) = \nu_X^+(0) + \nu_X^-(0)$
- These peak factors work relatively well for wide-band processes and high thresholds because the CDF was derived based on ____________ assumption.
- Der Kiureghian (1980) proposed improved versions that work for general cases based on Vanmarcke's formula (discussed later)

◎ **First-passage probability by Vanmarcke (1975)**

Recall, the first-passage probability was derived in the form

$$p_X(a;\tau) = 1 - A\cdot\exp\left(-\int_0^\tau \alpha(a;t)dt\right) = 1 - A\cdot L_X(a;t)$$

where A denotes the probability of the "safe start" and $L_X(a;t) = \exp\left(-\int_0^\tau \alpha(a;t)dt\right)$ represents the conditional probability of the first-passage failure given "safe start"

When the first-passage probability is described as above, one can show that $\alpha(a;t)$ is interpreted as (See L&S)

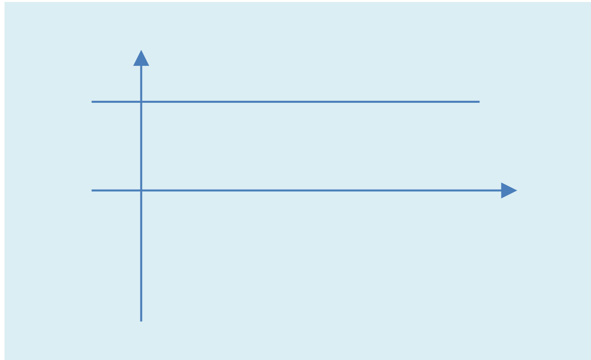
$$\alpha(a;t) = \lim_{\Delta t\to 0}\frac{E[\text{No. of crossings in } (t, t+\Delta t)|\text{no prior crossings up to } t]}{\Delta t}$$

In words, $\alpha(a;t)$ in the formulation above should be "   " mean crossing rate given _______________

※ In the Poisson assumption based approach, $\alpha(a;t)$ is approximated by __________, which is "   " mean crossing rate. This means the Poisson approach neglects __________

___________ between crossing events. This is why the approach works well when the threshold is high and the process is ________-band.

Vanmarcke (1975) took into account the statistical dependence between the crossing events by introducing the envelope process and the "clump" size, i.e. the average number of crossings of the original process per a crossing of the envelope process.



For example, the clump size of a stationary Gausssian process with zero-mean is

$$\mathrm{E}[\mathrm{CS}] = \frac{1}{1 - \exp\left(-\sqrt{2\pi}\delta^{1.2} r\right)}$$

where δ is the bandwidth parameter and $r = a/\sigma_X$ is the normalized threshold.

- $\delta \cong 0$ (narrow band): $\mathrm{E}[\mathrm{CS}]$ large (envelope crossing → many process crossings)
- $\delta \cong 1$ (wide band): $\mathrm{E}[\mathrm{CS}] \cong 1$ (one crossing per one envelope crossing)

As a result,

$$p_X(a;\tau) = 1 - B \cdot \exp\left(-\int_0^{\tau} \eta^+(a;t)dt\right)$$

$$\boxed{\begin{aligned} B &= P(E(0) < a) = \int_0^a f_E(e;0)de \\ \eta^+(a;t) &= \frac{P(E(t) \ge a)\nu_X^+(0;t)}{P(E(t) < a)}\left[1 - \exp\left(\frac{-\nu_E^+(a;t)}{P(E(t) \ge a)\nu_X^+(0;t)}\right)\right] \end{aligned}}$$

For a stationary Gaussian process with zero-mean, using the envelope process by Cramer and Leadbetter (1967), the first-passage probability is expressed using

$$\boxed{\begin{aligned} B &= 1 - \exp(-r^2/2) \\ \eta_X^+(a;t) &= \nu_X^+(a;t)\frac{1 - \exp\left(-\sqrt{2\pi}\delta^{1.2} r\right)}{1 - \exp(-r^2/2)} \end{aligned}}$$

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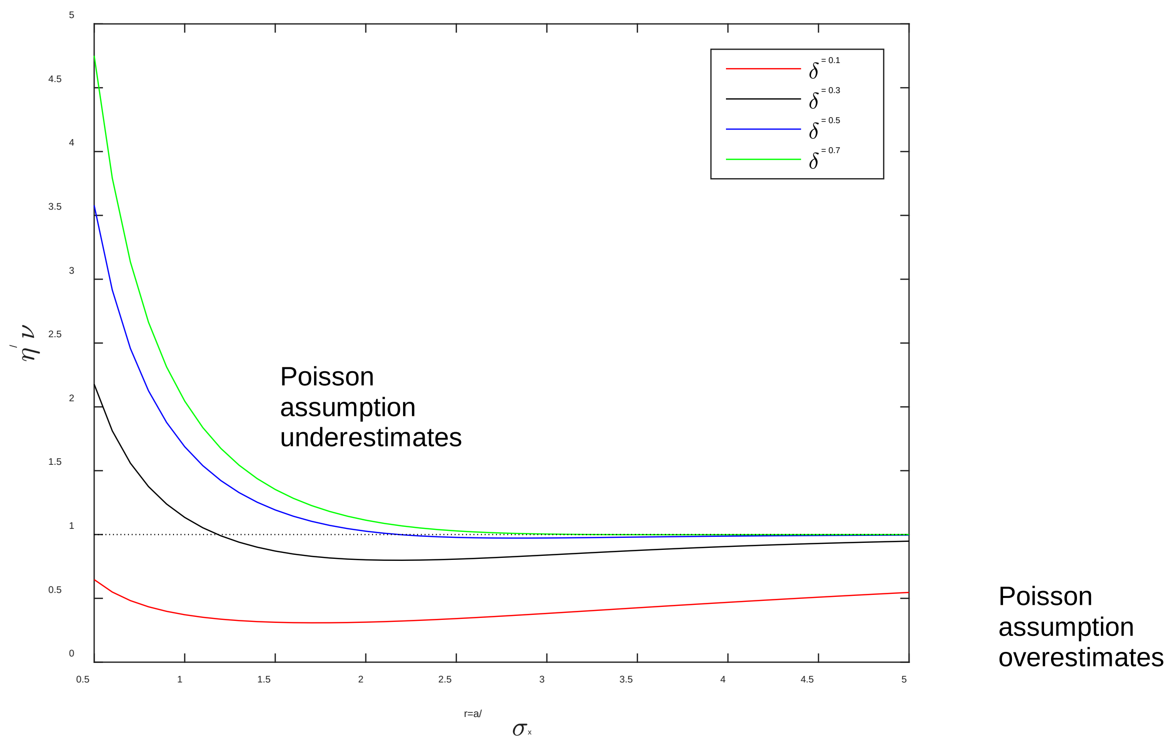
Note: For two-sided crossings, use $\nu_X(a;t)$ instead of $\nu_X^+(a;t)$, and $\sqrt{\pi/2}$ instead of $\sqrt{2\pi}$

