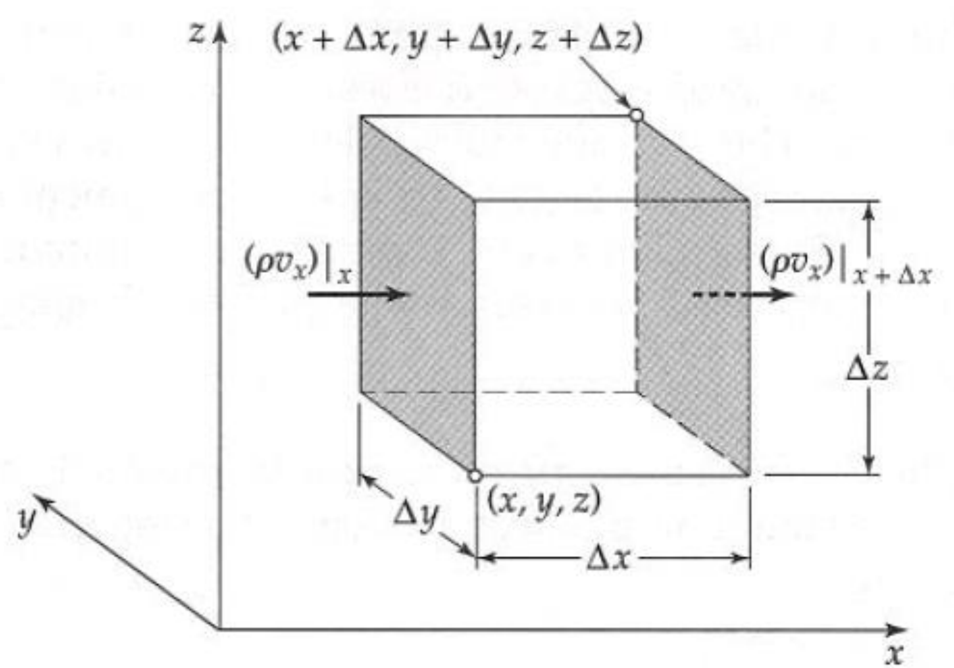
Chapter 3. The Equation of Change for Isothermal System

- The equation of continuity, of motion
- The equation of mechanical energy
- The equation of angular momentum
- The equations of change in terms of the substantial derivative
- Dimensional analysis of the equations of change

Chapter 3. The Equation of Change for Isothermal System

- Equation of continuity



"Transport Phenomena" 2nd ed.,
R.B. Bird, W.E. Stewart, E.N. Lightfoot

3.1. Equation of continuity

- Transient !

$$\begin{bmatrix}\text{rate of}\\ \text{increase}\\ \text{of mass}\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{mass in}\end{bmatrix} - \begin{bmatrix}\text{rate of}\\ \text{mass out}\end{bmatrix}$$

$$\begin{aligned}\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \cdot \left[\left(\rho v_x\right)\big|_x - \left(\rho v_x\right)\big|_{x+\Delta x} \right] \\ + \Delta z \Delta x \cdot \left[\left(\rho v_y\right)\big|_y - \left(\rho v_y\right)\big|_{y+\Delta y} \right] \\ + \Delta x \Delta y \cdot \left[\left(\rho v_z\right)\big|_z - \left(\rho v_z\right)\big|_{z+\Delta z} \right]\end{aligned}$$

