

A 3D visualization of a reservoir with multiple wells. The reservoir is shown as a layered block with a color gradient from blue to green. Several wells are shown as vertical lines with pie charts at the top, representing different fluid phases. The pie charts are divided into three colors: blue, green, and red. The background is dark, and the entire scene is framed by large white brackets on the left and right sides.

Numerical Solution of 1D Single Phase Flow Equation

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Flow Equation

- $\frac{\partial}{\partial x} \left(\frac{k}{\mu B} \frac{\partial P}{\partial x} \right) + \frac{Q}{A \partial x} = \frac{\partial}{\partial t} \left(\frac{\phi}{B} \right)$
- We can't move k, μ, B, ϕ out of $\partial / \partial x$ and $\partial / \partial t$
Because k, μ, B, ϕ depends on x and t .
 - ✓ ϕ, B : functions of $P, P(x, t)$
 - ✓ Heterogeneous : $k(x)$
 - ✓ μ, B : functions of $P(x, t)$
 - $\frac{k}{\mu B}$ is dependent of x → $f(x)$
- Here, we assume a fluid viscosity is constant

How to Numerically Approximate Derivatives?

$$\frac{\partial}{\partial x} \left(\frac{k}{\mu B} \frac{\partial P}{\partial x} \right) + \frac{Q}{A \partial x} = \frac{\partial}{\partial t} \left(\frac{\phi}{B} \right)$$

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Approximation of Spatial Term

$$\begin{aligned}\frac{\partial}{\partial x} \left[f(x) \frac{\partial P}{\partial x} \right]_i &= \frac{f_{i+1/2}}{\Delta x^2} (P_{i+1} - P_i) + \frac{f_{i-1/2}}{\Delta x^2} (P_{i-1} - P_i) \\ &= \frac{1}{\Delta x^2} \left(\frac{k}{\mu B} \right)_{i+1/2} (P_{i+1} - P_i) + \frac{1}{\Delta x^2} \left(\frac{k}{\mu B} \right)_{i-1/2} (P_{i-1} - P_i) \\ &= \frac{1}{\Delta x^2} \frac{1/(\mu B)_{i+1} + 1/(\mu B)_i}{1/k_{i+1} + 1/k_i} (P_{i+1} - P_i) + \frac{1}{\Delta x^2} \frac{1/(\mu B)_{i-1} + 1/(\mu B)_i}{1/k_{i-1} + 1/k_i} (P_{i-1} - P_i)\end{aligned}$$

This is called transmissibility $T_{i+1/2}$

$$= T_{i+1/2} (P_{i+1} - P_i) + T_{i-1/2} (P_{i-1} - P_i)$$

Approximation of Temporal Term

$$\begin{aligned}\frac{\partial}{\partial t} \left[\frac{\phi}{B} \right]_i &= \left[\frac{1}{B} \frac{\partial \phi}{\partial t} + \phi \frac{\partial(1/B)}{\partial t} \right]_i \\ &= \left[\frac{1}{B} \frac{\partial \phi}{\partial P} \frac{\partial P}{\partial t} + \phi \frac{\partial(1/B)}{\partial P} \frac{\partial P}{\partial t} \right]_i \\ &= \left[\frac{\phi C_r}{B} \frac{\partial P}{\partial t} + \frac{\phi C_f}{B} \frac{\partial P}{\partial t} \right]_i \\ &= \left[\frac{\phi(C_r + C_f)}{B} \frac{\partial P}{\partial t} \right]_i = \left[\frac{\phi C_t}{B} \frac{\partial P}{\partial t} \right]_i \\ &= \frac{\phi_i C_{t,i}}{B_i} \frac{P_i^{t+\Delta t} - P_i^t}{\Delta t} = \frac{\phi_i C_{t,i}}{B_i \Delta t} (P_i^{t+\Delta t} - P_i^t) = C_{p,i} (P_i^{t+\Delta t} - P_i^t)\end{aligned}$$

$C_r = \frac{1}{\phi} \frac{d\phi}{dP}$

$C_f = \frac{1}{\rho} \frac{d\rho}{dP} = \frac{1}{\text{constant}/B} \frac{d(\text{constant}/B)}{dP} = B \frac{d(1/B)}{dP}$

Chain rule

Explicit and Implicit Methods

- $$T_{i+1/2}(P_{i+1} - P_i) + T_{i-1/2}(P_{i-1} - P_i) + \frac{Q_i}{A\Delta x_i}$$
$$= C_{p,i}(P_i^{t+\Delta t} - P_i^t)$$

- P_1^t, \dots, P_N^t are given
we know pressure at current time(t) at all grid blocks.

