Chapter 10. Shell energy balances and temperature distributions in solids and laminar flow

- Shell energy balances, boundary conditions
- Heat conduction with a electrical heat source, a nuclear heat source, a viscous heat source, and a chemical heat source
- Heat conduction through composite walls
- Heat conduction in a cooling fin
- Forced and free convection

10.1 Shell energy balance

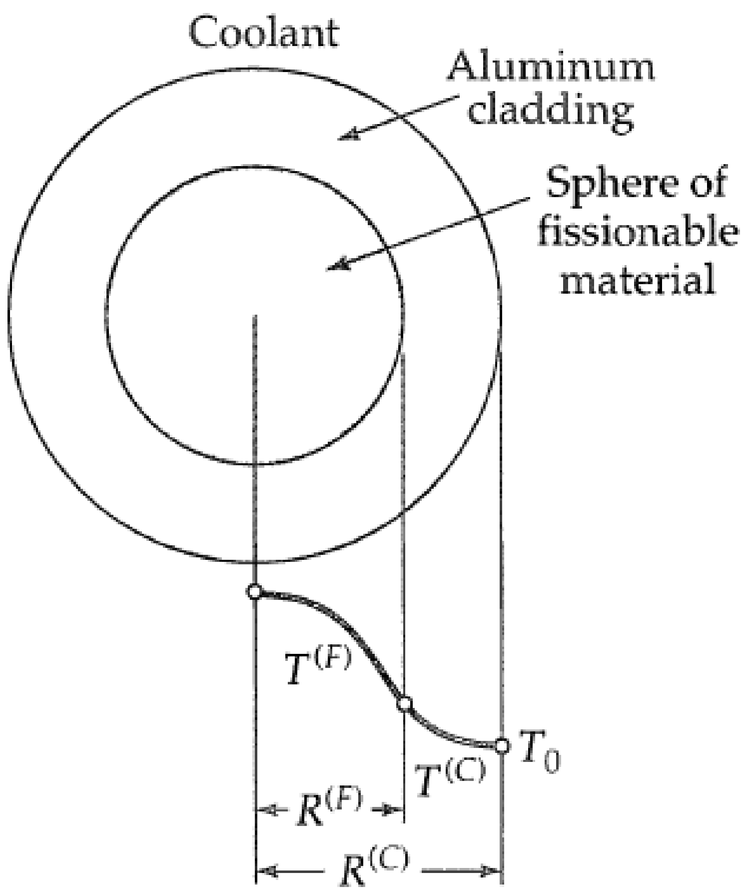
$$\begin{Bmatrix}\text{rate of}\\ \text{energy in}\\ \text{by convective}\\ \text{transport}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{energy out}\\ \text{by convective}\\ \text{transport}\end{Bmatrix} + \begin{Bmatrix}\text{rate of}\\ \text{energy in}\\ \text{by molecular}\\ \text{transport}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{energy out}\\ \text{by molecular}\\ \text{transport}\end{Bmatrix} +$$

$$\begin{Bmatrix}\text{rate of}\\ \text{work done}\\ \text{on system}\\ \text{by molecular}\\ \text{transport}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{work done}\\ \text{by system}\\ \text{by molecular}\\ \text{transport}\end{Bmatrix} + \begin{Bmatrix}\text{rate of}\\ \text{work done}\\ \text{on system}\\ \text{by external}\\ \text{forces}\end{Bmatrix} + \begin{Bmatrix}\text{rate of}\\ \text{energy}\\ \text{production}\end{Bmatrix} = 0$$

Boundary conditions

- Specified temperature at the surface
- Given heat flux normal to a surface
- At interfaces:
    - Continuity of temperature
    - Continuity of heat flux normal to the interface
- At solid-fluid interface: $q = h(T_0 - T_b)$
    - To, solid surface temperature
    - Tb, bulk fluid temperature

10.3 Heat conduction with a nuclear heat source



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- Spherical nuclear fuel element.
- The heat source

$$S_n = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right]$$

$$S\ \left[\frac{energy}{volume * time}\right]$$

- Spherical shell of thickness Δr

Shell balance for the nuclear fuel

- Rate of heat by conduction
  - In at r

$$q_r^{(F)}\big|_r \cdot 4\pi r^2 = (4\pi r^2 q_r^{(F)})\big|_r$$

  - out at r+Δr

$$q_r^{(F)}\big|_{r+\Delta r} \cdot 4\pi (r + \Delta r)^2 = (4\pi r^2 q_r^{(F)})\big|_{r+\Delta r}$$

