Abstract Priority Queues

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Outline

- This topic will:
  - Review queues
  - Discuss the concept of priority and priority queues
  - Look at two simple implementations:
    - Arrays of queues
    - AVL trees
  - Introduce heaps, an alternative tree structure which has better run-time characteristics

Background

- We have discussed Abstract Lists
  - Arrays, linked lists

- We saw three cases which restricted the operations:
  - Stacks, queues, deques

- Then, we studied search trees: Abstract Sorted Lists
  - Run times were generally $\Theta(\ln(n))$

- We will now look :
  - Priority queues
  - Restriction on Abstracted Sorted Lists

Definition

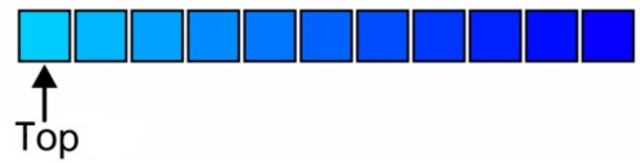
- With queues
  - The order may be summarized by *first in, first out*

- If each object is associated with a priority, we may wish to pop that object which has highest priority

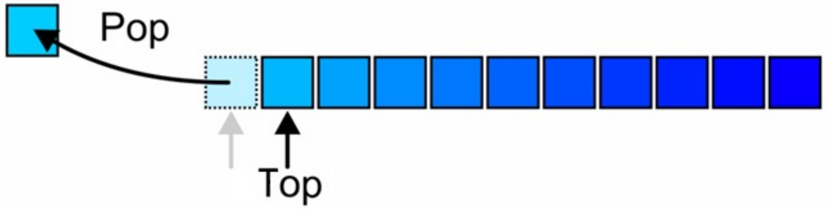
- With each pushed object, we will associate a nonnegative integer (0, 1, 2, …) where:
  - The value 0 has the *highest* priority, and
  - The higher the number, the lower the priority

Operations

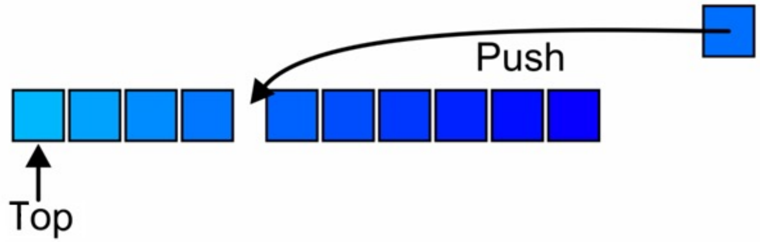
- The top of a priority queue is the object with highest priority



- Popping from a priority queue removes the current highest priority object:



- Push places a new object into the appropriate place



Process Priority in Linux

- This is the scheme used by Linux, *e.g.*,

```
% nice -15 ./a.out
```

sets the priority of the execution of a.out as -15

(priority range [-20 20], -20: the highest, 20: the lowest)

- The kernel will schedule processes according to the priority

$ man nice

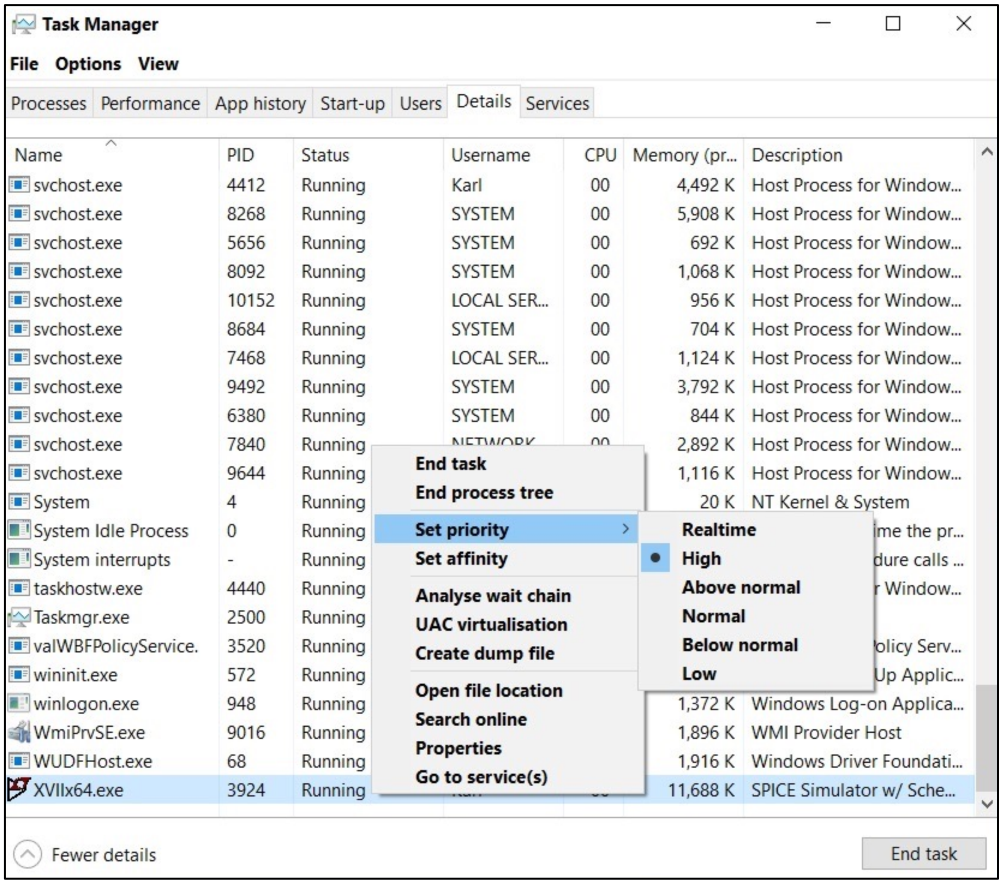
```
NICE(1)                                                    User Com
ands                                                       NICE(1)

NAME
       nice - run a program with modified scheduling priority

SYNOPSIS
       nice [OPTION] [COMMAND [ARG]...]

DESCRIPTION
       Run COMMAND with an adjusted niceness, which affects process scheduling.  With no COMMAND, print the current niceness.
Niceness values range from -20 (most favorable to the process) to 19 (least favorable to the process).
```

Process Priority in Windows



Implementations

- Our goal is to make the run time of each operation as close to $\Theta(1)$ as possible

- We will look at two naïve implementations using data structures we already know:
  - Multiple queues—one for each priority
  - An AVL tree

Multiple Queues

- Assume there is a fixed number of priorities, say M
  - Create an array of M queues
  - Push a new object onto the queue corresponding to the priority
  - Top and pop find the first empty queue with highest priority

Multiple Queues

- The run times are reasonable:
  - Push is $\Theta(1)$
  - Top and pop are both $O(M)$

- Unfortunately:
  - It restricts the range of priorities
  - The memory requirement is $\Theta(M + n)$

AVL Trees

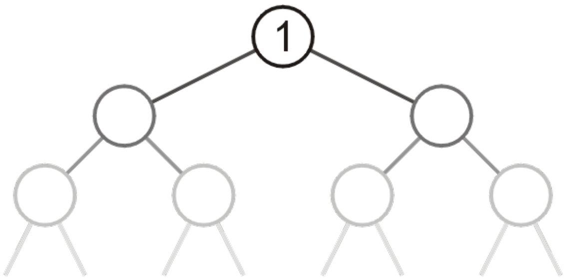
- We could simply insert the objects into an AVL tree where the order is given by the stated priority:
  - Insertion is $\Theta(\ln(n))$
  - Top is $\Theta(\ln(n))$
  - Remove is $\Theta(\ln(n))$

- There is significant overhead for maintaining both the tree and the corresponding balance

Better Idea: Heaps

- Can we do better?
  - That is, can we reduce some (or all) of the operations down to $\Theta(1)$?