Equation of continuity

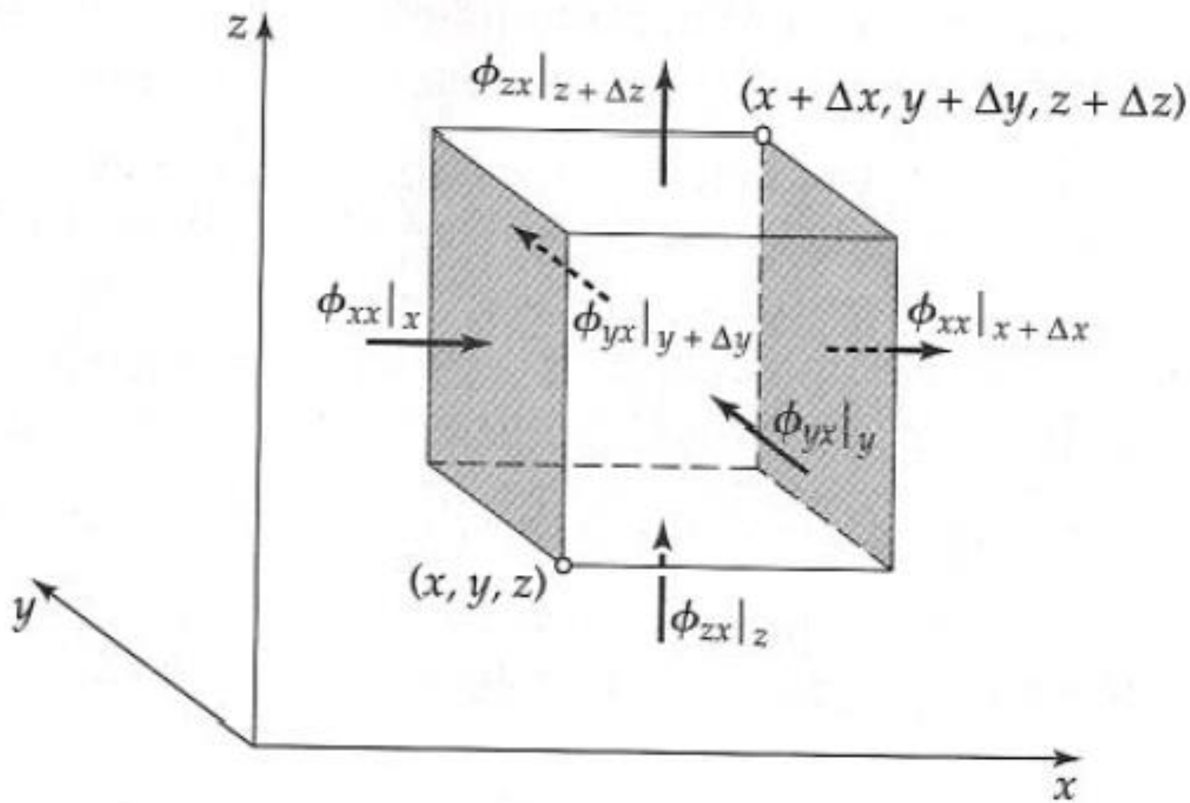
$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z\right)$$

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho \underline{v})$$

Rate of increase of mass per unit volume

Net rate of mass addition per unit volume by convection

3.2 The Equation of Motion



"Transport Phenomena" 2nd ed.,
R.B. Bird, W.E. Stewart, E.N. Lightfoot

Equation of motion: Balance

$$\begin{bmatrix}\text{rate of}\\ \text{increase of}\\ \text{momentum}\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{external}\\ \text{forces on}\\ \text{the fluid}\end{bmatrix}$$

- Momentum by convective and molecular transport: net rate of addition of x-component

$$\Delta y \Delta z \left(\phi_{xx}\Big|_{x} - \phi_{xx}\Big|_{x+\Delta x} \right) + \Delta z \Delta x \left(\phi_{yx}\Big|_{y} - \phi_{yx}\Big|_{y+\Delta y} \right) + \Delta x \Delta y \left(\phi_{zx}\Big|_{z} - \phi_{zx}\Big|_{z+\Delta z} \right)$$

Equation of motion

$$\frac{\partial}{\partial t}\rho v_x = -\left(\frac{\partial}{\partial x}\phi_x + \frac{\partial}{\partial y}\phi_y + \frac{\partial}{\partial z}\phi_z\right) + \rho g_x$$

- Vector notation $$\frac{\partial}{\partial t}\rho \underline{v} = -\left(\nabla \cdot \underline{\phi}\right) + \rho \underline{g}$$

where

$$\underline{\phi} = \underline{\pi} + \rho \underline{vv} = p\underline{\delta} + \underline{\tau} + \rho \underline{vv}$$

$\underline{\phi}$: combined momentum flux tensor

$\rho \underline{vv}$: convective momentum flux tensor

$\underline{\pi}$: Molecular momentum flux tensor

Equation of motion

$$\frac{\partial}{\partial t}\rho \underline{v} = -\left(\nabla \cdot \rho \underline{vv}\right) - \nabla \cdot \underline{p} - \left(\nabla \cdot \underline{\tau}\right) + \rho \underline{g}$$

