# Numerical Solution of
# 1D Single Phase Flow Equation

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# Flow Equation

- $\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right)+\frac{Q}{A\partial x}=\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right)$
- We can't move $k, \mu, B, \phi$ out of $\partial/\partial x$ and $\partial/\partial t$
  Because $k, \mu, B, \phi$ depends on $x$ and $t$.
  - ✓$\phi, B$ : functions of $P, P(x, t)$
  - ✓Heterogeneous : k$(x)$
  - ✓$\mu, B$ : functions of $P(x, t)$
  - → $\frac{k}{\mu \mathrm{B}}$ is dependent of $x$ → f$(x)$
- Here, we assume a fluid viscosity is constant

# How to Numerically Approximate Derivatives?

$$\frac{\partial}{\partial x}\left(\frac{k}{\mu B}\frac{\partial P}{\partial x}\right)+\frac{Q}{A\partial x}=\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right)$$

# Approximation of Spatial Term

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial P}{\partial x}\right]_i = \frac{f_{i+1/2}}{\Delta x^2}(P_{i+1} - P_i) + \frac{f_{i-1/2}}{\Delta x^2}(P_{i-1} - P_i)$$

$$= \frac{1}{\Delta x^2}\left(\frac{k}{\mu B}\right)_{i+1/2}(P_{i+1} - P_i) + \frac{1}{\Delta x^2}\left(\frac{k}{\mu B}\right)_{i-1/2}(P_{i-1} - P_i)$$

$$= \frac{1}{\Delta x^2}\frac{1/(\mu B)_{i+1} + 1/(\mu B)_i}{1/k_{i+1} + 1/k_i}(P_{i+1} - P_i) + \frac{1}{\Delta x^2}\frac{1/(\mu B)_{i-1} + 1/(\mu B)_i}{1/k_{i-1} + 1/k_i}(P_{i-1} - P_i)$$

This is called transmissibility $T_{i+1/2}$

$$= T_{i+1/2}(P_{i+1} - P_i) + T_{i-1/2}(P_{i-1} - P_i)$$

# Approximation of Temporal Term

$$\frac{\partial}{\partial t}\left[\frac{\phi}{B}\right]_i = \left[\frac{1}{B}\frac{\partial \phi}{\partial t} + \phi\frac{\partial(1/B)}{\partial t}\right]_i$$

$$= \left[\frac{1}{B}\frac{\partial \phi}{\partial P}\frac{\partial P}{\partial t} + \phi\frac{\partial(1/B)}{\partial P}\frac{\partial P}{\partial t}\right]_i$$

Chain rule

$$= \left[\frac{\phi C_r}{B}\frac{\partial P}{\partial t} + \frac{\phi C_f}{B}\frac{\partial P}{\partial t}\right]_i$$

$$= \left[\frac{\phi(C_r + C_f)}{B}\frac{\partial P}{\partial t}\right]_i = \left[\frac{\phi C_t}{B}\frac{\partial P}{\partial t}\right]_i$$

$$= \frac{\phi_i C_{t,i}}{B_i}\frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} = \frac{\phi_i C_{t,i}}{B_i \Delta t}\left(P_i^{t+\Delta t} - P_i^t\right) = C_{p,i}\left(P_i^{t+\Delta t} - P_i^t\right)$$

$$C_r = \frac{1}{\phi}\frac{d\phi}{dP}$$

$$C_f = \frac{1}{\rho}\frac{d\rho}{dP} = \frac{1}{constant/B}\frac{d(constant/B)}{dP} = B\frac{d(1/B)}{dP}$$

# Explicit and Implicit Methods

- $$T_{i+1/2}(P_{i+1} - P_i) + T_{i-1/2}(P_{i-1} - P_i) + \frac{Q_i}{A\Delta x_i} = C_{p,i}\left(P_i^{t+\Delta t} - P_i^t\right)$$

- $P_1^t, \cdots, P_N^t$ are given
  we know pressure at current time(t) at all grid blocks.