- Rate of thermal energy produced by nuclear fission

$$S_n \cdot 4\pi r^2 \, \Delta r$$

Differential equations

- Introducing these terms into the shell balance for the nuclear fuel and dividing by 4πΔr

$$\frac{d}{dr}\left(r^2 q_r^{(F)}\right) = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right] r^2$$

- For the cladding

$$\frac{d}{dr}\left(r^2 q_r^{(C)}\right) = 0$$

- Integrating both equations
  - For the nuclear fuel
$$q_r^{(F)} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\frac{r^3}{5}\right) + \frac{C_1^{(F)}}{r^2}$$
  - For the cladding
$$q_r^{(C)} = +\frac{C_1^{(C)}}{r^2}$$

- Boundary conditions

B.C. 1:	at $r = 0$,	$q_r^{(F)}$ is not infinite
B.C. 2:	at $r = R^{(F)}$,	$q_r^{(F)} = q_r^{(C)}$
B.C. 3:	at $r = R^{(F)}$,	$T^{(F)} = T^{(C)}$
B.C. 4:	at $r = R^{(C)}$,	$T^{(C)} = T_0$

Solving the equations

- Using BCs 1 and 2

$$q_r^{(F)} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\frac{r^3}{5}\right) \qquad q_r^{(C)} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2}$$

- Substituting the Fourier's law and integrating

$$T^{(F)} = -\frac{S_{n0}}{k^{(F)}}\left(\frac{r^2}{6} + \frac{b}{R^{(F)2}}\frac{r^4}{20}\right) + C_2^{(F)}$$

$$T^{(C)} = +\frac{S_{n0}}{k^{(C)}}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r} + C_2^{(C)}$$

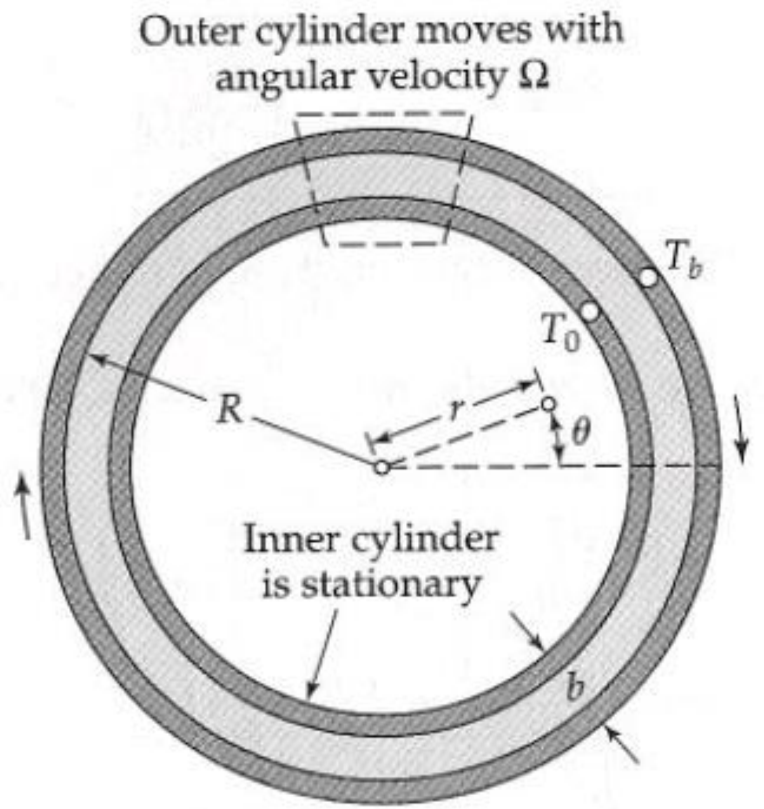
Results

- Using BCs 3 and 4

$$T^{(F)} = \frac{S_{n0}R^{(F)2}}{6k^{(F)}}\left\{\left[1 - \left(\frac{r}{R^{(F)}}\right)^2\right] + \frac{3}{10}b\left[1 - \left(\frac{r}{R^{(F)}}\right)^4\right]\right\} + \frac{S_{n0}R^{(F)2}}{3k^{(C)}}\left(1 + \frac{3}{5}b\right)\left(1 - \frac{R^{(F)}}{R^{(C)}}\right)$$

$$T^{(C)} = \frac{S_{n0}R^{(F)2}}{3k^{(C)}}\left(1 + \frac{3}{5}b\right)\left(\frac{R^{(F)}}{r} - \frac{R^{(F)}}{R^{(C)}}\right)$$