- $\frac{\partial}{\partial t}\rho \underline{v}$: rate of increase of momentum per unit volume
- $-\left(\nabla \cdot \rho \underline{vv}\right)$: rate of momentum addition by convection
- $-\nabla \cdot \underline{p} - \left(\nabla \cdot \underline{\tau}\right)$: rate of momentum addition by molecular transport
- $\rho \underline{g}$: external force on fluid

3.3. The equation of mechanical energy

- Dot product between velocity and equation of motion:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\left(\nabla \cdot \frac{1}{2}\rho v^2 \underline{v}\right) - (\nabla \cdot p\underline{v}) - p(-\nabla \cdot \underline{v})$$

$$-\left(\nabla \cdot (\underline{\tau} \cdot \underline{v})\right) - (-\underline{\tau} : \nabla \underline{v}) + \rho\left(\underline{v} \cdot \underline{g}\right)$$

The equation of mechanical energy

- Terms: rate of.................. per unit volume
  - increase of kinetic energy
  - addition of kinetic energy by convection
  - work done by pressure of surroundings on the fluid
  - reversible conversion of kinetic energy into internal energy
  - work done by viscous forces on the fluid
  - irreversible conversion from kinetic energy to internal energy
  - work by external force on the fluid

3.4. The equation of angular momentum

- Cross product between position and equation of motion:

$$\frac{\partial}{\partial t}\rho(\underline{r}\times\underline{v}) = -\left(\nabla\cdot\rho\underline{v}(\underline{r}\times\underline{v})\right) - \left(\nabla\cdot(\underline{r}\times p\underline{\delta})^{\dagger}\right)$$

$$-\left(\nabla\cdot(\underline{r}\times\underline{\tau}^{\dagger})^{\dagger}\right) + \left(\underline{r}\times\rho\underline{g}\right) - (\underline{\varepsilon}:\underline{\tau})$$

3.5. The equations of change in terms of the substantial derivative

- The partial time derivative $\frac{\partial c}{\partial t}$
- The total time derivative

$$\frac{dc}{dt} = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} + \frac{dx}{dt}\left(\frac{\partial c}{\partial x}\right)_{y,z,t} + \frac{dy}{dt}\left(\frac{\partial c}{\partial y}\right)_{z,x,t} + \frac{dz}{dt}\left(\frac{\partial c}{\partial z}\right)_{x,y,t}$$

- The substantial time derivative

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} + v_z \frac{\partial c}{\partial z} = \frac{\partial c}{\partial t} + \left(\underline{v} \cdot \nabla c\right)$$

Special cases

- For constant density and viscosity.
  Navier-Stokes equation

$$\rho \frac{D}{Dt}\vec{v} = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \quad \text{or} \quad \rho \frac{D}{Dt}\vec{v} = -\nabla P + \mu \nabla^2 \vec{v}$$

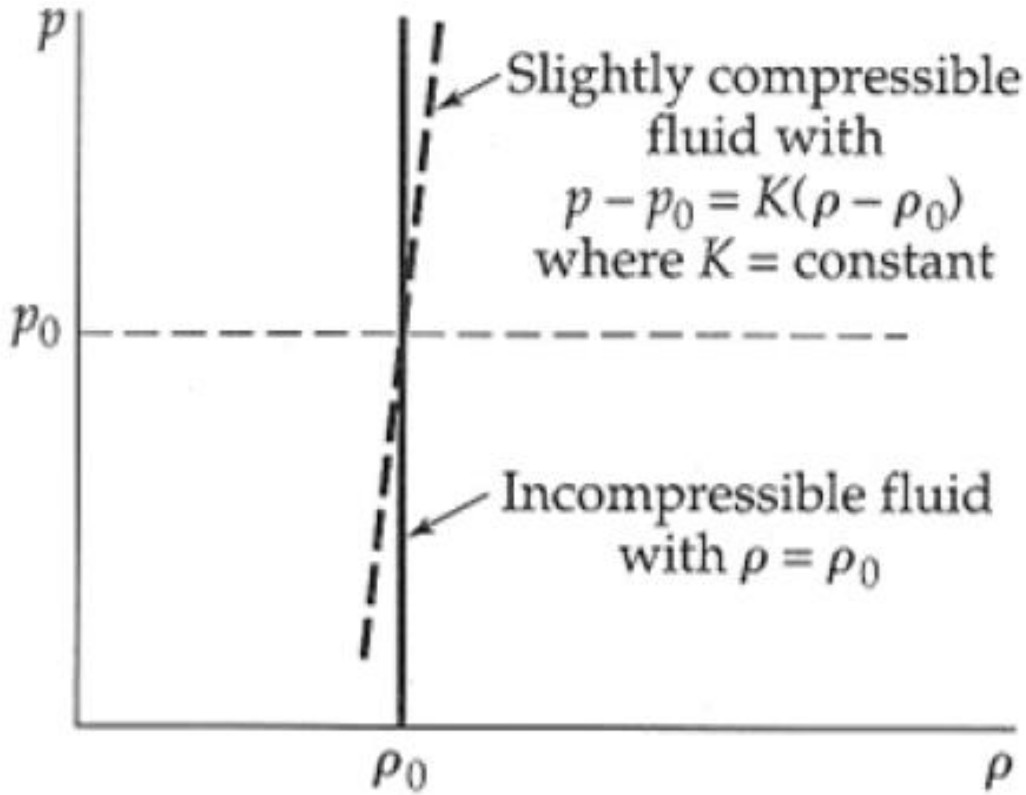
- Negligible acceleration terms:
  Stokes flow equation