※ $\eta_X^+(a)/\nu_X^+(a)$ for a stationary Gaussian process with zero-mean:



※ See Figure 4(a) in Song and Der Kiureghian (2006) ($\delta = 0.26$)

To account for the effect of the statistical dependence between crossing events, Der Kiureghian (1980) derived peak factors based on Vanmarcke's formula (for two-sided peak):

$$p = 1.253 + 0.209\nu_e\tau \qquad 0 < \nu_e\tau \le 2.1$$

$$= \sqrt{2\ln(\nu_e\tau)} + \frac{0.5772}{\sqrt{2\ln(\nu_e\tau)}} \qquad 2.1 < \nu_e\tau$$

$$q = 0.658 \quad 0 < \nu_e\tau \le 2.1$$

$$= \frac{1.20}{\sqrt{2\ln(\nu_e\tau)}} - \frac{5.40}{13 + [2\ln(\nu_e\tau)]^{3.2}} \qquad 2.1 < \nu_e\tau$$

where $\nu_e = 2\delta\nu_X(0) \quad 0 < \delta \le 0.1$

$$= (1.63\delta^{0.45} - 0.38)\nu_X(0) \quad 0.1 < \delta \le 0.69$$

$$= \nu_X(0) \quad 0.69 < \delta < 1$$

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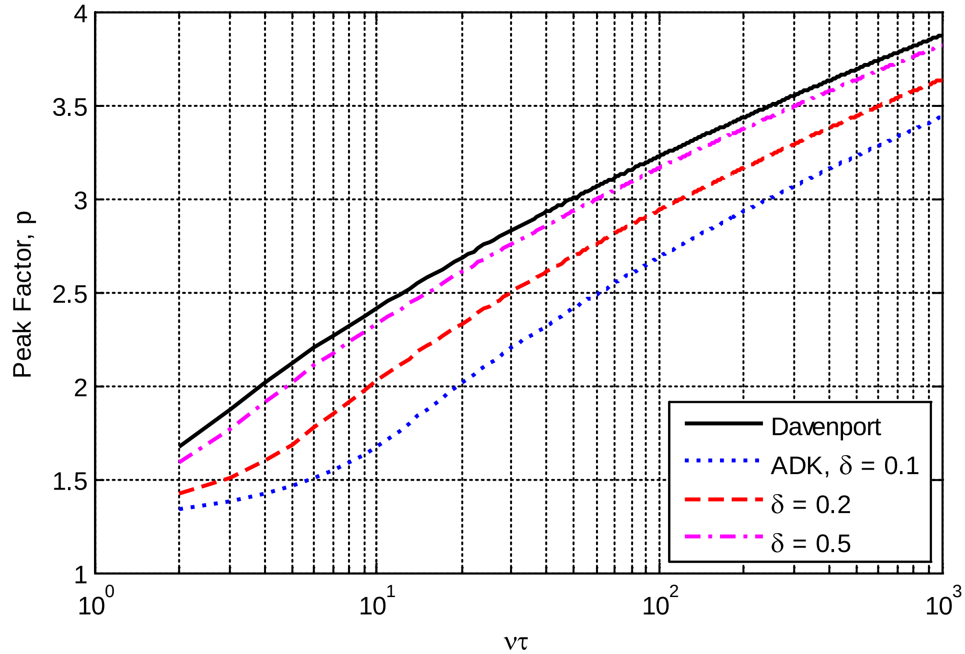
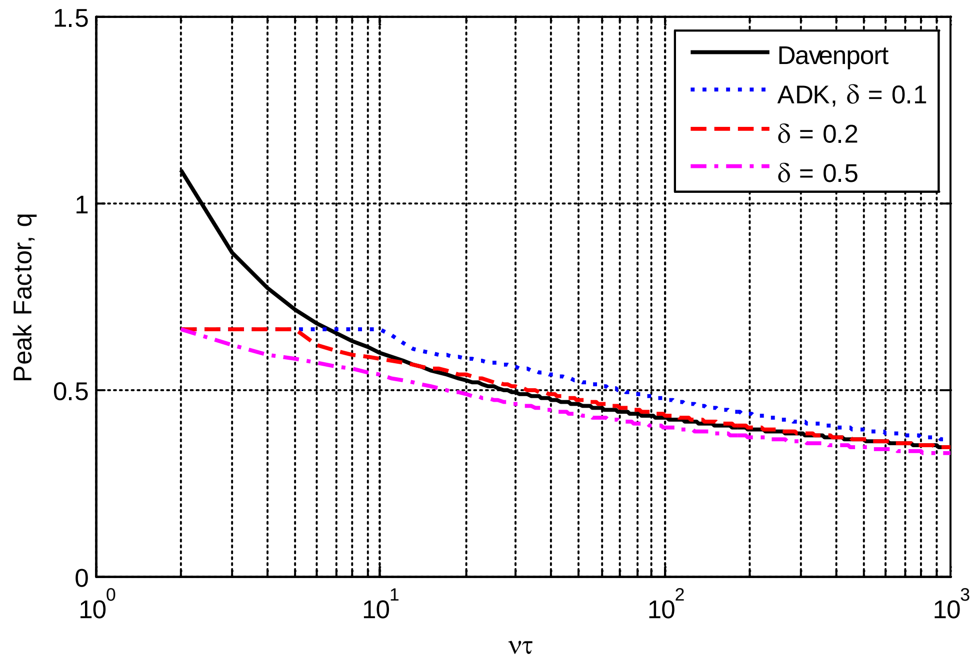
For the one-sided peak, replace $\nu_X(0)$ by $\nu_X^+(0)$, and δ by 2δ.