- The next topic defines a *heap*
  - A tree with the top object at the root
  - We will look at binary heaps
  - Numerous other heaps exists:
    - *d*-ary heaps
    - Leftist heaps
    - Skew heaps
    - Binomial heaps
    - Fibonacci heaps
    - Bi-parental heaps



Summary

- This topic:
  - Introduced priority queues
  - Considered two obvious implementations:
    - Arrays of queues
    - AVL trees
  - Discussed the run times and claimed that a variation of a tree, a heap, can do better

References

[1] Cormen, Leiserson, Rivest and Stein, *Introduction to Algorithms*, The MIT Press, 2001, §6.5.

[2] Mark A. Weiss, *Data Structures and Algorithm Analysis in C++*, 3rd Ed., Addison Wesley, 2006.

[3] Joh Kleinberg and Eva Tardos, Algorithm Design, Pearson, 2006, §2.5.

[4] Elliot B. Koffman and Paul A.T. Wolfgang, *Objects, Abstractions, Data Structures and Design using C++*, Wiley, 2006, §8.5.

Binary Heaps

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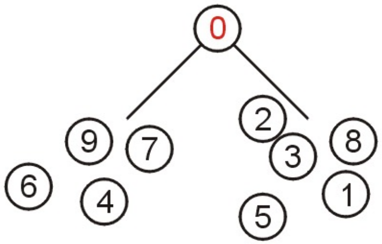
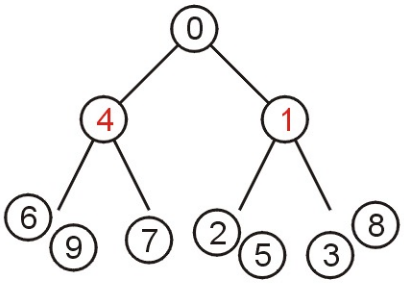
Outline

- In this topic, we will:
  - Define a binary min-heap
  - Look at some examples
  - Operations on heaps:
    - Top
    - Pop
    - Push
  - An array representation of heaps
  - Define a binary max-heap
  - Using binary heaps as priority queues

Definition

- A non-empty binary tree is a min-heap if
  - The key of the root is less than or equal to all the keys in both sub-trees
  - Both of the sub-trees (if any) are also binary min-heaps

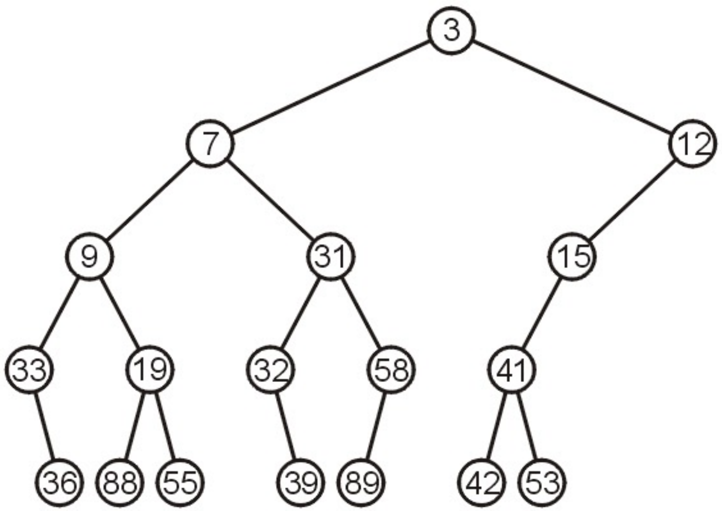




- From this definition:
  - A single node is a min-heap
  - All keys in either sub-tree are greater than the root key

Example

- This is a binary min-heap:

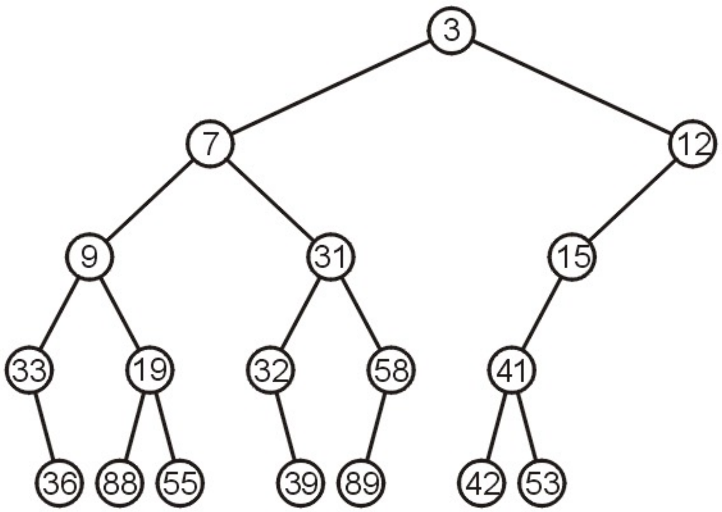


Operations

- We will consider three operations:
  - Top
  - Pop
  - Push

Top

- We can find the top object in $\Theta(1)$ time: 3

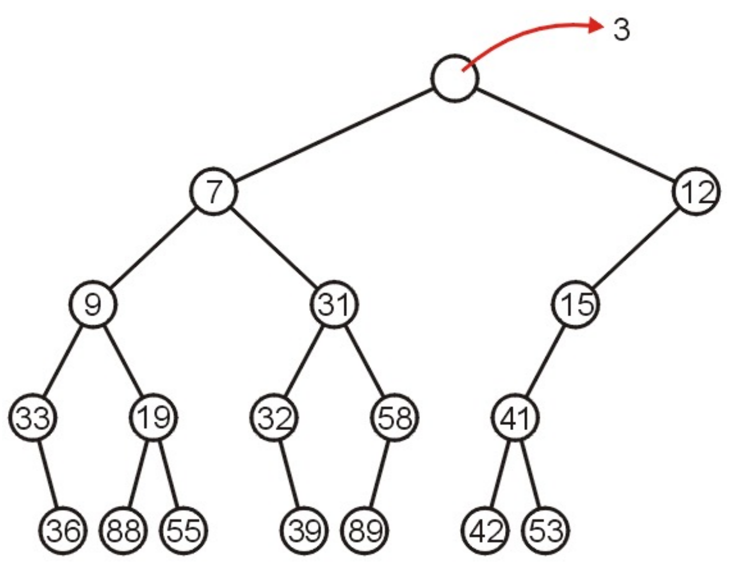


Pop

- To remove the minimum object:
  - Promote the node of the sub-tree which has the least value
  - Recurs down the sub-tree from which we promoted the least value

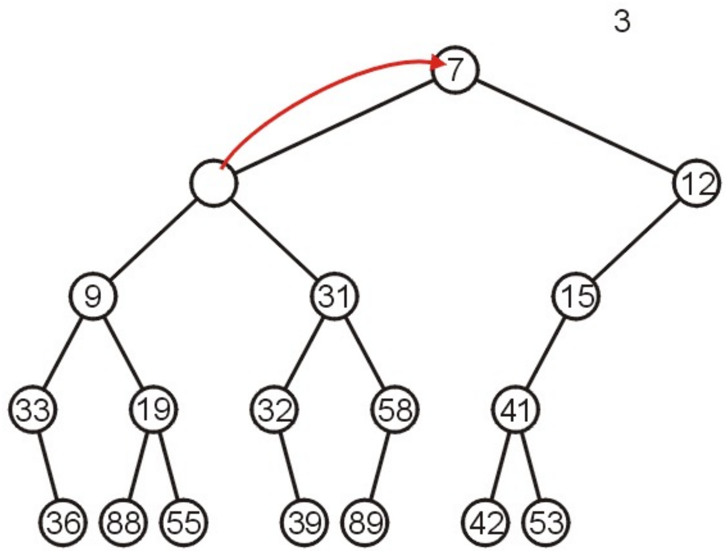
Pop: 3

- Using our example, we remove 3:



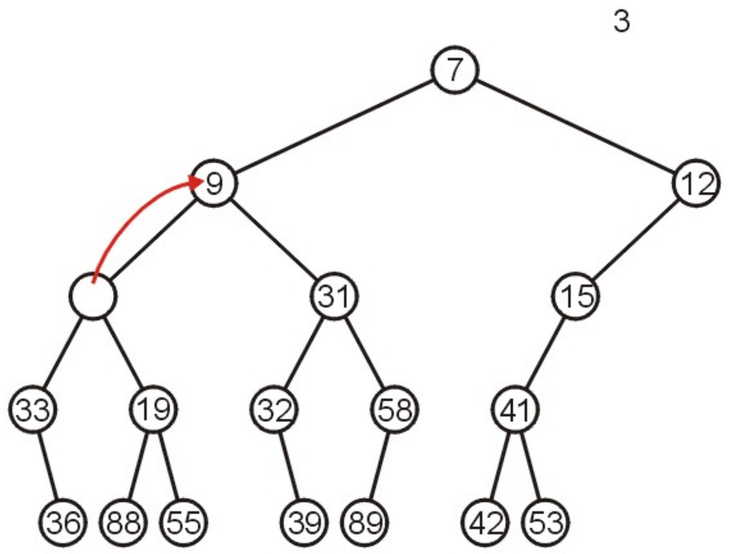
Pop: 3

- We promote 7 (the minimum of 7 and 12) to the root:



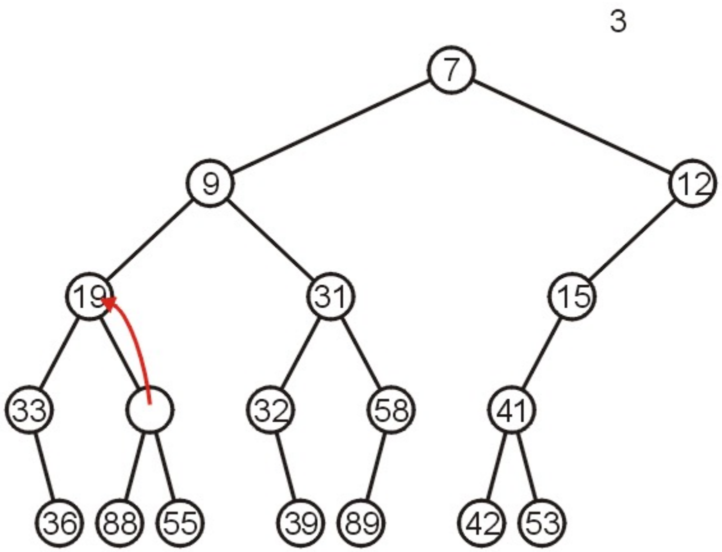
Pop: 3

- In the left sub-tree, we promote 9:



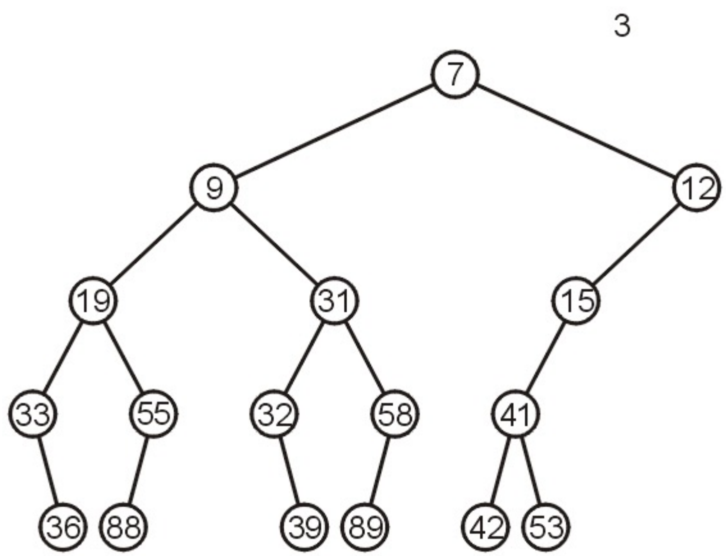
Pop: 3

- Recursively, we promote 19:



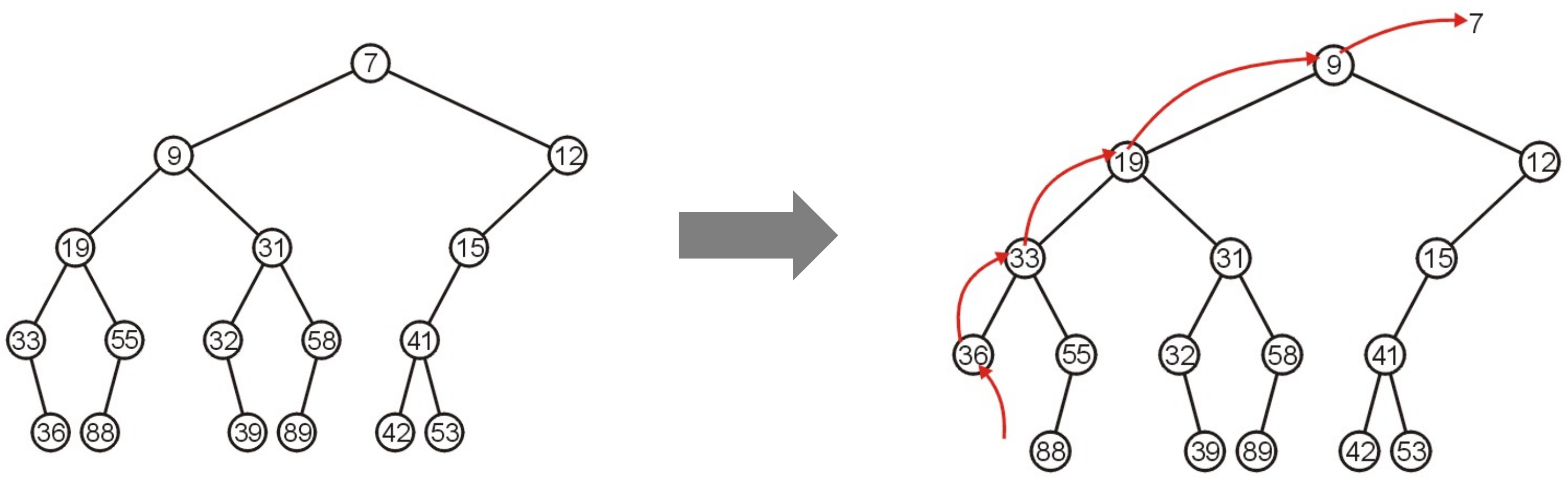
Pop: 3

- Finally, 55 is a leaf node, so we promote it and delete the leaf



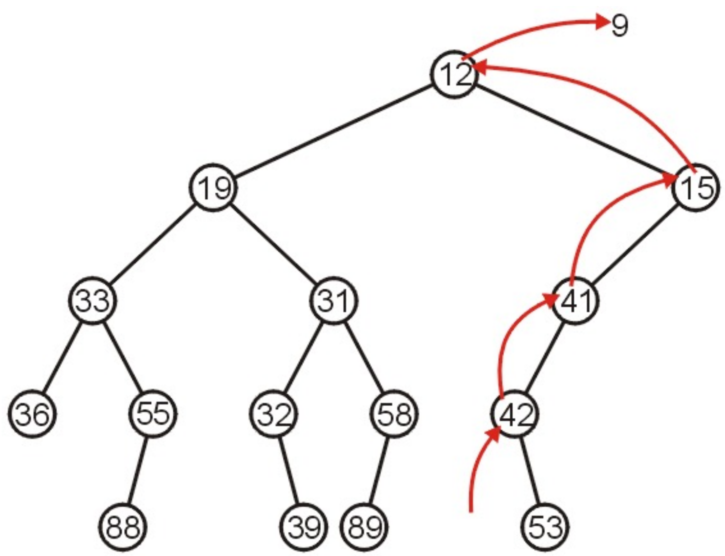
Pop: 7

- Repeating this operation again, we can remove 7:



Pop: 9

- If we remove 9, we must now promote from the right sub-tree:



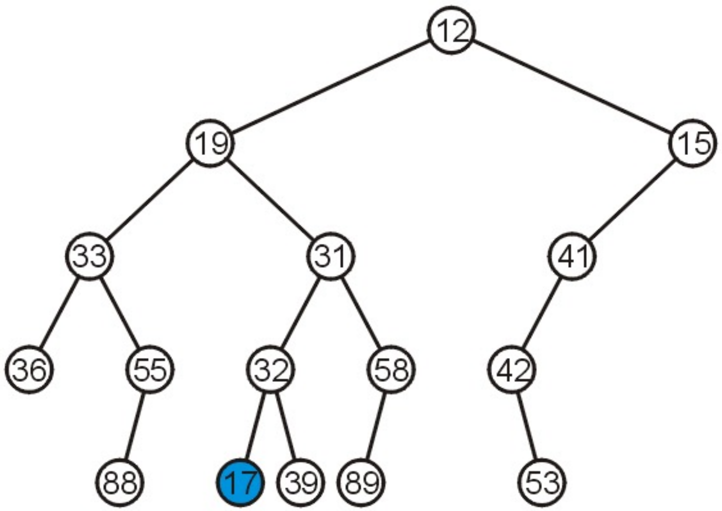
Push

- Inserting into a heap may be done either:
  - **Bottom-up:** At a leaf (move it up if it is smaller than the parent)
  - **Top-down:** At the root (insert the larger object into one of the subtrees)

- We will use **the bottom-up approach** with binary heaps

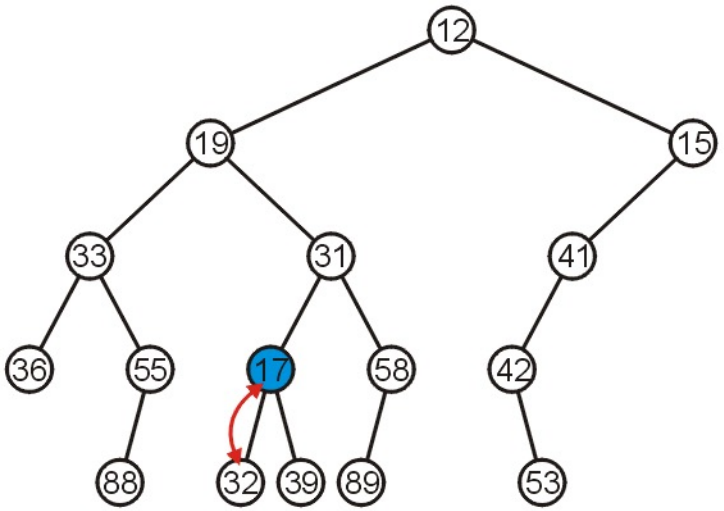
Push: 17

- Inserting 17 into the last heap
  - Select an arbitrary node to insert a new leaf node:



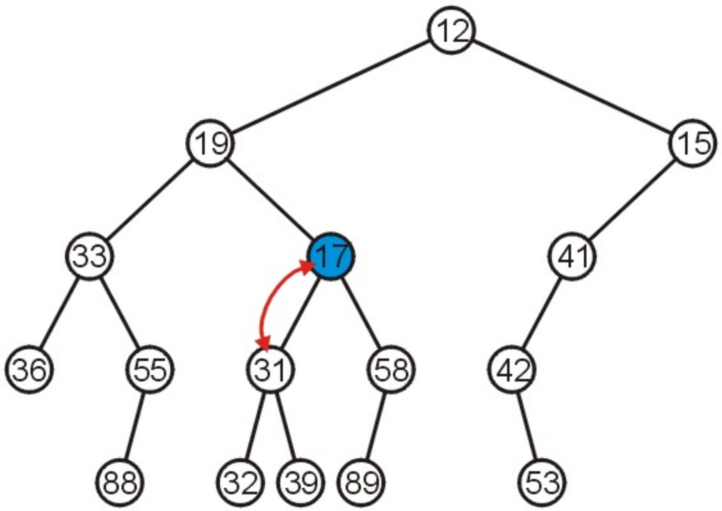
Push: 17

- The node 17 is less than the node 32, so we swap them



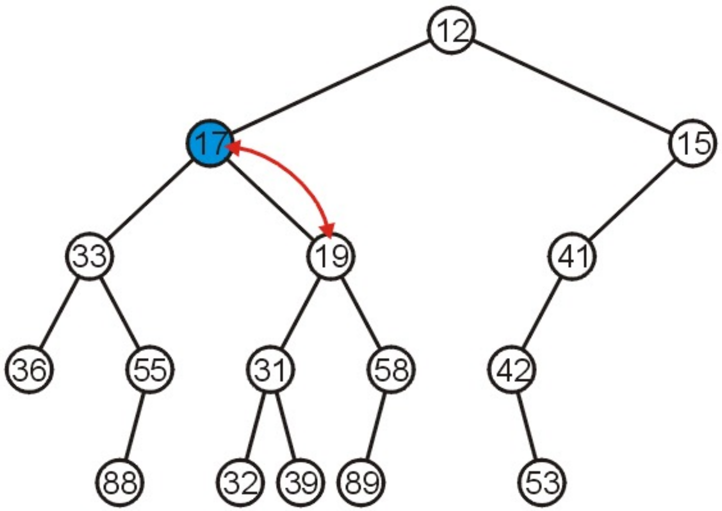
Push: 17

- The node 17 is less than the node 31; swap them



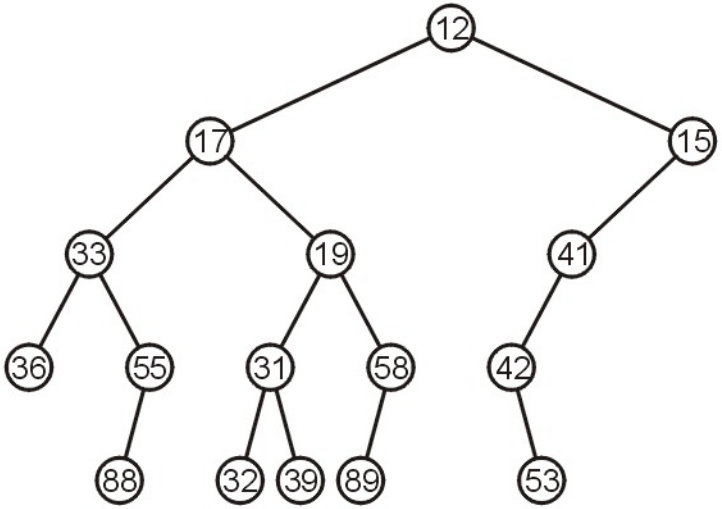
Push: 17

- The node 17 is less than the node 19; swap them



Push: 17

- The node 17 is greater than 12 so we are finished



Push Observation: One-way Percolation up/down

- Observation: both the left and right subtrees of 19 were greater than 19, thus we are guaranteed that we don't have to send the new node down (to the other subtree)

- This process is called *percolation up*, that is, the lighter (smaller) objects move up from the bottom of the min-heap

Keeping Balance

- With binary search trees, we introduced the concept of *balance*
  - AVL Trees
  - B-Trees
  - Red-black Trees

- How do we maintain the balance of binary heap?