$$0 = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

- Negligible viscous forces:
  Euler equation

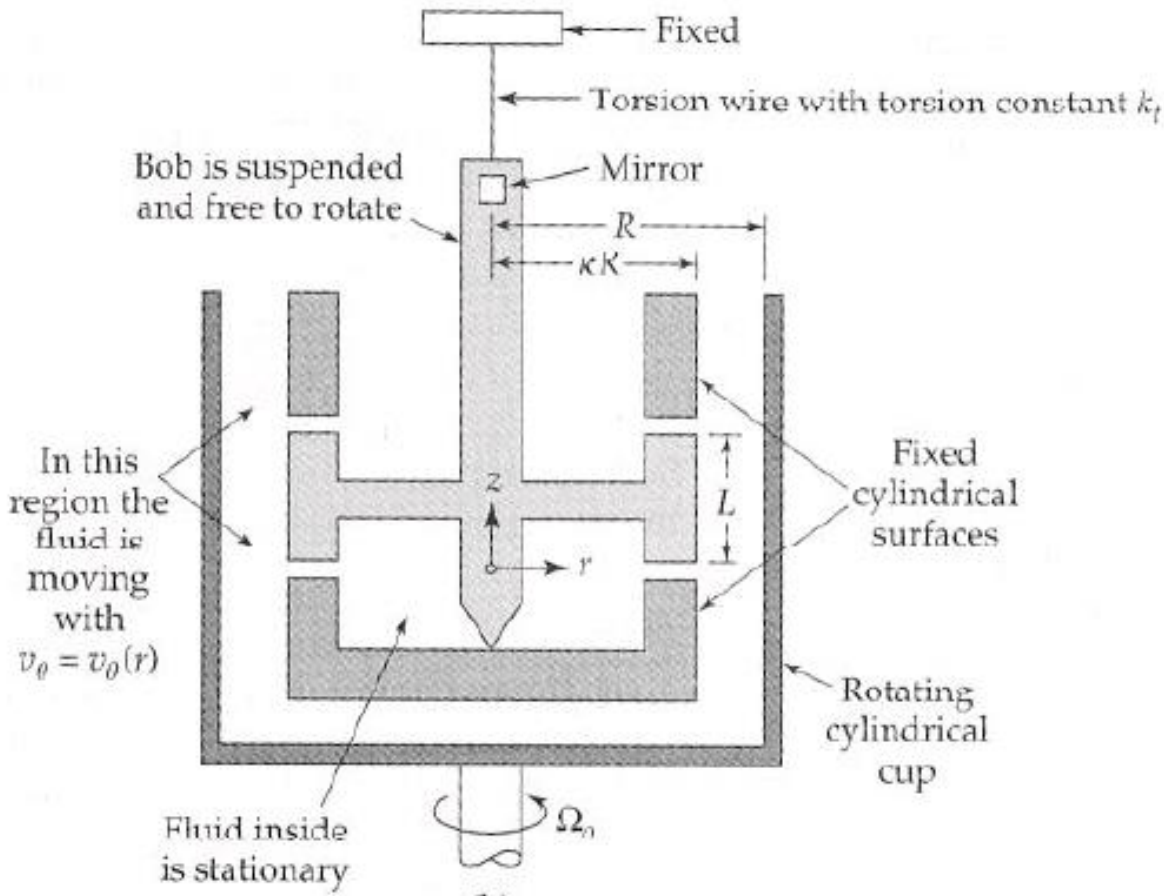
$$\rho \frac{D}{Dt}\vec{v} = -\nabla p + \rho \vec{g}$$