Example: two-sided peak factors for stationary Gaussian with zero-mean





Note: When $\nu\tau = 10 \times 20 = 200$ (a rough upperbound for typical earthquake responses), $p = 2.93 \sim 3.43$ and $q = 0.37 \sim 0.43$

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◎ First-passage probability concept to multiple stochastic processes

$$P\left(\max_{0\le t<\tau} X_1(t) > a_1 \cap \max_{0\le t<\tau} X_2(t) > a_2\right)?$$

Song, J., and A. Der Kiureghian (2006). Joint first-passage probability and reliability of systems under stochastic excitation. Journal of Engineering Mechanics. ASCE, 132(1), 65-77.

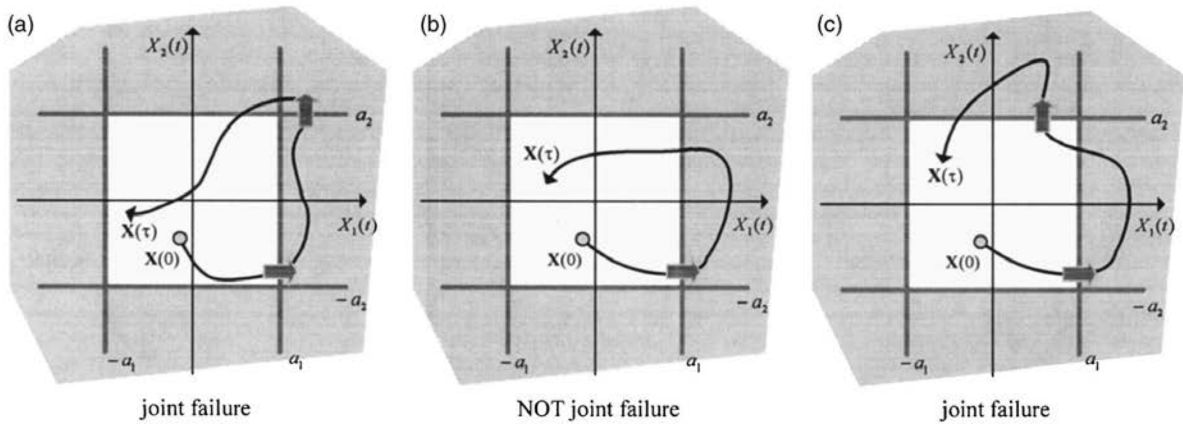


Fig. 1. Trajectories of a vector process and relation to the joint failure event

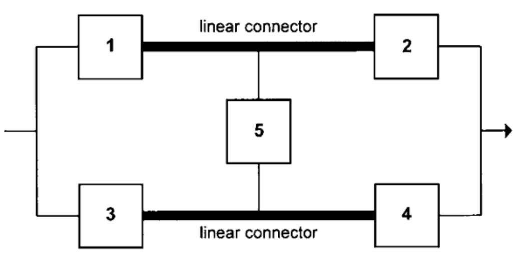


Fig. 8. System with five interconnected equipment items

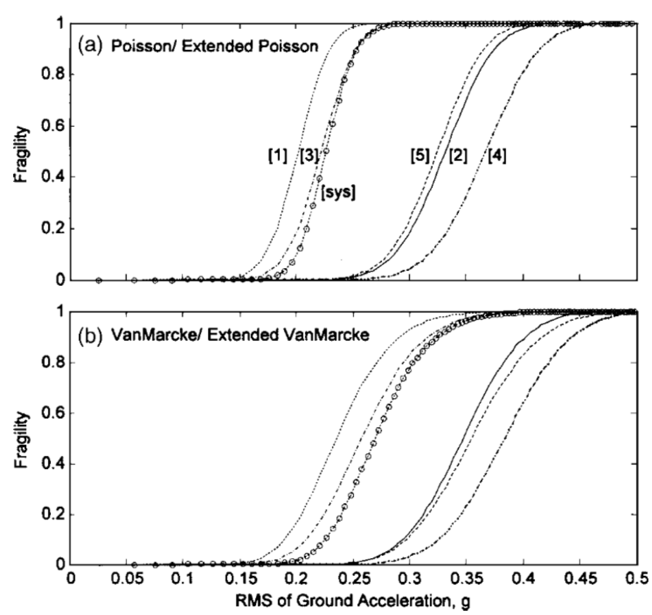


Fig. 9. Equipment and system fragility estimates by (a) extended Poisson approximation and (b) extended VanMarcke approximation

457.643 Structural Random Vibrations
In-Class Material: Class 25

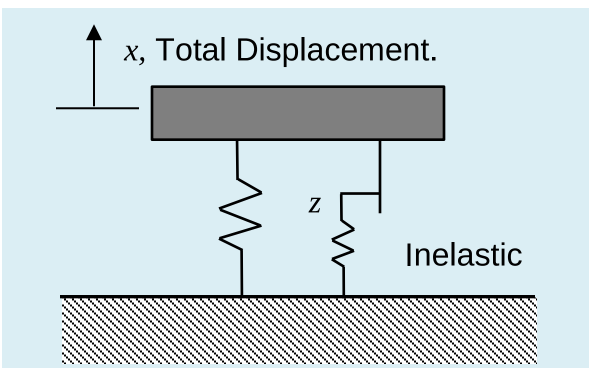
VI. Introduction to Nonlinear Random Vibration Analysis

◎ (Differential equation based) hysteretic constitutive models in structural dynamics

"Hyteresis"

- Origin: ferromagnetic materials
- Memory-based multi-valued relation between an input signal & output (generally only "rate-independent" relationship (viscous materials X)

Mechanical model for description by differential equation based hysteresis model



z: Auxiliary variable representing inelastic behavior ("internal variable" – Capecchi & de Felice 2001, JEM) ~ displacement of inelastic spring

- $z = x$: no slide
- $z = 0$: slide

(nonlinearity determined by difference between z and x)

Resisting force:

$$\boxed{f_s(x,z) = \alpha k_0 x + (1-\alpha) k_0 z}$$

- α: post-to-pre-yield stiffness ratio
  - ✓ $\alpha = 0$: perfect plastic
  - ✓ $\alpha = 1$: linear elastic
- k_0: initial stiffness

Evolution of z follows a nonlinear differential equation

$$\boxed{\dot{z} = \dot{x} \cdot h(x, \dot{x}, z)}$$

Meaning of the nonlinear function $h(\cdot)$?