Easy Solution: Complete Tree

- To keep the balance, we maintain the shape of complete tree structure

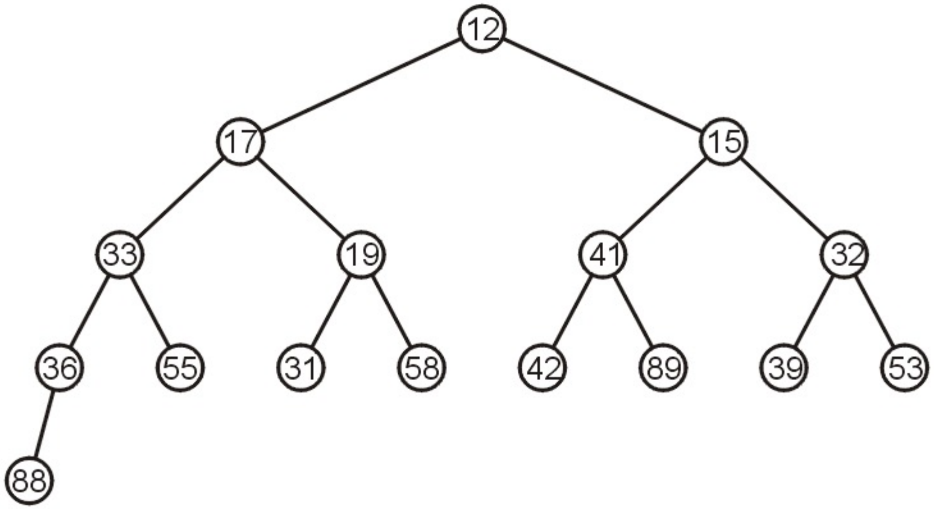
- We have already seen
  - It is easy to store a complete tree as an array

- If we can store a heap of size *n* as an array of size $\Theta(n)$, this would be great!

- We now need to think about how to support push and pop.

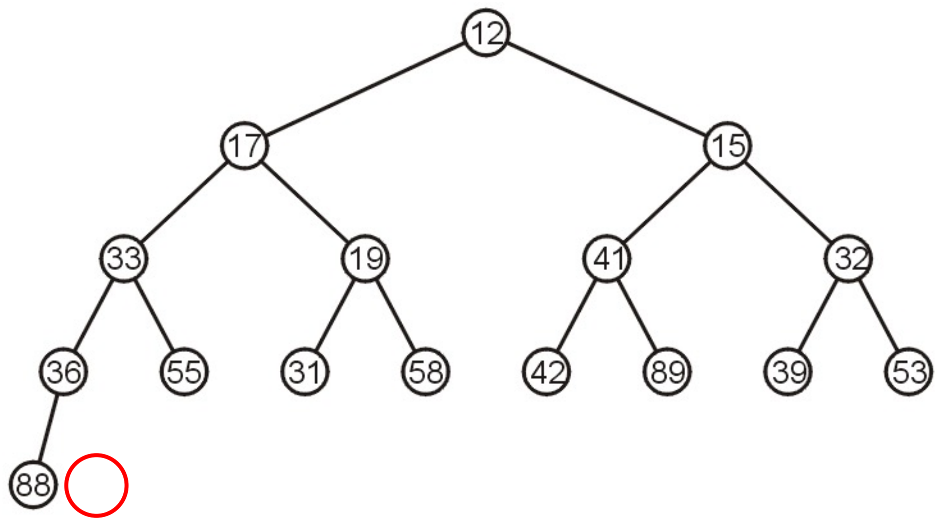
Complete Trees

- For example, the previous heap may be represented as the following complete tree:



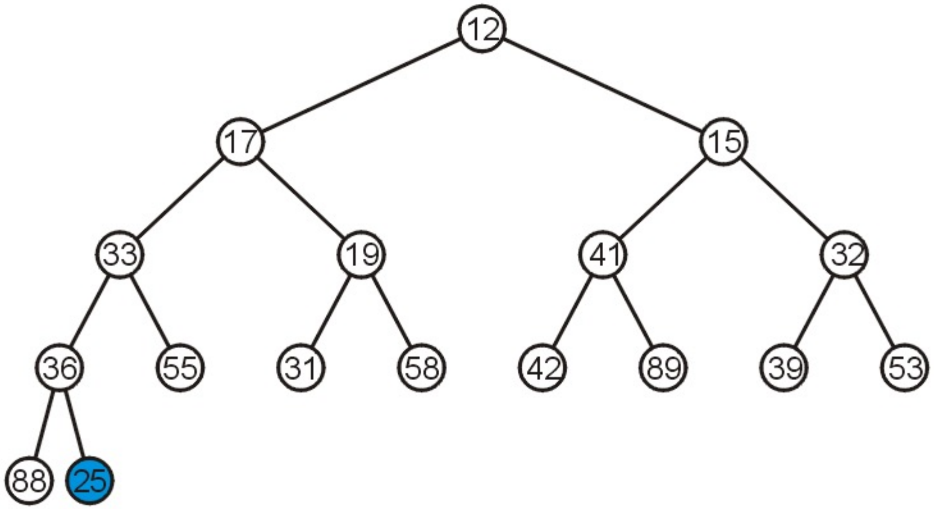
Complete Trees: Push

- If we insert into a complete tree, we only need to place the new node as a leaf node in the appropriate location and percolate up



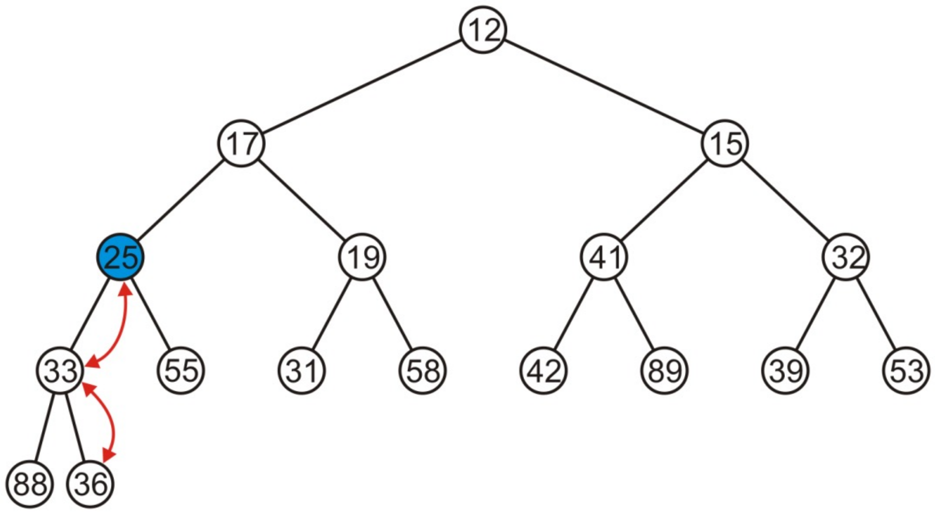
Complete Trees: Push

- For example, push 25:



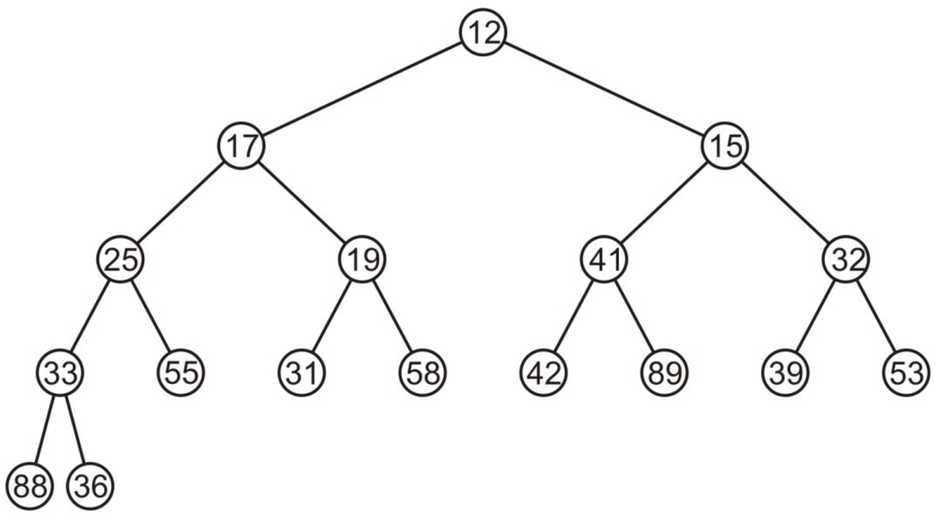
Complete Trees: Push

- We have to percolate 25 up into its appropriate location
  - The resulting heap is still a complete tree