10.4 Heat conduction with a viscous heat source



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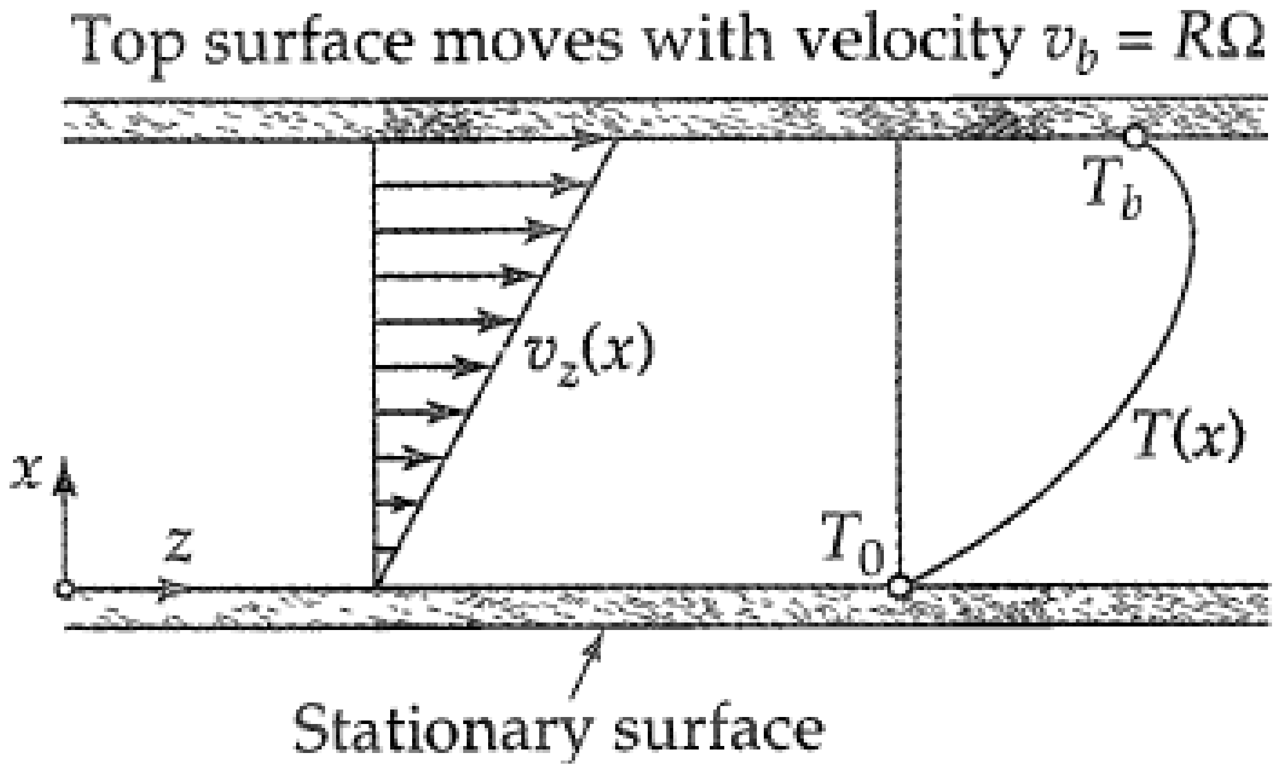
- Friction in the fluid produces heat
- Mechanical energy degraded to thermal energy
- Fluid temperature T only a function the of radius

Our system for small b



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Momentum and temperature balances

- Combined energy flux vector

$$\mathbf{e} = (\tfrac{1}{2}\rho v^2 + \rho\hat{H})\mathbf{v} + [\boldsymbol{\tau} \cdot \mathbf{v}] + \mathbf{q}$$

- Velocity distribution

$$v_z = v_b(x/b), \text{ where } v_b = \Omega R$$

Momentum and temperature balances

- Energy balance over a shell with dimensions Δx, W, and L

$$WLe_x|_x - WLe_x|_{x+\Delta x} = 0$$

$$e_x = \left(\frac{1}{2}\rho v^2 + \rho \widehat{H}\right) \cdot v_x +$$

$$\tau_{xx} \cdot v_x + \tau_{xy} \cdot v_y + \tau_{xz} \cdot v_z + q_x$$

Differential Equations

- Dividing by Δx WL and taking limit Δx →0

$$\frac{de_x}{dx} = 0$$

- Integrating, no possible to evaluate C_1

$$e_x = C_1$$

- Introducing e_x. Convective transport is zero $(v_x = 0)$

$$e_x = \tau_{xz} \cdot v_z + q_x$$

Differential Equations

- Work by molecular mechanisms, v_z only velocity no zero and heat transport by molecular mechanism