Explicit Method

- Explicit : $P_{i+1}^t, P_i^t, P_{i-1}^t$, Only $P_i^{t+\Delta t}$ is unknown
- $T_{i+1/2}(P_{i+1}^t - P_i^t) + T_{i-1/2}(P_{i-1}^t - P_i^t) + \frac{Q_i}{A\Delta x_i} = C_{p,i}(P_i^{t+\Delta t} - P_i^t)$
- Very Easy to solve!
 - ✓ John Von Neumann stability analysis for homogeneous rock
 - ✓ $\Delta t \leq \frac{1}{2} \left(\frac{\phi \mu C_t}{k} \right) \Delta x^2$
 - ✓ Conditionally Stable

Implicit Method

- Implicit : $P_{i+1}^{t+\Delta t}$, $P_i^{t+\Delta t}$, $P_{i-1}^{t+\Delta t}$
 - ✓ Should solve inversion of a matrix
 - ✓ but theoretically stable,
 - ✓ but unstable for too large $\Delta t, \Delta x$,
 - ✓ but much more stable than explicit

$$\begin{aligned} & T_{i+1/2} (P_{i+1}^{t+\Delta t} - P_i^{t+\Delta t}) + T_{i-1/2} (P_{i-1}^{t+\Delta t} - P_i^{t+\Delta t}) + \frac{Q_i}{A\Delta x_i} = \\ & C_{p,i} (P_i^{t+\Delta t} - P_i^t) \end{aligned}$$

Formulation of Implicit Method

Implicit

$$T_{i+1/2}(P_{i+1}^{t+\Delta t} - P_i^{t+\Delta t}) + T_{i-1/2}(P_{i-1}^{t+\Delta t} - P_i^{t+\Delta t}) + \frac{Q_i}{A\Delta x_i}$$

$$= C_{p,i}(P_i^{t+\Delta t} - P_i^t)$$

$$a_i P_{i-1}^{t+\Delta t} + b_i P_i^{t+\Delta t} + c_i P_{i+1}^{t+\Delta t} = d_i, \quad i = 1, 2, \dots, N \quad (\# \text{ of grid blocks})$$

$$a_i T_{i-1/2} P_{i-1}^{t+\Delta t} + (-T_{i+1/2} - T_{i-1/2} - C_{p,i}) P_i^{t+\Delta t} + T_{i+1/2} P_{i+1}^{t+\Delta t} = -C_{p,i} P_i^t - Q_i/A\Delta x \quad d_i$$

$$T_{i+1/2} = \frac{1}{\Delta x^2} \frac{1/(\mu B)_{i+1} + 1/(\mu B)_i}{1/k_{i+1} + 1/k_i}, \quad Q_i \begin{cases} \text{injection : +} \\ \text{production : -} \end{cases}$$

$$C_{p,i} = \frac{\phi_i C_{t,i}}{B_i \Delta t}, \quad C_{t,i} = C_r + C_f$$

Simultaneous Linear Equations

- Assume we know P_i^t , calculate $P_i^{t+\Delta t}$

$$i = 1, 2, 3, 4, 5$$

$$i = 1, \quad a_1 P_0 + b_1 P_1 + c_1 P_2 = d_1$$

$$i = 2, \quad a_2 P_1 + b_2 P_2 + c_2 P_3 = d_2$$

$$i = 3, \quad a_3 P_2 + b_3 P_3 + c_3 P_4 = d_3$$

$$i = 4, \quad a_4 P_3 + b_4 P_4 + c_4 P_5 = d_4$$

$$i = 5, \quad a_5 P_4 + b_5 P_5 + c_5 P_6 = d_5$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 \\ 0 & 0 & 0 & a_5 & b_5 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}$$

A

x

d

$$x = A^{-1}d$$