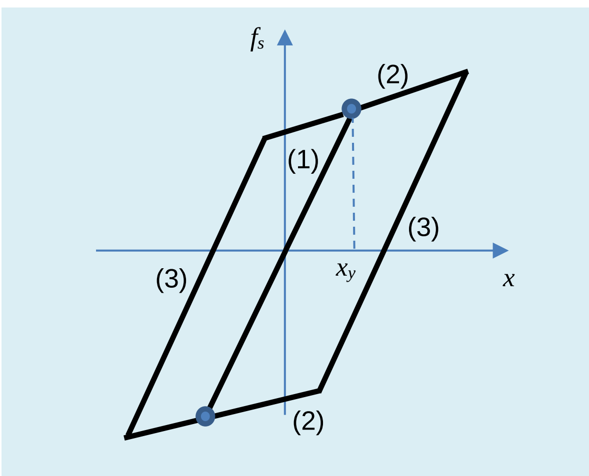
$$\frac{dz}{dt} = \frac{dx}{dt} \cdot h(\cdot)$$

Therefore,

$h(\cdot)\left(=\frac{dz}{dx}\right)$ determines the slope of z with respect to x at a given time.

◎ Bilinear model (Kaul & Penzien 1974 JEMD; Asano & Iwan 1984 EESD)

Main idea: describe inelastic spring in the mechanical model by Coulomb slider (i.e. no slide until it reaches the yield displacement)



(1) $-x_y < z < x_y$:

the Coulomb slider does not slide, i.e.

$z = x$ and $\dot{z} = \dot{x}$

$f_s(x, z) = \alpha k_0 x + (1 - \alpha) k_0 x = k_0 x$ (linear)

(2) $z > x_y, \dot{x} > 0$ or $z < -x_y, \dot{x} < 0$:

Coulomb slider slides (i.e. $\dot{z} = 0$)

(3) $z > x_y, \dot{x} < 0$ or $z < -x_y, \dot{x} > 0$:

Coulomb slider stops sliding $\dot{z} = \dot{x}$

Differential-equation model by Kaul & Penzien (1974):

$$\dot{z} = \dot{x} \cdot \{U(z + x_y) - U(z - x_y) + U(z - x_y) \cdot U(-\dot{x}) + U(-z - x_y) \cdot U(\dot{x})\}$$

where $U(\cdot)$ denotes the step function.

How to solve the nonlinear system differential equation, i.e.

E.O.M. with $f_s = \alpha k_0 x + (1 - \alpha) k_0 z$ plus $\dot{z} = \dot{x} \cdot h(x, \dot{x}, z)$

e.g. Runge-Kutta method (after transforming to state-space formulation $\dot{\mathbf{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$

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◎ Bouc-Wen class model

Bouc (1967) first proposed and Wen (1976) modified to the form

$$\boxed{\dot{z} = \dot{x} \cdot [A - |z|^n \psi(x, \dot{x}, z)]}$$

where

- A: scale of hysteresis loop
- n: smoothness of transition from pre-yielding to post-yielding
- $\psi(x, \dot{x}, z)$: "shape-control" function

Reviews are available in Song & ADK (2006, JEM), and Ismail et al. (2009, Archi. Comp. Meth. Engrg.)

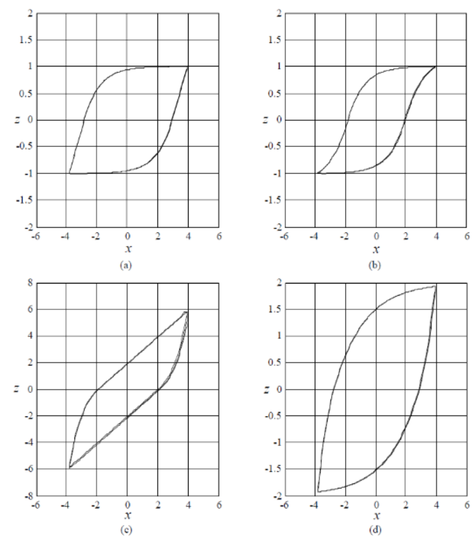

Figure 3.3 Hysteresis loops by Bouc-Wen model ($A=1, n=1$) (a) $\gamma=0.5$, $\beta=0.5$, (b) $\gamma=0.1$, $\beta=0.9$, (c) $\gamma=0.5$, $\beta=-0.5$ and (d) $\gamma=0.75$, $\beta=-0.25$

1) Bouc (1967, 1971)
   - $n = 1$
   - $\psi(x, \dot{x}, z) = \gamma + \beta \mathrm{sgn}(\dot{x}z)$
2) Wen (1976)
   - n: generalized
   - $\psi(x, \dot{x}, z) = \gamma + \beta \mathrm{sgn}(\dot{x}z)$

   The parameters γ and β in the "shape-control" function determine the shapes of the hysteresis loops (Song and ADK 2006)

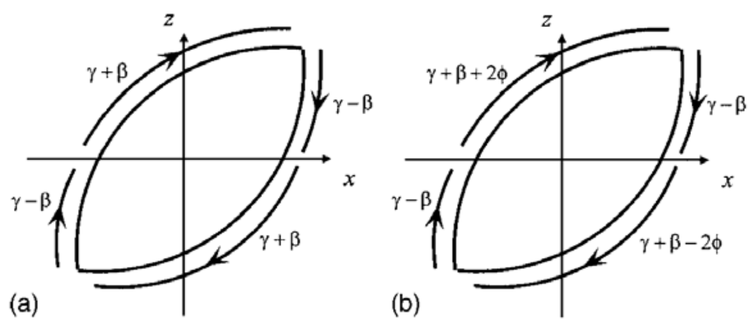

Fig. 2. Values of shape-control function for: (a) original Bouc–Wen model; and (b) model by Wang and Wen