$$\tau_{xz} = -\mu(dv_z/dx) \qquad\qquad -k\frac{dT}{dx}$$

- Energy balance

$$-k\frac{dT}{dx} - \mu v_z \frac{dv_z}{dx} = C_1$$

- Introducing v_z.
  - Second term: rate viscous heat production per unit volume

$$-k\frac{dT}{dx} - \mu x\left(\frac{v_b}{b}\right)^2 = C_1$$

Differential Equations

- Integrating

$$T = -\left(\frac{\mu}{k}\right)\left(\frac{v_b}{b}\right)^2 \frac{x^2}{2} - \frac{C_1}{k} x + C_2$$

- Using the BCs:

$$\text{at } x = 0, \qquad T = T_0$$
$$\text{at } x = b, \qquad T = T_b$$

for $T_b \neq T_0$

$$\left(\frac{T - T_0}{T_b - T_0}\right) = \frac{1}{2} \mathrm{Br} \frac{x}{b}\left(1 - \frac{x}{b}\right) + \frac{x}{b}$$

Temperature distribution

- Brinkman number

$$\mathrm{Br} = \mu v_b^2 / k(T_b - T_0)$$

$$\mathrm{Br} = \frac{\text{viscous dissipation}}{\text{Heat transport by molecular mechanisms}}$$

