Chapter 18

Bose-Einstein Gases

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18.1 Black Body Radiation

- Black body

  Black body is an idealized physical body that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence.

- Black body radiation

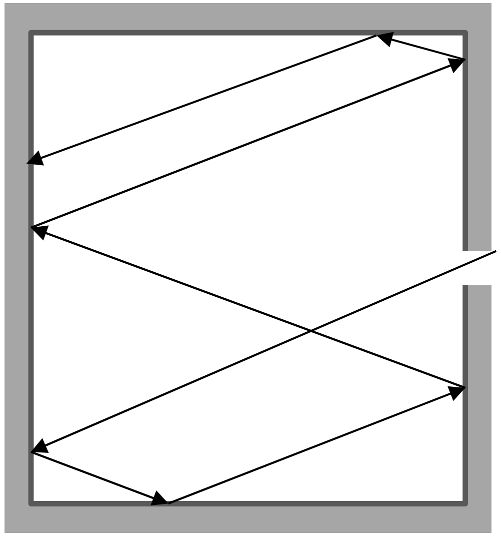
  A black body in thermal equilibrium emit black body radiation (electromagnetic waves) whose spectrum is only regarded with temperature.

18.1 Black Body Radiation

- Black body and Photon gas

  Consider a volume, V enclosed by insulated wall with small hole.

  Photons injected from the hole nearly re-emitted so that the inner surface of the volume can be regarded as a black body while inner space is treated to be filled with photon gas.



$\sum N_i \neq N$, because photons continue to be absorbed and emitted.

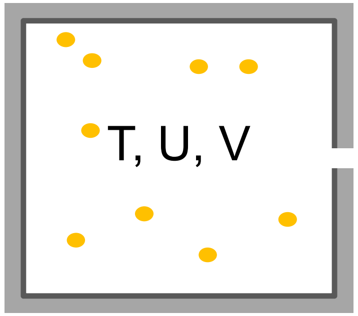
$\sum N_i \epsilon_i = U$, because the wall is isolated.

18.1 Black Body Radiation

- Photon gas with Bose Einstein statistics

  Photon gas enclosed with black body surface follows Boson statistics while having no constraint about particle numbers.

  Photons are bosons of spin 1 and obey Bose-Einstein statistics.



18.1 Black Body Radiation

- The photons emitted by one energy level may be absorbed at another, so the number of photons is not constant

$$\sum N_i \neq N$$

- Bose-Einstein distributions

From Stirling's approximation, $\ln(N!) = N\ln(N) - N$

$$\ln(w_{BE}) = \sum[(N_i + g_i - 1)\ln(N_i + g_i - 1) - N_i\ln(N_i) - (g_i - 1)\ln(g_i - 1)]$$

N_i for i^{th} energy level is undetermined yet

→ Method of Lagrange multiplier is used to obtain the most probable macro state under two constraints,
$\sum N_i \neq N, \sum N_i\epsilon_i = E$

$$\frac{\partial(\ln(w_{BE}))}{\partial N_i} + \cancel{\alpha}\frac{\partial(\sum N_i - N)}{\partial N_i} + \beta\frac{\partial(\sum N_i\epsilon_i - E)}{\partial N_i} = 0$$

the Lagrange multiplier $\alpha = 0$, and $e^{-\alpha} = 1$

18.1 Black Body Radiation

- Distribution function

Applying method of Lagrange multipliers to Bose-Einstein distributions,

$$\ln(N_i + g_i - 1) + \frac{g_i + N_i - 1}{g_i + N_i - 1} - \ln(N_i) - \frac{N_i}{N_i} + \beta\epsilon_i = 0$$

Then, the Bose-Einstein distribution function becomes as

$$\ln\left(\frac{N_i + g_i - 1}{N_i}\right) = -\beta\epsilon_i$$

$$N_i = g_i \frac{1}{e^{-\beta} \quad - 1} \qquad \left(\beta = -\frac{1}{kT}\right)$$

$$f(\epsilon_i) \equiv \frac{N_i}{g_i} = \frac{1}{e^{-\beta\epsilon_i} - 1} = \frac{1}{e^{\epsilon_i/kT} - 1}$$