The equation of state



"Transport Phenomena" 2nd ed.,
R.B. Bird, W.E. Stewart, E.N. Lightfoot

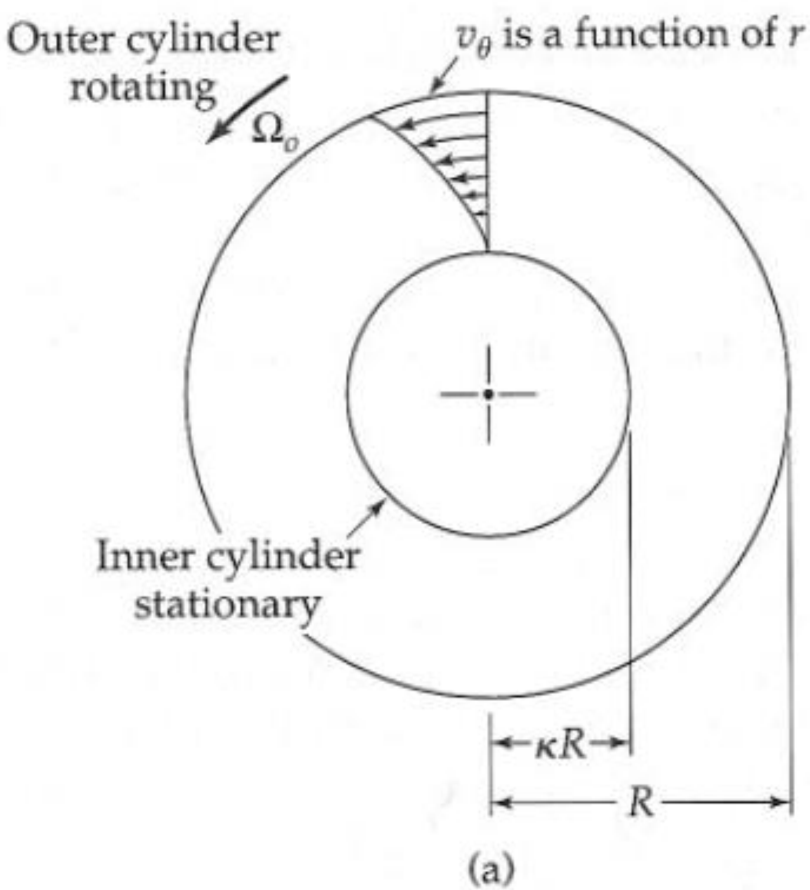
3.6 Use of the Equations of Change



(b)

"Transport Phenomena" 2nd ed.,
R.B. Bird, W.E. Stewart, E.N. Lightfoot

Operation of a Couette viscosimeter



- Constant ρ and μ

$$v_\theta = v_\theta(r)$$

$$v_r = v_z = 0$$

$$p = p(r, z)$$

"Transport Phenomena" 2nd ed.,
R.B. Bird, W.E. Stewart, E.N. Lightfoot

Equations

- r-component

$$-\rho\frac{v_o^2}{r} = -\frac{\partial p}{\partial r}$$

- θ-component

$$0 = \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)$$

- z-component

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

Solving equations

- Integrating

$$\frac{1}{r}\frac{d}{dr}\left(rv_\theta\right) = C_1 \qquad \frac{d}{dr}\left(rv_\theta\right) = C_1 r$$

$$rv_\theta = \frac{1}{2}C_1 r^2 + C_2 \qquad v_\theta = \frac{1}{2}C_1 r + \frac{C_2}{2}$$