Temperature distribution

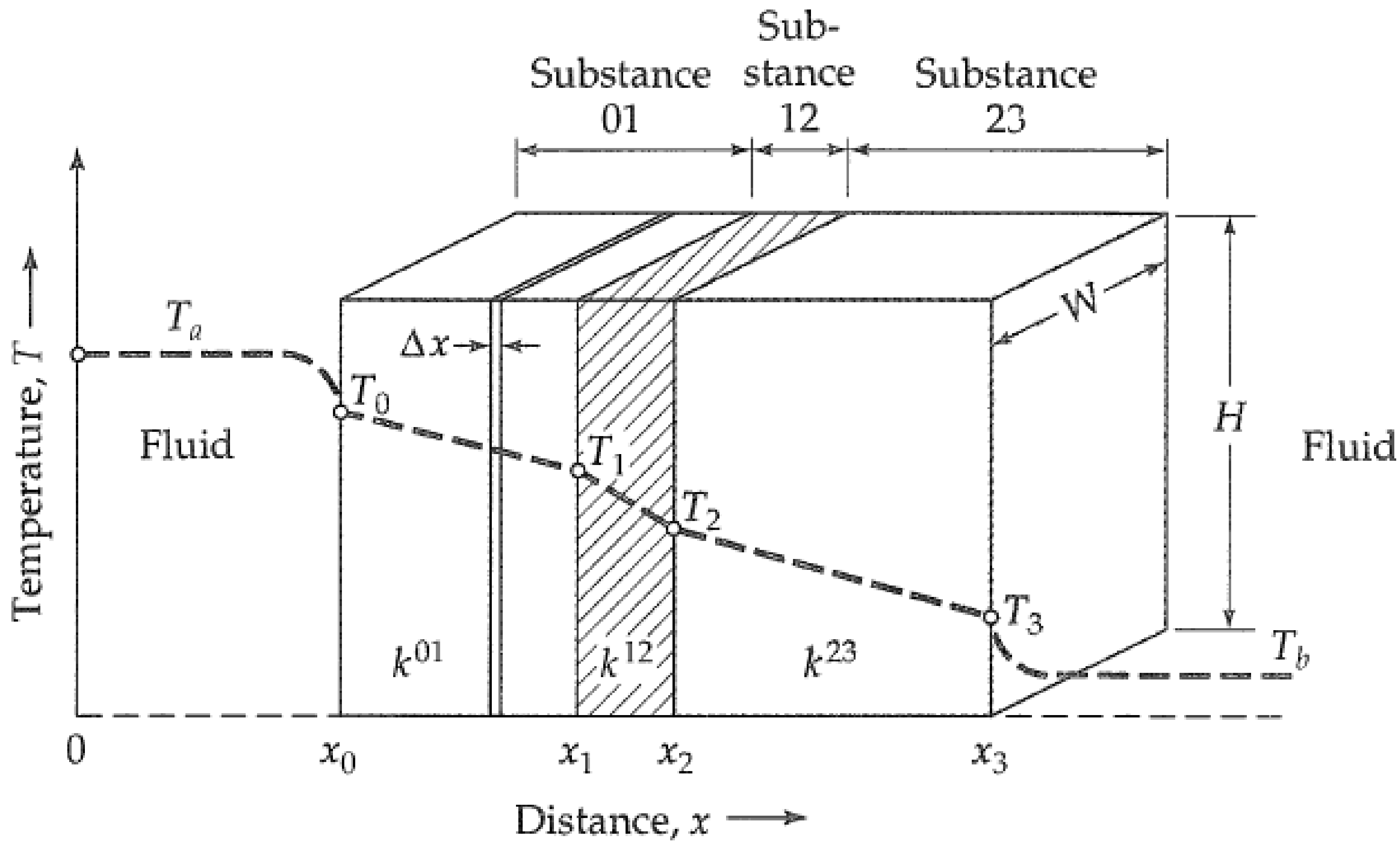
$$T_b = T_0$$

$$\frac{T - T_0}{T_0} = \frac{1}{2}\frac{\mu v_b^2}{kT_0}\frac{x}{b}\left(1 - \frac{x}{b}\right) + \frac{x}{b}$$

the maximum temperature is at $x/b = \frac{1}{2}$

→ temperature dependence of the viscosity

10.6 Heat conduction through composite walls



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Results

- Heat flux

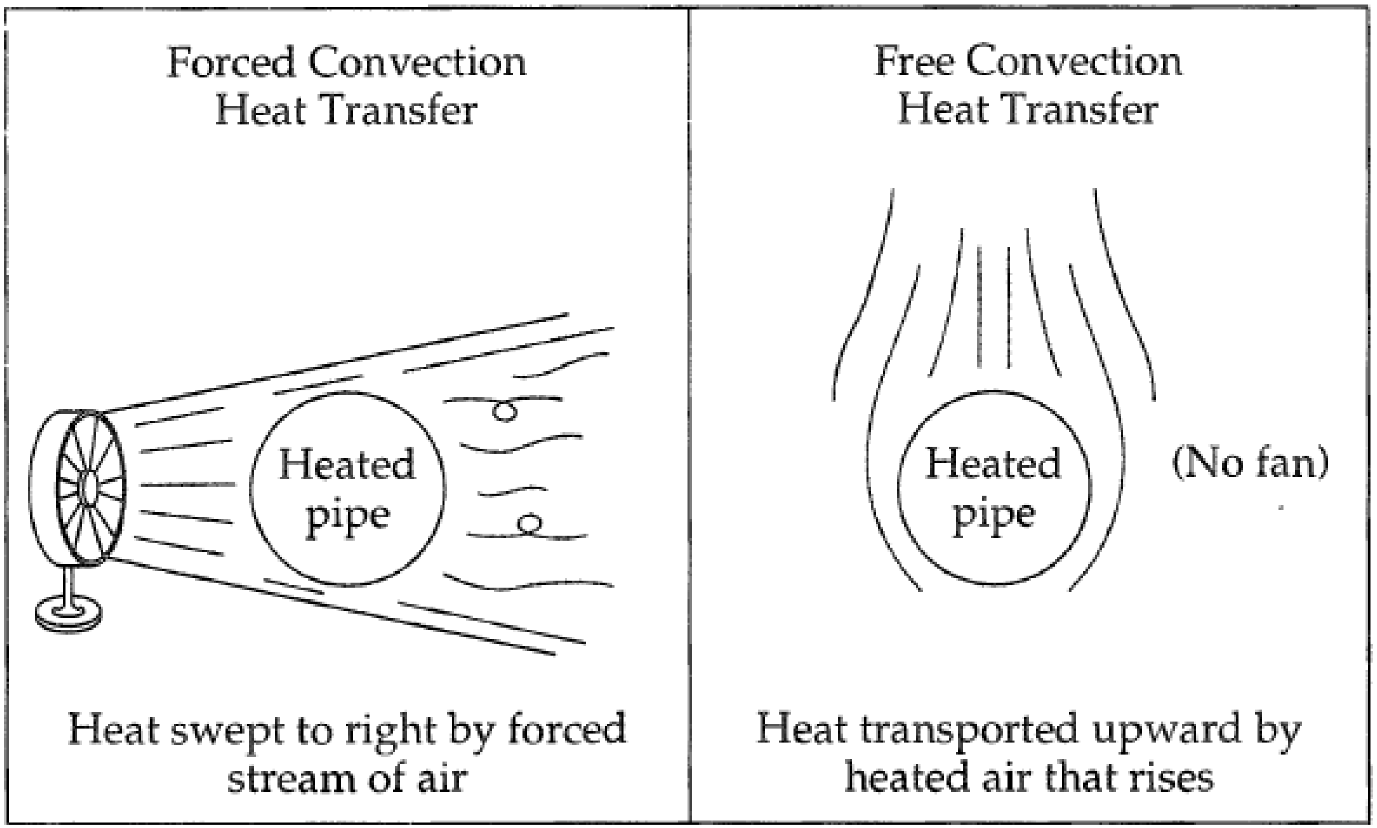
$$q_0 = \frac{T_a - T_b}{\left(\frac{1}{h_0} + \sum_{j=1}^{3} \frac{x_j - x_{j-1}}{k_{j-1,j}} + \frac{1}{h_3}\right)}$$