18.1 Black Body Radiation

- The number of photons per quantum state

$$f_j = \frac{N_j}{g_j} = \frac{1}{e^{\varepsilon_j/kT} - 1}$$

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{\varepsilon/k} \ \ - 1}$$

$$f(\nu) = \frac{N(\nu)}{g(\nu)} = \frac{1}{e^{h\nu/kT} - 1}$$

- The number of quantum states with frequencies in the range $\nu\ to\ \nu + d\nu$

$$g(\nu)d\nu = \underset{\substack{\downarrow \\ \text{Polarization}}}{2} \times \frac{4\pi V}{c^3}\nu^2 d\nu \qquad c : the\ speed\ of\ light$$

- The energy in the range $\nu\ to\ \nu + d\nu$

$$\begin{aligned} u(\nu)d\nu &= N(\nu)d\nu \times h\nu \\ &= g(\nu)f(\nu)d\nu \times h\nu \\ &= \boldsymbol{\frac{8\pi V\nu^2 d\nu}{c^3} \cdot \frac{h\nu}{e^{h\nu/kT-1}}} \end{aligned}$$

Plank radiation formula

18.1 Black Body Radiation

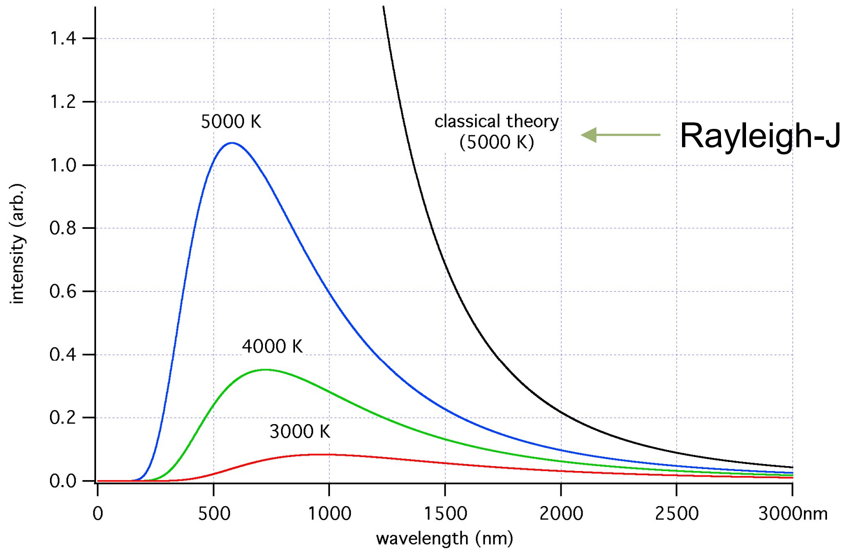
- The blackbody spectrum is often expressed in terms of the wavelength.

  Then, $u(\nu)d\nu \propto u(\lambda)d\lambda$

$$\nu = \frac{c}{\lambda} \qquad d\nu = -\frac{c}{\lambda^2}d\lambda \qquad |d\nu| = \frac{c}{\lambda^2}|d\lambda|$$

$$\underline{u(\lambda)}d\lambda = \frac{8\pi V\frac{c^2}{\lambda^2}\frac{c}{\lambda^2}d\lambda}{c^3}\frac{\frac{hc}{\lambda}}{e^{hc/\lambda kT}-1} = 8\pi hcV\frac{d\lambda}{\lambda^5(e^{hc/\lambda k}\ -1)}$$

The energy per unit wavelength in the range $\lambda\ to\ \lambda + d\lambda$ (wavelength spectrum)



Rayleigh-Jeans law

Black body radiation spectrum
(http://alfalfasurvey.wordpress.com)