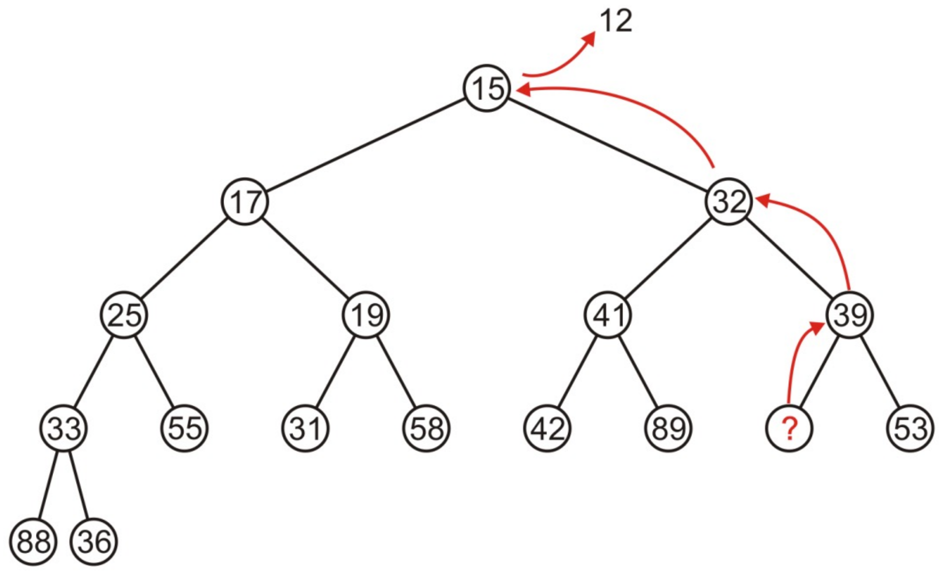
Complete Trees: Pop

- Suppose we want to pop the top entry: 12



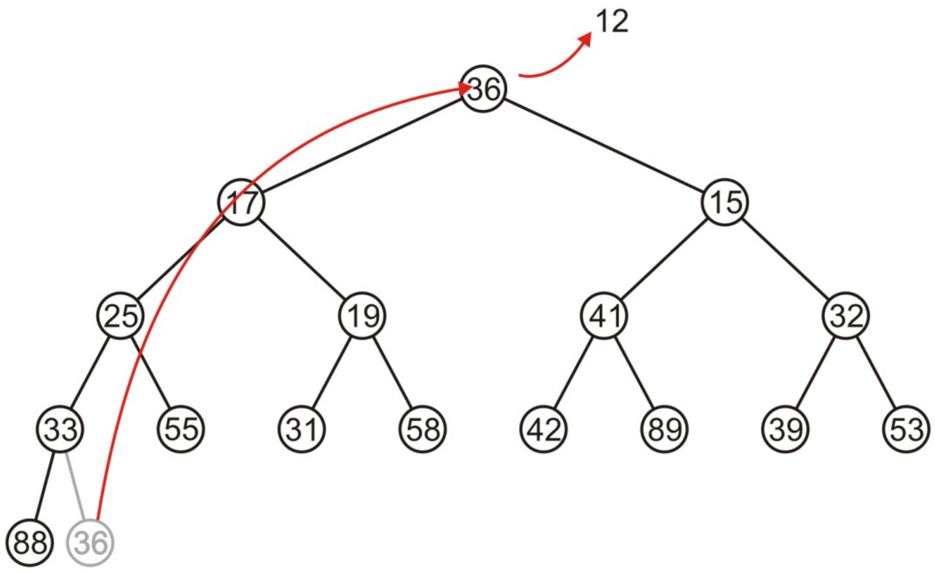
Complete Trees: Pop

- Percolating up creates a hole leading to a non-complete tree



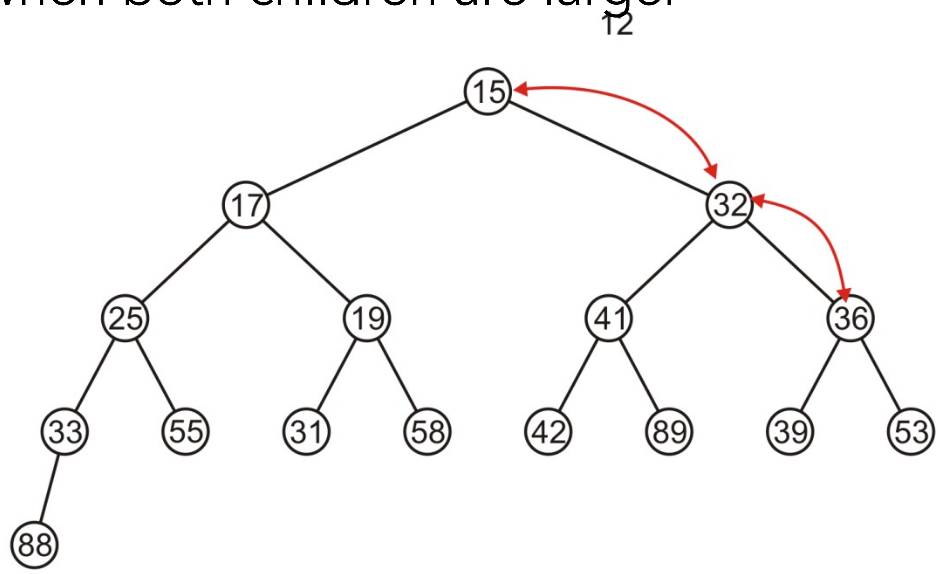
Complete Trees: Pop

- Alternatively, copy the last entry in the heap to the root



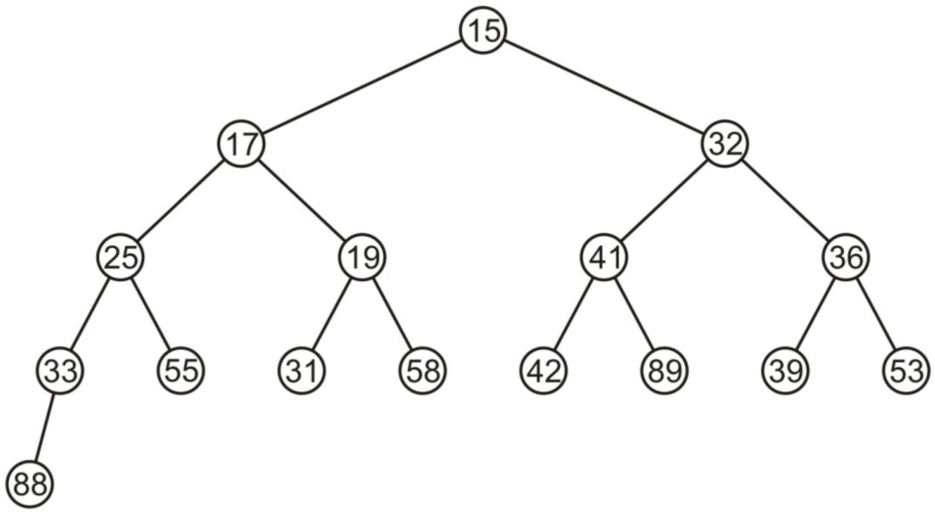
Complete Trees: Pop

- Now, percolate 36 down swapping it with the smallest of its children
  - We halt when both children are larger



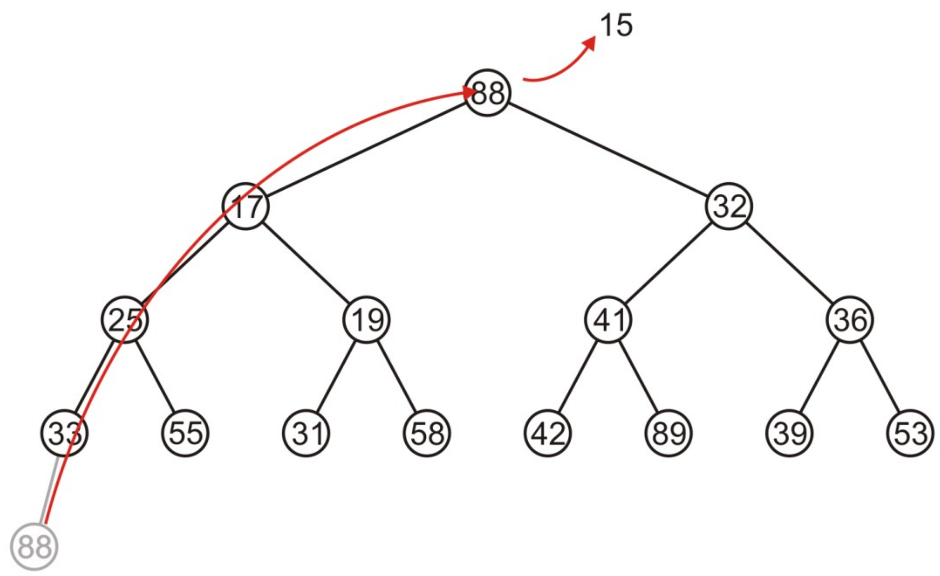
Complete Trees: Pop

- The resulting tree is now still a complete tree:



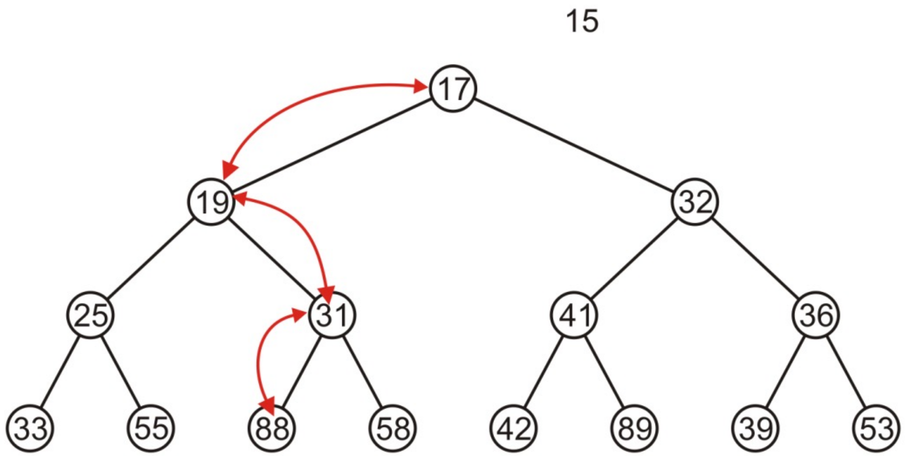
Complete Trees: Pop

- Again, popping 15, copy up the last entry: 88



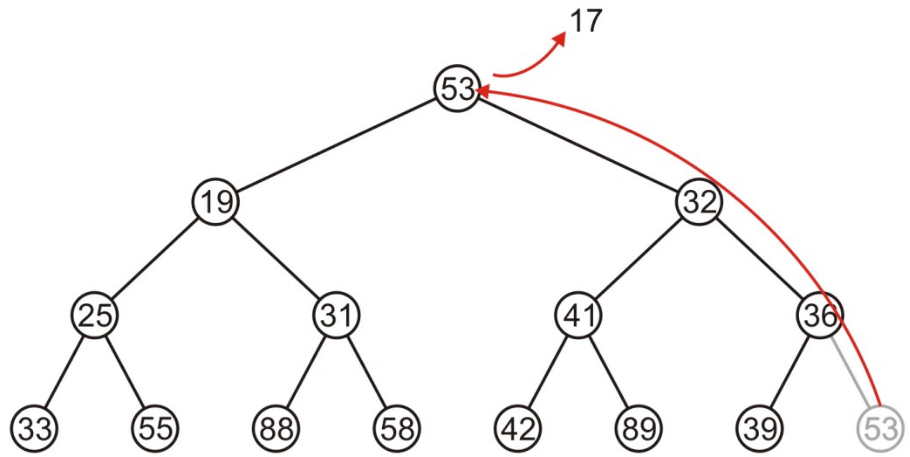
Complete Trees: Pop

- This time, it gets percolated down to the point where it has no children



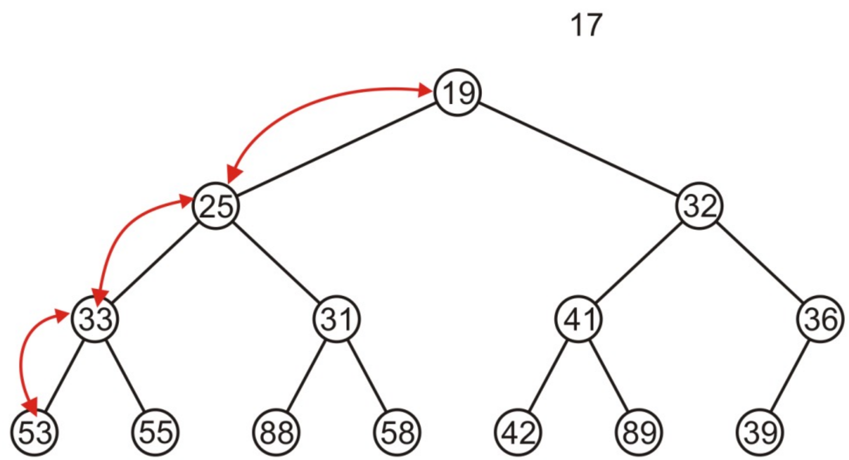
Complete Trees: Pop

- In popping 17, 53 is moved to the top



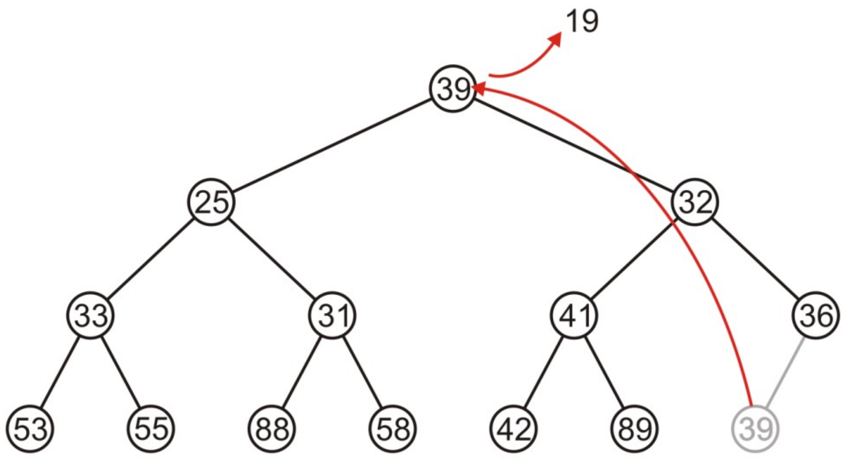
Complete Trees: Pop

- And percolated down, again to the deepest level



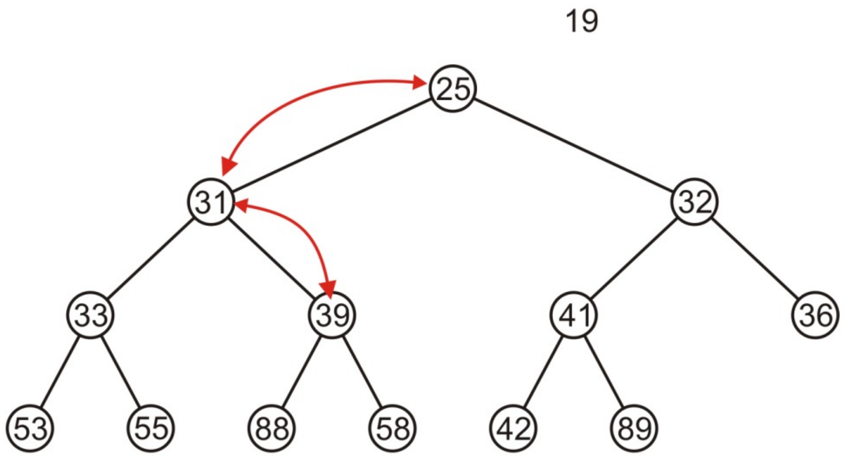
Complete Trees: Pop

- Popping 19 copies up 39



Complete Trees: Pop

- Which is then percolated down to the second deepest level



Run-time Analysis

- Accessing the top object is $\Theta(1)$

- Popping the top object is $O(\ln(n))$
  - We copy something that is already in the lowest depth—it will likely be moved back to the lowest depth

- How about push?

Run-time Analysis

- Recall our insertion works bottom-up (percolation up)

- **Worst case**: If we are inserting an object less than the root (at the front), then the run time will be **O**(ln(n))

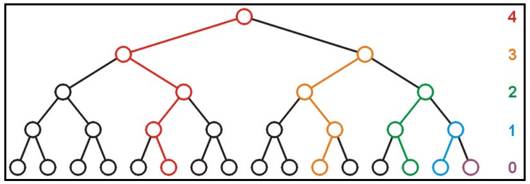
- **Best case**: If we insert an object greater than any object (at the back), then the run time will be **O**(1)

- **Average Case?** This is tricky to answer
  - Will it be **O**(ln(n))?

Run-time Analysis

- Assumption
  - Previously inserted n values were drawn from a distribution $\mathbf{U}$
  - To be inserted value x is also drawn from the same distribution $\mathbf{U}$

- Analysis
  - $n/2$ nodes are at height h (the leaves)
    - At the ½ probability, x is less than $n/2$ nodes
    - At the ½ probability, we need at least one percolation up
  - $n/4$ nodes are at height h-1
    - At the ¼ probability, x is less than $n/4$ nodes
    - At the ¼ probability, we need at least two percolation up
  - …
  - 1 node is at height 0 (the root)



So the expected number of percolation up is

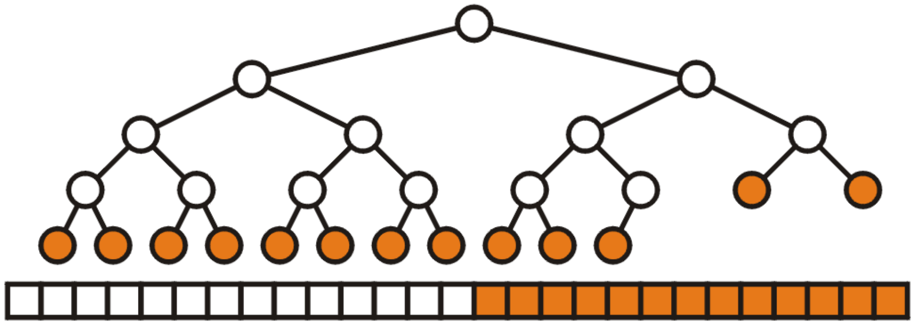
$$\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 3+\cdots=\sum_{k=1}^{\infty}\frac{k}{2^k}=2$$

Therefore, we have an average run time of O(1)

Run-time Analysis

- An arbitrary removal requires that all entries in the heap be checked: O(n)

- A removal of the largest object in the heap still requires all leaf nodes to be checked – there are approximately $n/2$ leaf nodes: O(n)



Run-time Analysis

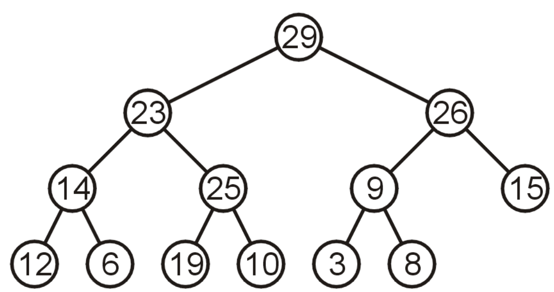
- To summarize, our grid of run times is given by:

	Average	Worst
Top (Find min)	**O**(1)	**O**(1)
Pop (Delete min)	**O**(ln(n))	**O**(ln(n))
Insert	**O**(1)	**O**(ln(n))

Binary Max Heaps

- A binary max-heap is identical to a binary min-heap except that the parent is always larger than either of the children

- For example, the same data as before stored as a max-heap yields



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	29	23	26	14	25	9	15	12	6	19	10	3	8		

Other Heaps

- Other heaps have its own unique run-time characteristics
  - Leftist, skew, binomial and Fibonacci heaps all use a node-based implementation requiring $\Theta(n)$ additional memory
  - For Fibonacci heaps, the run-time of all operations (including merging two Fibonacci heaps) except pop are $\Theta(1)$

Summary

- In this talk, we have:
  - Discussed binary heaps
  - Looked at an implementation using arrays
  - Analyzed the run time:
    - Head $\Theta(1)$
    - Push $\Theta(1)$ average
    - Pop $O(\ln(n))$
  - Discussed implementing priority queues using binary heaps

References

[1] Donald E. Knuth, *The Art of Computer Programming, Volume 3: Sorting and Searching*, 2nd Ed., Addison Wesley, 1998, §7.2.3, p.144.

[2] Cormen, Leiserson, and Rivest, *Introduction to Algorithms*, McGraw Hill, 1990, §7.1-3, p.140-7.

[3] Weiss, *Data Structures and Algorithm Analysis in C++, 3rd Ed.*, Addison Wesley, §6.3, p.215-25.