- Transfer equation $q_0 = U(T_a - T_b)$ or $Q_0 = U(WH)(T_a - T_b)$

The quantity U, called the "overall heat transfer coefficient,"

- What controls the flux?

10.8 Forced and free convection



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Comparison

1. The flow patterns are determined primarily by some external force	1. The flow patterns are determined by the buoyant force on the heated fluid
2 First, the velocity profiles are found; then they are used to find the temperature profiles (usual procedure for fluids with constant physical properties)	2. The velocity profiles and temperature profiles are interdependent
3. The Nusselt number depends on the Reynolds and Prandtl numbers (see Chapter 14)	3. The Nusselt number depends on the Grashof and Prandtl numbers (see Chapter 14)

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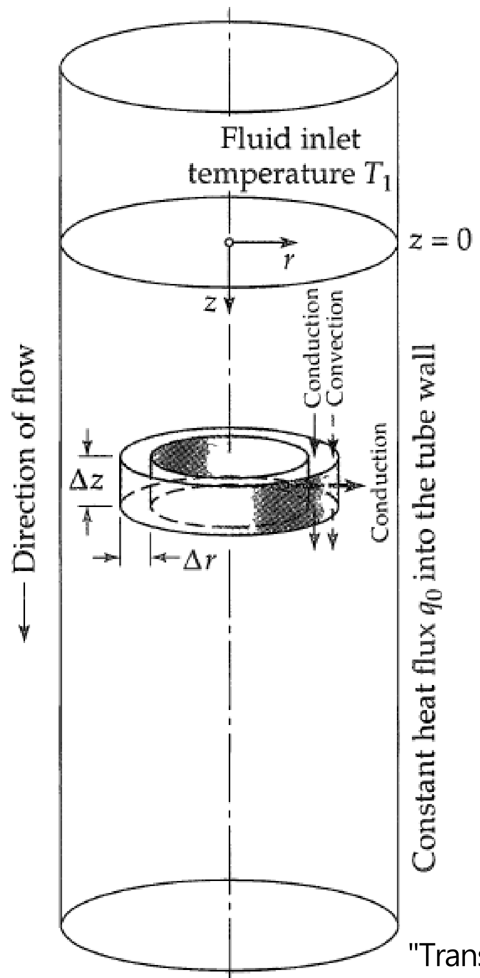
Forced Convection



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- Laminar flow in a circular tube of radius R
- Constant physical properties
- Inlet temperature T_1
- Constant radial heat flux, q_0

Shell balance

- Energy in at r and energy out in r+ Δr and energy in at z and energy out in z+ Δz

$$e_r|_r \cdot 2\pi r \Delta z = (2\pi r e_r)|_r \, \Delta z$$

$$e_r|_{r+\Delta r} \cdot 2\pi(r + \Delta r)\Delta z = (2\pi r e_r)|_{r+\Delta r} \, \Delta z$$

$$e_z|_z \cdot 2\pi r \Delta r$$

$$e_z|_{z+\Delta z} \cdot 2\pi r \Delta r$$

- Work done on fluid by gravity

$$\rho v_z g_z \cdot 2\pi r \Delta r \Delta z$$

- Adding these terms

$$\frac{(re_r)|_r - (re_r)|_{r+\Delta r}}{\Delta r} + r\,\frac{e_z|_z - e_z|_{z+\Delta z}}{\Delta z} + \rho v_z g_z r = 0$$

- Taking limit as Δr and Δz go to zero

$$-\frac{1}{r}\frac{\partial}{\partial r}(re_r) - \frac{\partial e_z}{\partial z} + \rho v_z g = 0$$