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3) Baber & Wen (1981): Considered the degradation effect by making the model parameters functions of ϵ, "the dissipated energy"

4) Baber & Noori (1984): Introduce additional parameters to describe "pinching" effect

5) Wang & Wen (1998): Aim to describe "asymmetric" shape by adding additional terms

$$\psi(x,\dot{x},z) = \gamma + \beta\mathrm{sgn}(\dot{x}z) + \phi[\mathrm{sgn}(\dot{x}) + \mathrm{sgn}(z)]$$

➔ Added more DOFs (see the figure above)

6) Generalized Bouc-Wen (Song & ADK, 2006)

Generalize the "shape-control" function to describe highly asymmetric behavior

$$\psi(x,\dot{x},z) = \beta_1\mathrm{sgn}(\dot{x}z) + \beta_2\mathrm{sgn}(x\dot{x}) + \beta_3\mathrm{sgn}(xz) + \beta_4\mathrm{sgn}(\dot{x}) + \beta_5\mathrm{sgn}(z) + \beta_6\mathrm{sgn}(x)$$

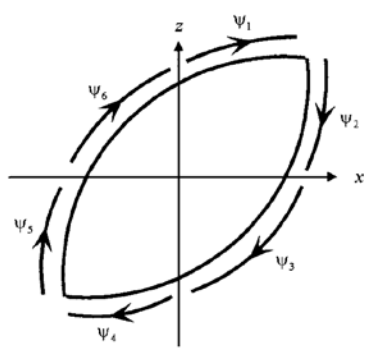


Fig. 3. Values of shape-control function for generalized Bouc–Wen model

Six phases can now have all different values, and the values are determined as

$$\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\\\psi_5\\\psi_6\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\\ -1 & -1 & 1 & -1 & 1 & 1\\ 1 & -1 & -1 & -1 & -1 & 1\\ 1 & 1 & 1 & -1 & -1 & -1\\ -1 & -1 & 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 & 1 & -1\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\\\beta_5\\\beta_6\end{bmatrix}$$

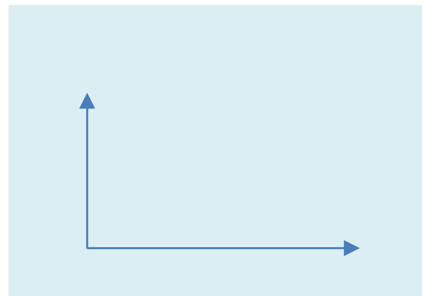
The model parameters $\beta_i, i = 1, \ldots, 6$ can be fitted by use of

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}$$

※ Weakness of Bouc-Wen class models:

- can violate the requirement of classical plasticity theory ("Drucker's postulate"; Bažant 1978); can create negative dissipative energy when "loading-unloading" occurs without load reversal

- But this problem is not critical if $\mathrm{E}[f_s] \cong 0$ (Wen 1989, Hurtado & Barbat 1996)



※ Bouc-Wen class models are widely-used in structural dynamics and earthquake engineering because

1) Can describe a wide-class of phenomena (pinching, degradation, etc.)

2) Facilitates efficient time history analysis (no IF or THEN)

3) Facilitates efficient random vibration analysis

   e.g. Nonlinear random vibration analysis for Bouc-Wen model by Equivalent Linearization Method (Wen 1980)

◎ Nonlinear time-history analysis of structural system with Bouc-Wen class models

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{R}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{z}) = -\mathbf{M}\mathbf{1}\ddot{x}_g$$

where $\mathbf{R}(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{z})$ uses $f_s = \alpha k_0 x + (1-\alpha) k_0 z$ to describe the resistant force of each B-W element. The auxiliary variable follows the nonlinear differential equation $\dot{z} = \dot{x} \cdot h(x, \dot{x}, z)$.

Transformed to state-space formulation, i.e. $\mathbf{y} = \{x_1, \dot{x}_1, x_2, \dot{x}_2, \dots, z_1, \dots, z_m\}$

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Example: Two connected equipment items in an electrical substation (Song, 2004)

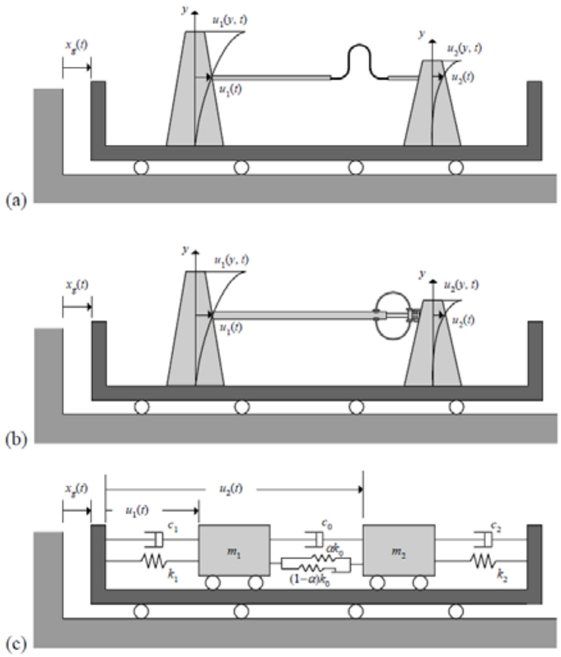

Figure 2.1 Mechanical models of equipment items connected by rigid bus connectors: (a) RB-FSC-connected system, (b) Bus-slider-connected system, and (c) idealized system with SDOF equipment models

$$\dot{\mathbf{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$$

where

$$\mathbf{y} = \{u_1, \dot{u}_1, u_2, \dot{u}_2, z\}^{\mathrm{T}}$$

$$\mathbf{g}(\mathbf{y}) = \left\{ \begin{array}{c} \dot{u}_1 \\ -\left(\dfrac{k_1 + \alpha k_0}{m_1}\right) u_1 - \left(\dfrac{c_1 + c_0}{m_1}\right) \dot{u}_1 + \dfrac{\alpha k_0}{m_1} u_2 + \dfrac{c_0}{m_1} \dot{u}_2 + \dfrac{(1-\alpha) k_0}{m_1} z \\ \dot{u}_2 \\ \dfrac{\alpha k_0}{m_2} u_1 + \dfrac{c_0}{m_2} \dot{u}_1 - \left(\dfrac{k_2 + \alpha k_0}{m_2}\right) u_2 - \left(\dfrac{c_2 + c_0}{m_2}\right) \dot{u}_2 - \dfrac{(1-\alpha) k_0}{m_2} z \\ \Delta\dot{u} \cdot h(\Delta u, \Delta\dot{u}, z) \end{array} \right\}$$

$$\mathbf{f} = \left\{ 0 \quad -\frac{l_1}{m_1}\ddot{x}_g \quad 0 \quad -\frac{l_2}{m_2}\ddot{x}_g \quad 0 \right\}^{\mathrm{T}}$$

Can solve the differential equation by a numerical method such as the fourth and fifth order Runge-Kutta-Fehlberg (RKF) algorithm.