18.1 Black Body Radiation

- The total energy density

$$U = \int_0^\infty u(\lambda)d\lambda$$

$$\frac{U}{V} = 8\pi hc \int_0^\infty \frac{d\lambda}{\lambda^5(e^{hc/\lambda kT} - 1)}$$

$$x = \frac{hc}{\lambda kT}$$

$$\frac{U}{V} = \frac{8\pi}{h^3c^3}(kT)^4 \underset{= \frac{\pi^4}{15}}{\underline{\int_0^\infty \frac{x^3dx}{e^x - 1}}}$$

$$Thus, \quad \boldsymbol{\frac{U}{V} = aT^4} \qquad a = \frac{8\pi^5k^4}{15h^3c^3} = 7.55 \times 10^{-16}\,\mathrm{J/(m^3K^4)}$$

- The energy flux

$$\boldsymbol{e = \frac{c}{4}\left(\frac{U}{V}\right) = \sigma T^4} \qquad \boldsymbol{\sigma = \frac{ca}{4} = 5.67 \times 10^{-8}}\,\mathrm{W/m^2K^4} \qquad \textbf{Stefan – Boltzmann law}$$

18.1 Black Body Radiation

- The wavelength λ_{max} at which $u(\lambda)$ is **a maximum** satisfies a relation known as Wien's displacement law.

$$u(\lambda) = 8\pi hcV \frac{1}{\lambda^5(e^{hc/\lambda kT} - 1)}$$

$$\frac{d}{d\lambda}\left[\lambda^5\left(e^{hc/\lambda k} \;\; - 1\right)\right] = 0$$

$$x = \frac{hc}{\lambda kT} \qquad \frac{x}{5} = 1 - e^{-x}$$

$$\frac{1}{x^2}\frac{d}{dx}[x^{-5}(e^x - 1)] = \frac{1}{x^2}[x^{-5}e^x - 5x^{-6}(e^x - 1)] = 0$$

$$\therefore x = 4.96$$

$$\frac{hc}{\lambda_{max}kT} = 4.96$$

$$\boldsymbol{\lambda_{max}T = \frac{hc}{4.96k} = 2.90 \times 10^{-3}}\text{mK} \qquad \textbf{Wien's displacement law}$$

18.1 Black Body Radiation

- For long wavelengths, $\frac{hc}{\lambda kT} \ll 1$

$$e^{hc/\lambda k} \quad \approx 1 + \frac{hc}{\lambda kT} \quad \text{Taylor series}$$

$$u(\lambda)d\lambda = 8\pi hcV \frac{1}{\lambda^5\left(\frac{hc}{\lambda kT}\right)}$$

$$= \boldsymbol{V \frac{8\pi kT}{\lambda^4} d\lambda} \quad \textbf{Rayleigh} - \textbf{Jeans Formula}$$

- For short wavelengths, $e^{\frac{hc}{\lambda kT}} \gg 1$

$$\boldsymbol{u(\lambda)d\lambda = 8\pi hcV \frac{e^{-hc/\lambda kT}}{\lambda^5} d\lambda} \quad \textbf{Wien's law}$$

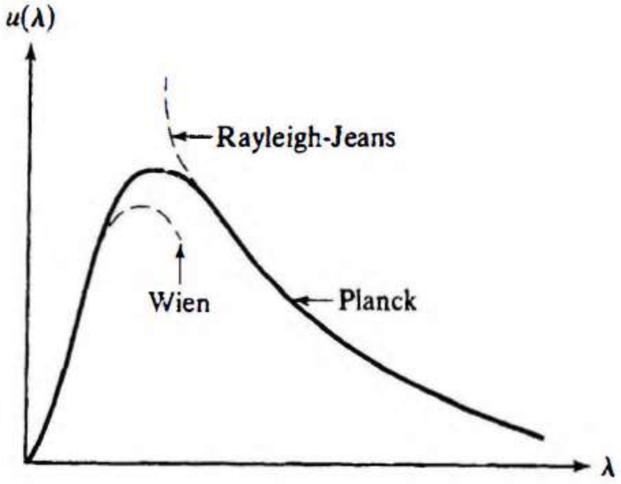


Fig. Sketch of Planck’s law, Wien’s law and the Rayleigh-Jeans law

18.2 Properties of a Photon Gas

- The heat capacity

$$\frac{U}{V} = aT^4 = \frac{8\pi^5 k^4}{15h^3c^3}T^4 \qquad (a = \frac{8\pi^5 k^4}{15h^3c^3})$$

$$C_V = \left.\frac{\partial U}{\partial T}\right)_V = \frac{32\pi^5 k^4}{15h^3c^3}T^3 V$$