Differential equations

$$e_r = \tau_{rz} v_z + q_r = -\left(\mu \frac{\partial v_z}{\partial r}\right) v_z - k \frac{\partial T}{\partial r}$$

$$e_z = (\tfrac{1}{2}\rho v_z^2) v_z + \rho \hat{H} v_z + \tau_{zz} v_z + q_z$$

$$= (\tfrac{1}{2}\rho v_z^2) v_z + (p - p^\circ) v_z + \rho \hat{C}_p (T - T^\circ) v_z - \left(2\mu \frac{\partial v_z}{\partial z}\right) v_z - k \frac{\partial T}{\partial z}$$

Substituting the differential equation

$$\rho \hat{C}_p v_z \frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \mu\left(\frac{\partial v_z}{\partial r}\right)^2 + v_z\left[-\frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \rho g\right]$$

Viscous heating (neglected)

The equation of motion
For the Poiseuille flow

Differential equation

$$\rho \hat{C}_p v_{z,\max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

- BCs

$$\text{at } r = 0, \qquad T = \text{finite}$$

$$\text{at } r = R, \qquad k\frac{\partial T}{\partial r} = q_0 \text{ (constant)}$$

$$\text{at } z = 0, \qquad T = T_1$$

Dimensionless equation

- Dimensionless variables

$$\Theta = \frac{T - T_1}{q_0 R/k} \qquad \xi = \frac{r}{R} \qquad \zeta = \frac{z}{\rho \hat{C}_p v_{z,\max} R^2/k}$$

- Equation and boundary conditions

$$(1 - \xi^2)\frac{\partial \Theta}{\partial \zeta} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \Theta}{\partial \xi}\right)$$

at $\xi = 0$, $\Theta = \text{finite}$

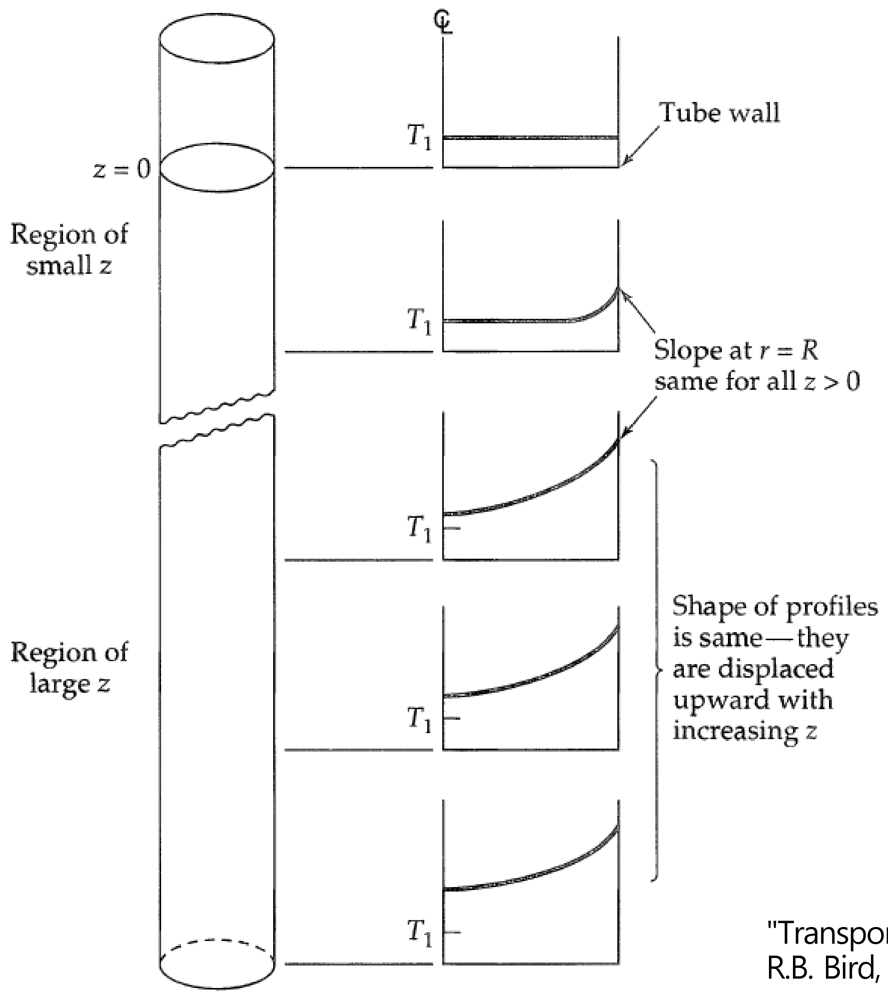
at $\xi = 1$, $\frac{\partial \Theta}{\partial \xi} = 1$