# Explicit Method

- Explicit : $P_{i+1}^{t}$ , $P_i^t$, $\mathrm{P}_{\mathrm{i}-1}^{\mathrm{t}}$, Only $P_i^{t+\Delta t}$ is unknown
- $T_{i+1/2}\left(P_{i+1}^{t} - P_i^{t}\right) + T_{i-1/2}\left(P_{i-1}^{t} - P_i^{t}\right) + \frac{Q_i}{A\Delta x_i} = C_{p,i}\left(P_i^{t+\Delta t} - P_i^{t}\right)$
- Very Easy to solve!
  - ✓John Von Neumann stability analysis for homogeneous rock
  - ✓$\Delta t \leq \frac{1}{2}\left(\frac{\phi\mu C_t}{k}\right)\Delta x^2$
  - ✓Conditionally Stable

# Implicit Method

- Implicit : $P_{i+1}^{t+\Delta t}, P_{i}^{t+\Delta t}, \mathrm{P}_{\mathrm{i-1}}^{t+\Delta t}$
  - ✓Should solve inversion of a matrix
  - ✓but theoretically stable,
  - ✓but unstable for too large $\Delta t, \Delta x$,
  - ✓but much more stable than explicit
- $T_{i+1/2}\left(P_{i+1}^{t+\Delta t} - P_{i}^{t+\Delta t}\right) + T_{i-1/2}\left(P_{i-1}^{t+\Delta t} - P_{i}^{t+\Delta t}\right) + \frac{Q_i}{A\Delta x_i} = C_{p,i}\left(P_{i}^{t+\Delta t} - P_{i}^{t}\right)$

# Formulation of Implicit Method

Implicit

$$T_{i+1/2}\left(P_{i+1}^{t+\Delta t} - P_i^{t+\Delta t}\right) + T_{i-1/2}\left(P_{i-1}^{t+\Delta t} - P_i^{t+\Delta t}\right) + \frac{Q_i}{A\Delta x_i}$$

$$= C_{p,i}\left(P_i^{t+\Delta t} - P_i^t\right)$$

$$a_i P_{i-1}^{t+\Delta t} + b_i P_i^{t+\Delta t} + c_i P_{i+1}^{t+\Delta t} = d_i\,, \qquad i = 1,2,\ldots,N$$ (# of grid blocks)

$$T_{i-1/2} P_{i-1}^{t+\Delta t} + \left(-T_{i+1/2} - T_{i-1/2} - C_{p,i}\right) P_i^{t+\Delta t} + T_{i+1/2} P_{i+1}^{t+\Delta t}$$

$a_i$ $b_i$ $c_i$

$$= -C_{p,i} P_i^t - Q_i / A\Delta x$$ $d_i$

$$T_{i+1/2} = \frac{1}{\Delta x^2}\frac{1/(\mu B)_{i+1} + 1/(\mu B)_i}{1/k_{i+1} + 1/k_i}\,, \qquad Q_i \begin{cases} injection: + \\ production: - \end{cases}$$

$$C_{p,i} = \frac{\phi_i C_{t,i}}{B_i \Delta t}\,, \qquad C_{t,i} = C_r + C_f$$

# Simultaneous Linear Equations

- Assume we know $P_i^t$, calculate $P_i^{t+\Delta t}$

$$i = 1,2,3,4,5$$

$$
\begin{aligned}
i = 1, \quad & a_1 P_0 + b_1 P_1 + c_1 P_2 = d_1 \\
i = 2, \quad & a_2 P_1 + b_2 P_2 + c_2 P_3 = d_2 \\
i = 3, \quad & a_3 P_2 + b_3 P_3 + c_3 P_4 = d_3 \\
i = 4, \quad & a_4 P_3 + b_4 P_4 + c_4 P_5 = d_4 \\
i = 5, \quad & a_5 P_4 + b_5 P_5 + c_5 P_6 = d_5
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 \\ 0 & 0 & 0 & a_5 & b_5 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}}_{x}
=
\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}}_{d}
$$

$$x = A^{-1} d$$