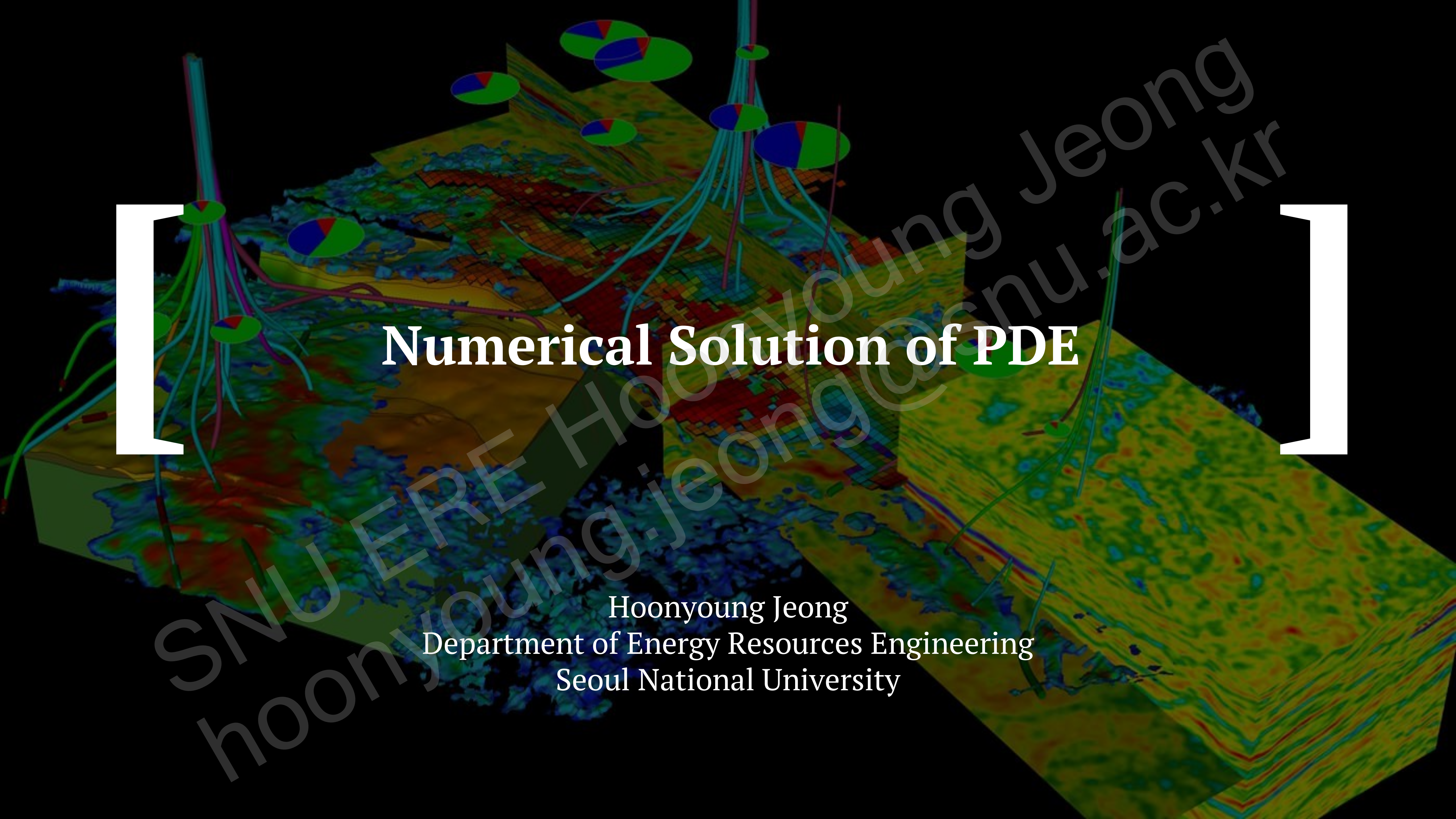

# Numerical Solution of PDE

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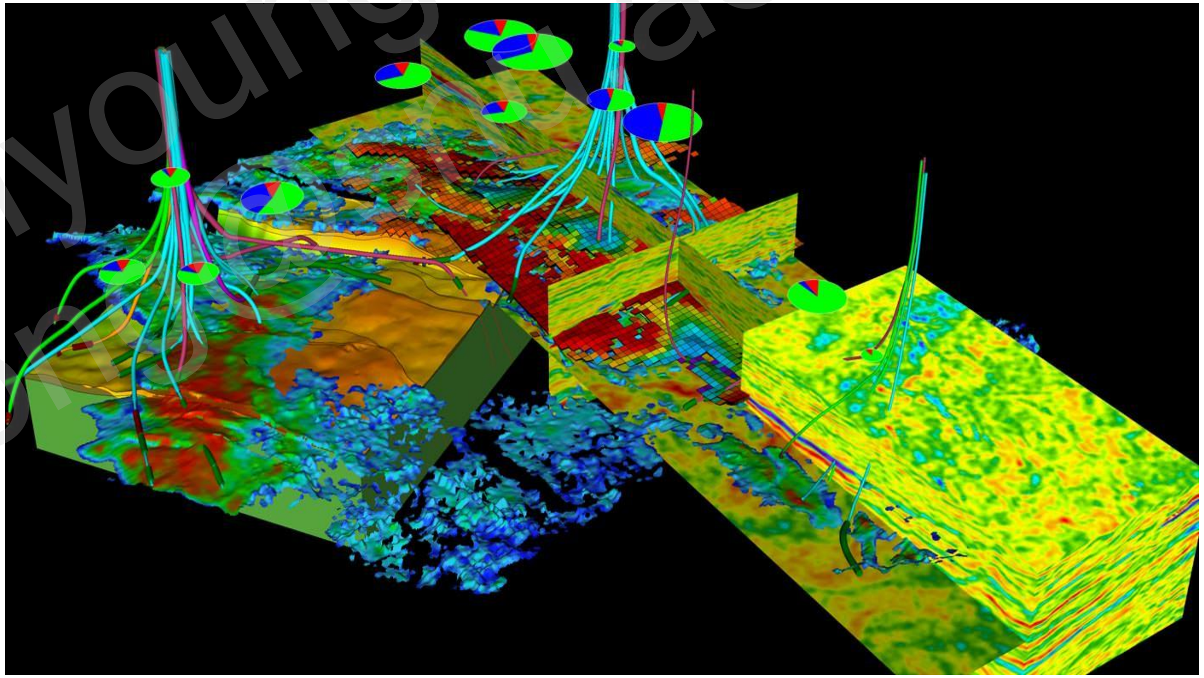
# Why Numerical Solution?

- Heterogeneous rock
  - ✓ Porosity and permeability are a function of a location
- Multiphase fluid
  - ✓ Liquid, gas
  - ✓ Multi-components
  - ✓ Immiscibility
- Fluid properties change over pressure
  - ✓ Pressure changes over location and time
  - ✓ Fluid properties change over location and time
- Complicated boundary conditions
  - ✓ Variable well operating conditions
  - ✓ Irregular reservoir shape
  - ✓ Faults
  - ✓ …



# Flow Equation

- $\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right)+\frac{Q}{A\partial x}=\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right)$
- We can't move $k, \mu, B, \phi$ out of $\partial/\partial x$ and $\partial/\partial t$
  Because $k, \mu, B, \phi$ depends on $x$ and $t$.
  - ✓$\phi, B$ : functions of $P, P(x, t)$
  - ✓Heterogeneous : k$(x)$
  - ✓$\mu, B$ : functions of $P(x, t)$
  - → $\frac{k}{\mu \mathrm{B}}$ is dependent of $x$ → f$(x)$
- Here, we assume a fluid viscosity is constant

# How to Numerically Approximate Derivatives?

$$\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right)+\frac{Q}{A\partial x}=\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right)$$

# Taylor Series

- Univariate function
  - ✓ $f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
  - ✓ You can approximate the function values near $x$ if you know $f(x)$ and the derivatives
- Multivariate function
  - ✓ $f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T\Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T\mathbf{H}(\mathbf{x})\Delta\mathbf{x} + \cdots$