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457.643 Structural Random Vibrations
In-Class Material: Class 26 (Final)

VI. Introduction to Nonlinear Random Vibration Analysis (contd.)

◎ Equivalent linearization method (ELM; aka stochastic linearization method)

Among various methods such as Fokker-Planck equation, stochastic averaging, moment closure, perturbation (Lutes and Sarkani 2004), ELM is considered as a nonlinear random vibration approach with the highest potential for practical use (Pradlwarter & Schuëller 1991)

- Applicable to both stationary and nonstationary processes
- Applicable to a wide class of nonlinear behavior
- Can handle MDOF systems and FE models
- Takes significantly less efforts than Monte Carlo simulations (especially for low-probability events)

Consider an original nonlinear system: $\dot{\mathbf{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$:

One can find an "equivalent linear" system: $\dot{\mathbf{y}}_e = \mathbf{A} \cdot \mathbf{y}_\mathrm{e} + \boldsymbol{f}$ such that the mean-square error (caused by linearization) $\mathrm{E}[(\boldsymbol{g}(\boldsymbol{y}) - \mathbf{A}\boldsymbol{y})^\mathrm{T}(\boldsymbol{g}(\boldsymbol{y}) - \mathbf{A}\boldsymbol{y})]$ is minimized.

Note: ELM based on the error definition above is considered "standard" ELM while the error measure $\mathrm{E}[(\boldsymbol{g}(\boldsymbol{y}_e) - \mathbf{A}\boldsymbol{y}_e)^\mathrm{T}(\boldsymbol{g}(\boldsymbol{y}_e) - \mathbf{A}\boldsymbol{y}_e)]$ is called "SPEC-alternative" ELM (Crandall 2001).

Crandall, S.H. (2001) Is stochastic equivalent linearization a subtly flawed procedure? *Probabilistic Engineering Mechanics*, 16:169-176

※ Other ELMs:

✓ Tail equivalent linearization method (**TELM**; Fujimura and ADK, 2007): equivalent linear system by unit impulse response function based on discrete representation of input stochastic process and first-order reliability method (FORM)

✓ Gaussian-mixture based equivalent linearization method (**GM-ELM**; Wang and Song, 2016): fit the response distribution by Gaussian mixture distribution. Each Gaussian density in the mixture represents an imaginary SDOF oscillator.

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◎ "Standard" ELM – how to find equivalent linear coefficients

1) In general, the equivalent linear coefficient (minimizing the mean-square error) matrix is derived as (Kozin 1987)

$$\boxed{\mathbf{A} = \frac{\mathrm{E}[\boldsymbol{g}(\boldsymbol{y})\boldsymbol{y}^{\mathrm{T}}]}{\mathrm{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}]}}$$

But, this formula is impractical because (1) the distribution of $\mathbf{y}$ is unknown, and (2) it is not straightforward to compute the expectation $\mathrm{E}[\cdot]$ that involves the nonlinear responses.

2) "Restricted" ELM: $\mathbf{y}$ is assumed to be nearly Gaussian (e.g. the input stochastic process is Gaussian, and the nonlinearity is not strong)

When $\dot{\mathbf{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$ is alternatively formulated as $\mathbf{q}(\mathbf{y}, \dot{\mathbf{y}}, \boldsymbol{f}) = \mathbf{0}$, the equivalent linear coefficient matrix is derived as (Atalik & Utku 1976)

$$\boxed{A_{ij} = \mathrm{E}\left[\frac{\partial q_i(\boldsymbol{y})}{\partial y_j}\right]}$$

Example: Application of this approach to standard MDOF system

$\mathbf{q}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \boldsymbol{f}$ can be linearized to $\mathbf{M}^e\ddot{\boldsymbol{x}} + \mathbf{C}^e\dot{\boldsymbol{x}} + \mathbf{K}^e\boldsymbol{x} = \boldsymbol{f}$ where

$$\boxed{M_{ij}^e = \mathrm{E}\left[\frac{\partial q_i}{\partial \ddot{x}_j}\right], C_{ij}^e = \mathrm{E}\left[\frac{\partial q_i}{\partial \dot{x}_j}\right], K_{ij}^e = \mathrm{E}\left[\frac{\partial q_i}{\partial x_j}\right]}$$

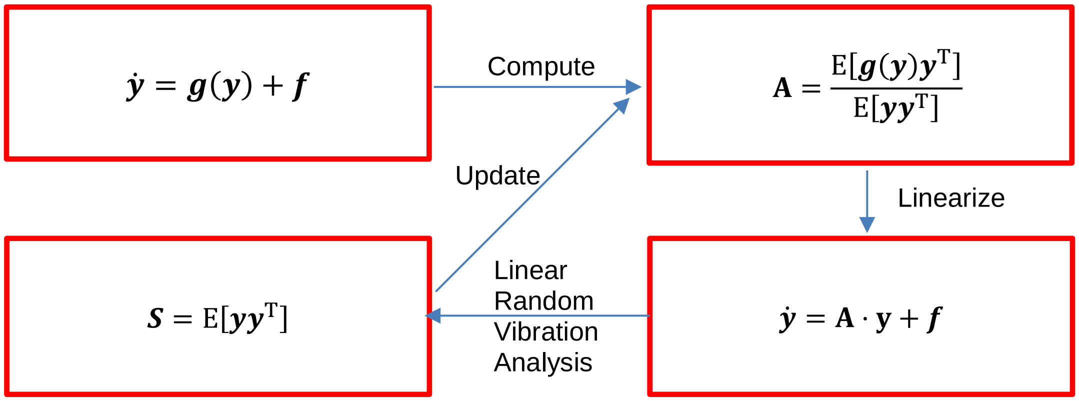
For the given type of a nonlinear system, one needs to derive the closed-form expressions of these expectations in terms of $\mathrm{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}]$ so that one can obtain the moments by solving (equivalent) linear random vibration problem iteratively (Details shown below for the Bouc-Wen class model).