at $\zeta = 0$, $\Theta = 0$

- Asymptotic solution for large ζ
  - It is expect a linear rise of the fluid temperature in ζ
  - Constant temperature profile for large ζ

$$\Theta(\xi, \zeta) = C_0 \zeta + \Psi(\xi)$$

Temperature profile for large ζ



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Boundary conditions

- Boundary condition 3 has to be changed. New condition comes from the energy balance

$$2\pi R z q_0 = \int_0^{2\pi}\int_0^R \rho \hat{C}_p (T - T_1) v_z r \, dr \, d\theta \quad \Leftrightarrow \quad \zeta = \int_0^1 \Theta(\xi, \zeta)(1-\xi^2)\xi \, d\xi$$

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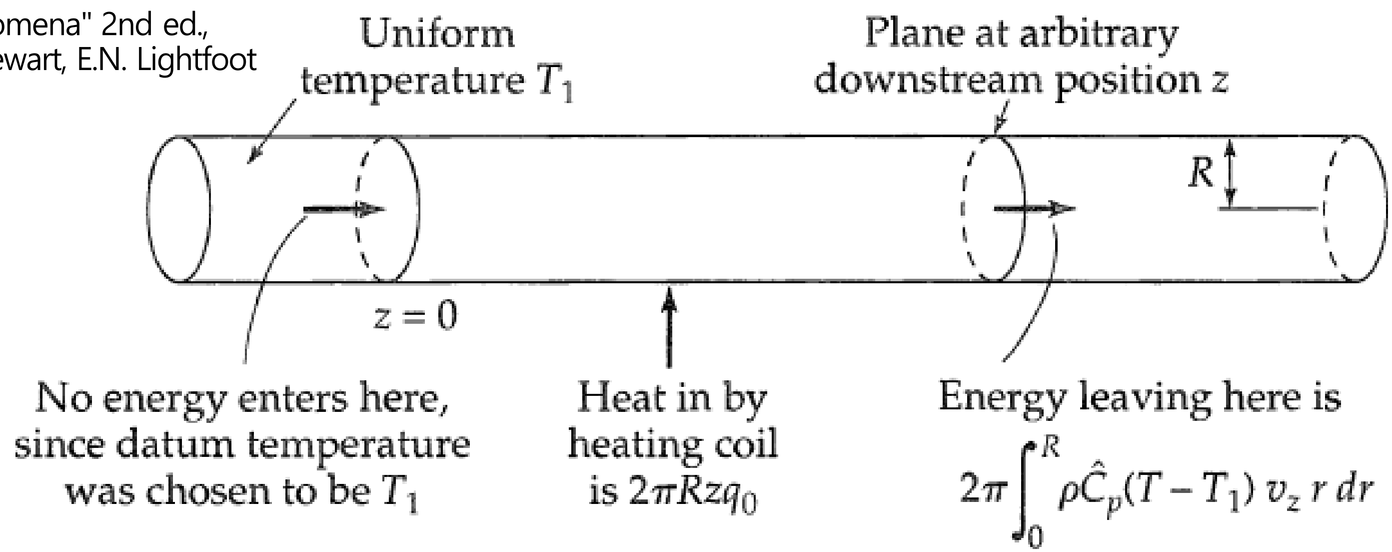

Solution

- Then, the differential equation for temperature profile

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Psi}{d\xi}\right) = C_0(1-\xi^2)$$

- Integrating twice

$$\Theta(\xi,\zeta) = C_0\zeta + C_0\left(\frac{\xi^2}{4}-\frac{\xi^4}{16}\right) + C_1 \ln \xi + C_2$$

Solution

- The integration constant are determined using the boundary conditions

$$\Theta(\xi, \zeta) = 4\zeta + \xi^2 - \tfrac{1}{4}\xi^4 - \tfrac{7}{24}$$

Arithmetic average temperature

$$\langle T \rangle = \frac{\int_0^{2\pi}\int_0^R T(r, z) r \, dr \, d\theta}{\int_0^{2\pi}\int_0^R r \, dr \, d\theta} = T_1 + (4\zeta + \tfrac{7}{24}) \frac{q_0 R}{k}$$

Bulk temperature at z

$$T_b = \frac{\langle v_z T \rangle}{\langle v_z \rangle} = \frac{\int_0^{2\pi}\int_0^R v_z(r) T(r, z) r \, dr \, d\theta}{\int_0^{2\pi}\int_0^R v_z(r) r \, dr \, d\theta} = T_1 + (4\zeta) \frac{q_0 R}{k}$$