# Forward Approximation of 1st Derivative

- $f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- If $\Delta x$ is small, $\Delta x^2$ is very small. So, ignore $\frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $f(x+\Delta x) = f(x) + f'(x)\Delta x$
- $f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$

# Backward Approximation of 1st Derivative

- $f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $-f'(x)\Delta x = f(x-\Delta x) - f(x) - \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $f'(x) = \frac{f(x-\Delta x)-f(x)}{-\Delta x} + \frac{f''(x)}{2!}\Delta x + \cdots$
- $f'(x) = \frac{f(x-\Delta x)-f(x)}{-\Delta x} + O(\Delta x)$

# Central Approximation of 1st Derivative

- $f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $f(x+\Delta x) - f(x-\Delta x) = 2f'(x)\Delta x + O(\Delta x^3)$
- $f'(x) = \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$

# Approximation of 2nd Derivative

- $f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots$
- $f(x+\Delta x) + f(x-\Delta x) = 2f(x) + f''(x)\Delta x^2 + O(\Delta x^4)$
- $f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$
- $f''(x) = \frac{\frac{f(x+\Delta x)-f(x)}{\Delta x} - \frac{f(x)-f(x-\Delta x)}{\Delta x}}{\Delta x} + O(\Delta x^2)$

# Smaller $\Delta x$ is Better?

- No
- Smaller $\Delta x$ makes $\partial f / \partial x$ more accurate, but if $\Delta x$ is too small, $f(x + \Delta x) - f(x)$ may be zero
- How to decide $\Delta x$?
  - ✓Not too large, not too small
  - ✓1%, 0.1%, 0.01%, … of $x$
  - ✓Depends on your problem
  - ✓Central approximation is less sensitive to a perturbation size ($\Delta x$) than forward and backward approximations