3) Unrestricted ELM (Pradlwarter & Schuëller 1991)

- Not limited to "Gaussian response" assumption

- Need to identify joint distribution model for the given class of nonlinear problem (and how to obtain the moments as well)

◎ Nonlinear random vibration analysis by standard ELM



◎ (Standard, restricted) ELM for Bouc-Wen model (Wen 1980)

Suppose a system with Bouc-Wen element(s) is subjected to a zero-mean Gaussian (filtered) white noise.

1) Derivation of analytical (closed-form) expression for equivalent linear coefficients

The nonlinear differential equation about the evolution of the auxiliary variable, i.e.

$\dot{z} = \dot{x} \cdot [A - |z|^n(\gamma + \beta \mathrm{sgn}(\dot{x}z))]$

This can be alternatively described as

$q(\dot{x}, z, \dot{z}) = \dot{z} - \dot{x} \cdot \left[A - |z|^n\left(\gamma + \beta \mathrm{sgn}(\dot{x}z)\right)\right] = 0$

This nonlinear differential equation is linearized to

$a_0\dot{z} + a_1\dot{x} + a_2 z = 0$

From Atalik & Utku (1976), i.e. $A_{ij} = \mathrm{E}\left[\dfrac{\partial q_i}{\partial (\,)_j}\right]$

a) $a_0 = \mathrm{E}\left[\dfrac{\partial q}{\partial \dot{z}}\right] = \mathrm{E}[1] = 1$

b) $a_1 = \mathrm{E}\left[\dfrac{\partial q}{\partial \dot{x}}\right] = \mathrm{E}[-A + \gamma|z|^n + |z|\beta \mathrm{sgn}(\dot{x}z) + \dot{x}|z|\beta 2\delta(\dot{x})\mathrm{sgn}(z)]$

One can show that $\mathrm{E}[|z|] = \sqrt{\dfrac{2}{\pi}}\sigma_z$ and

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$$\mathrm{E}[|z|\mathrm{sgn}(\dot{x}z)] = \mathrm{E}[|z|\mathrm{sgn}(\dot{x})\mathrm{sgn}(z)] = \mathrm{E}[\dot{z}\cdot\mathrm{sgn}(\dot{x})] = \sqrt{\frac{2}{\pi}}\frac{\mathrm{E}[\dot{x}z]}{\sigma_{\dot{X}}}$$

Here a useful formula for zero-mean Gaussian, introduced in Atalik & Utku (1976), $\mathrm{E}[\mathbf{y}h(\mathbf{y})] = \mathrm{E}[\mathbf{y}\mathbf{y}^{\mathrm{T}}]\cdot\mathrm{E}[\nabla h(\boldsymbol{y})]$ is used for the derivation.

Finally,

$$a_1 = \sqrt{\frac{2}{\pi}}\left[\beta\frac{\mathrm{E}[z\dot{x}]}{\sigma_{\dot{X}}} + \gamma\sigma_z\right] - A$$

c) $a_2 = \mathrm{E}\left[\frac{\partial q}{\partial z}\right] = \mathrm{E}[\dot{x}\mathrm{sgn}(z)\gamma + \dot{x}\mathrm{sgn}(z)\beta\mathrm{sgn}(\dot{x}z) + \dot{x}|z|\beta\mathrm{sgn}(\dot{x})2\delta(z)]$

$$= \sqrt{\frac{2}{\pi}}\left[\gamma\frac{\mathrm{E}[\dot{x}z]}{\sigma_z} + \beta\sigma_{\dot{X}}\right]$$

2) Construct an equivalent linear system

$$\mathbf{y} = \begin{Bmatrix} x \\ z \\ \dot{x} \end{Bmatrix}, \text{ and } \dot{\mathbf{y}} = \begin{Bmatrix} \dot{x} \\ \dot{z} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -a_2/a_0 & -a_1/a_0 \\ -\alpha\omega_0^2 & -(1-\alpha)\omega_0^2 & -2\xi\omega_0 \end{bmatrix}\begin{Bmatrix} x \\ z \\ \dot{x} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ f(t)/m \end{Bmatrix}$$

$$\dot{\mathbf{y}} = \boldsymbol{G}\boldsymbol{y} + \boldsymbol{f}$$

3) Perform linear random vibration analysis

e.g. if $f(t)$ is a white noise, the 2nd moment follows the Lyapunov equation (Lin 1967)