- BC's:

at $r = \kappa R \quad v_\theta = 0$

at $r = R \quad v_\theta = \Omega_o R$

Results

- Then

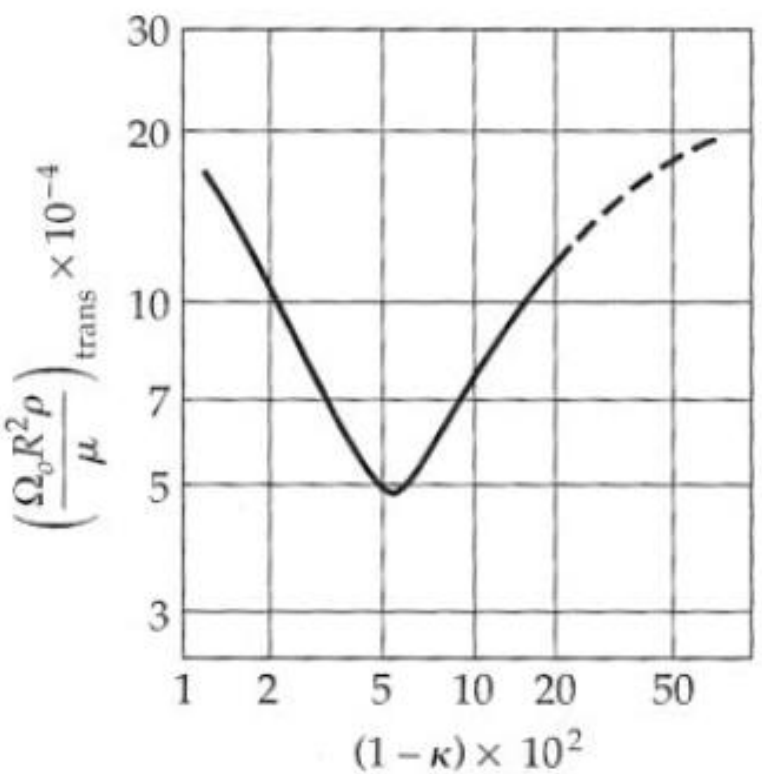
$$v_\theta = \Omega_o R \frac{\left(\frac{r}{\kappa R} - \frac{\kappa R}{r}\right)}{\left(\frac{1}{\kappa} - \kappa\right)}$$

- Torque

$$T_z = \left(-\tau_{r\theta}\right)\big|_{r=\kappa R} \cdot 2\pi\kappa RL \cdot \kappa R$$

$$= 4\pi\mu\Omega_o R^2 L\left(\frac{\kappa^2}{1-\kappa^2}\right)$$

More



"Transport Phenomena" 2nd ed., R.B. Bird, W.E. Stewart, E.N. Lightfoot

- Critical Reynolds numbers for the tangential flow in an annulus (above which the system becomes turbulent)
- Outer cylinder rotating
- Inner cylinder stationary

3.7. Dimensional analysis of the equations of change

- For simplicity, constant density and viscosity

$$(\nabla \cdot \underline{v}) = 0 \qquad \rho \frac{D}{Dt}\underline{v} = -\nabla \underline{P} + \mu \nabla^2 \underline{v}$$

- Defining dimensionless variables

$$\tilde{x} = \frac{x}{l_o} \qquad \tilde{y} = \frac{y}{l_o} \qquad \tilde{z} = \frac{z}{l_o} \qquad \tilde{t} = \frac{v_o t}{l_o}$$

$$\tilde{v} = \frac{v}{v_o} \qquad \tilde{P} = \frac{P - P_o}{\rho v_o}$$

3.7. Dimensional analysis of the equations of change

- Equation of continuity

$$(\tilde{\nabla} \cdot \underline{\tilde{v}}) = 0$$

- Equation of motion

$$\frac{D}{D\tilde{t}}\underline{\tilde{v}} = -\tilde{\nabla}\underline{\tilde{P}} + \frac{\mu}{l_o v_o \rho}\tilde{\nabla}^2\underline{\tilde{v}}$$

3.7. Dimensional analysis of the equations of change

- Reynolds number

$$\mathrm{Re} = \left[\frac{l_o v_o \rho}{\mu}\right] = \left[\frac{\text{inertial forces}}{\text{viscous forces}}\right]$$

- Other numbers are Froude and Weber numbers

$$\mathrm{Fr} = \left[\frac{v_o^2}{l_o g}\right] \qquad \mathrm{We} = \left[\frac{\sigma}{l_o v_o^2 \rho}\right]$$