$$\mathbf{GS} + \mathbf{SG}^{\mathrm{T}} + \mathbf{B} = \mathbf{0}$$

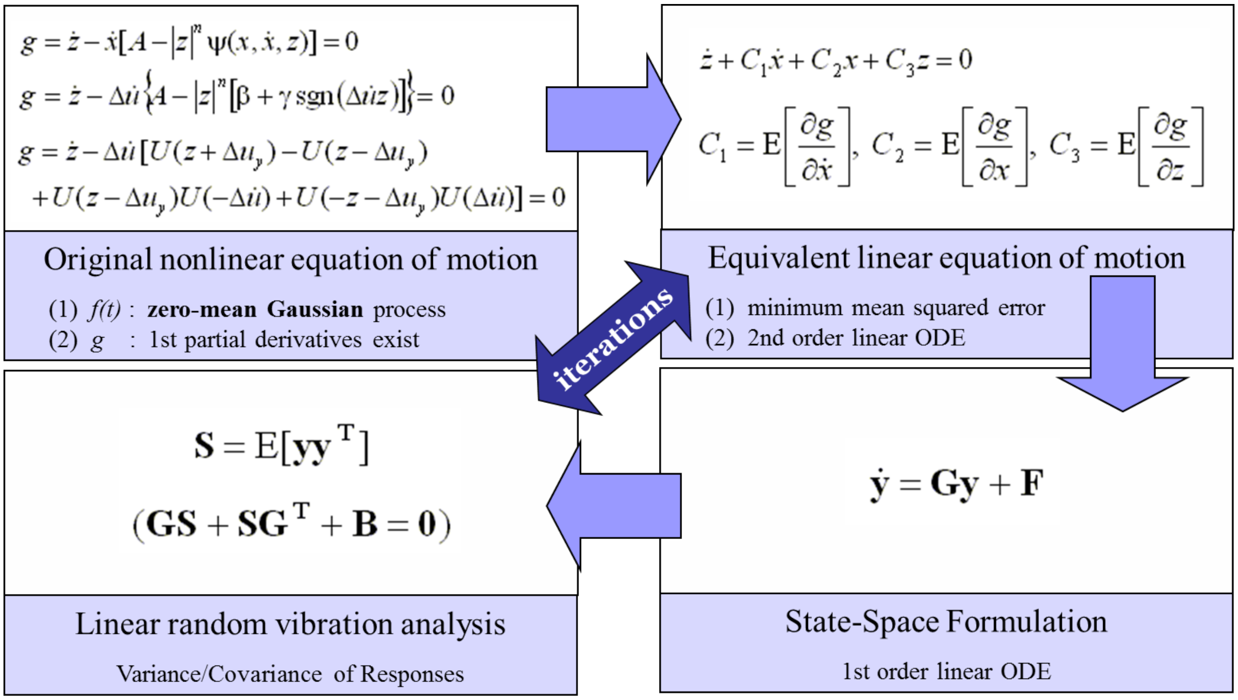
where $B_{ij} = 0$ except $B_{33} = 2\pi\Phi_0$ (→ Φ_0 is the PSD of the white noise $f(t)$) and

$$\mathbf{S} = \mathrm{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}] = \mathrm{E}\begin{bmatrix} x^2 & xz & x\dot{x} \\ xz & z^2 & z\dot{x} \\ x\dot{x} & z\dot{x} & \dot{x}^2 \end{bmatrix}$$

This random vibration analysis approach can be used for filtered white noise case as well by introducing the ground displacement x_g to the state-space vector, i.e. adding another DOF representing the filter.

4) Re-compute a_1 and a_2 based on new $\mathbf{S} = \mathrm{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}]$

5) Repeat 1)-4) until the solution converges.



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Song, J., A. Der Kiureghian, and J.L. Sackman (2007). Seismic interaction in electrical substation equipment connected by nonlinear rigid bus conductors. *Earthquake Engineering and Structural Dynamics*, Vol. 36, 167-190.

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VII. Random Vibration Analysis by Structural Reliability Analysis Methods

References:

Der Kiureghian, A. (2000). The geometry of random vibrations and solutions by FORM and SORM. *Probabilistic Engineering Mechanics*, 15(1), 81-90

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◎ Discrete representation of a random process

1) Discrete representation (in time domain)

$$\boxed{f(t) = \mu(t) + \sum_{i=1}^{n} u_i s_i(t) = \mu(t) + \mathbf{u}^{\mathrm{T}}\mathbf{s}(\mathrm{t})}$$

where $u_i, i = 1, \ldots, n$ are uncorrelated standard normal random variables

$\mathbf{s}(t)$ is a vector of deterministic time-varying basis function which is identified based on the correlation structure of the process, e.g. Karhunen-Loève expansion

2) Example: filtered white noise (EQ input)

$$f(t) = \int_0^{\mathrm{t}} u(\tau)s(t-\tau)d\tau \cong \sum u_i s_i(t)$$

where $s(\cdot)$ denotes the unit impulse response function of the filter.

3) Discrete representation (in frequency domain)

For example (Wang and Song, 2016), a white noise can be discretized as

$$\ddot{x}_g(t) = \sigma \sum_{j=1}^{n/2} \left[u_j \cos(\omega_j t) + \hat{u}_j \sin(\omega_j t)\right]$$

◎ Response of linear structure to Gaussian excitation

$$x(t) = \int_0^{\mathrm{t}} f(\tau)h(t-\tau)d\tau = \int_0^{\mathrm{t}} \sum_{i=1}^{n} u_i s_i(\tau)\, h(t-\tau)d\tau$$

where $h(\cdot)$ is the unit impulse response function of the structure, and thus the response is

$$\boxed{x(t) = \sum_{i=1}^{n} u_i a_i(t) = \boldsymbol{a}^{\mathrm{T}}(t)\boldsymbol{u}}$$

where $a_i(t) = \int_0^t s_i(\tau)h(t-\tau)d\tau$

In summary, the response of a linear structure to a Gaussian input can be described as a linear function of uncorrelated standard normal random variables (owing to the discrete representation).

◎ Instantaneous failure probability

The instantaneous failure probability of the linear response is

$$P(x(t_0) \geq x_0) = P\left(x_0 - \boldsymbol{a}^{\mathrm{T}}(t_0)\boldsymbol{u} \leq 0\right)$$

This is a structural reliability problem with a linear limit state function $\mathrm{g}(\mathbf{u}) = x_0 - \boldsymbol{a}^{\mathrm{T}}(t_0)\boldsymbol{u}$

From structural reliability theories, the failure probability is obtained by a closed-form solution

$$P(x(t_0) \geq x_0) = \Phi[-\beta(x_0, t_0)] = \Phi\left[-\frac{x_0}{\|\boldsymbol{a}(t_0)\|}\right]$$

One can also compute crossing rate, first-passage failure probability, etc. by structural reliability analysis in the standard normal space (Der Kiureghian, 2000).

This idea was utilized for efficient topology optimization with constraints on instantaneous failure probability (Chun et al. 2016).

Chun, J., J. Song, and G.H. Paulino (2016). Structural topology optimization under constraints on instantaneous failure probability. *Structural and Multidisciplinary Optimization*, 53(4): 773-799.

For nonlinear system and/or non-Gaussian process, first-order reliability method (FORM) or second-order reliability method (SORM) can be used to compute the probabilities approximately. This idea was further developed to propose the tail equivalent linearization method (TELM; Fujimura and ADK 2007).

--- End of Semester ---

Thanks a lot for your patience and great effort this semester.. J.S.

"There is no fear in love." – 1John 4:18a