- The absolute entropy

$$S = \int_0^T \frac{C_V}{T}dT = \frac{32\pi^5 k^4 V}{15h^3c^3}\cdot\frac{1}{3}T^3$$

- The Helmholtz function $F = U - TS$

$$F = U - TS = aT^4V - \frac{4}{3}aT^4\mathrm{V} = -\frac{1}{3}aT^4\mathrm{V}$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{1}{3}aT^4 = \frac{1}{3}\left(\frac{U}{V}\right) \qquad cf.\ Ideal\ gas\ \ P = \frac{2}{3}\frac{U}{V}$$

18.3 Bose Einstein Condensation

- The gas of noninteracting particles of large mass such that quantum effects only become important at very low temperatures. → **Ideal Bose-Einstein gas**
- ^{4}He undergoes a remarkable phase transition known as **Bose-Einstein condensation**.

- The Bose Einstein continuum distribution

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

Chemical potential $\mu(T) = ?$

18.3 Bose Einstein Condensation

- For Maxwell-Boltzmann distribution (Dilute gas)

BE: $f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}-1}$

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}}$$

$$\mu = -kT \ln \frac{Z}{N}$$

$$Z = \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} V$$

$$\therefore \boldsymbol{\frac{\mu}{kT} = -\ln\left[\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} \frac{V}{N}\right]}$$

- As an example, for ^{4}He at standard temperature and pressure,

$$\frac{\mu}{kT} = -12.43,\ \frac{\varepsilon}{kT} = 1.5,\ \frac{\varepsilon-\mu}{kT} = 13.9 \text{ and } f(\varepsilon) = 9 \times 10^{-7}$$

$$g(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V}{h^3} m^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon \qquad from\ Chap. 12$$

$$N = N_0 + N_{ex}$$

$$\boldsymbol{N_{excited} = \int N(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V m^{\frac{3}{2}}}{h^3} \int \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}}$$

18.3 Bose Einstein Condensation

$$N_{excited} = \int N(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V m^{\frac{3}{2}}}{h^3}\int \frac{\varepsilon^{\frac{1}{2}}d\varepsilon}{e^{(\varepsilon-\mu)/kT}-1}$$

- For the ground state (at very low temperature, this ground state becomes significant as bosons condense into this lowest state), $T \rightarrow 0,\ \varepsilon = 0$ and $\mathrm{N}(\varepsilon) \rightarrow N$

$$\frac{1}{e^{-\mu/kT}-1} \cong N$$

$$-\frac{\mu}{kT} = \ln\left(1+\frac{1}{N}\right) \cong \frac{1}{N} \sim 0$$

- For low temperature, $exp\left(-\frac{\mu}{kT}\right) \sim 1$

$$x = \frac{\varepsilon}{kT}$$

$$\boldsymbol{N_{ex} = V\frac{2}{\sqrt{\pi}}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\underbrace{\int_0^\infty \frac{x^{\frac{1}{2}}dx}{e^x-1}}_{=2.612\frac{\sqrt{\pi}}{2}} = 2.612V\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}}$$

18.3 Bose Einstein Condensation

- Bose temperature T_B is the temperature above which all the bosons should be in excited states. Thus $N_{ex} = N \;\; and \; T = T_B.$

$$N = 2.612V\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}$$

$$T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612V}\right)^{\frac{2}{3}}$$

- For $T > T_B,$ all the bosons are in excited states.

  For $T < T_B,$ increasing number of bosons occupy the ground state until at $T = 0$.

$$N = N_0 + N_{ex}$$

$$\frac{N_{ex}}{N} = \left(\frac{T}{T_B}\right)^{\frac{3}{2}} \qquad \frac{N_0}{N} = 1 - \left(\frac{T}{T_B}\right)^{\frac{3}{2}}$$

→ 6.02×10^{23} $^{4}\mathbf{He}$ atoms confined to a volume of $22.4 \times 10^{-3}\mathrm{m}^3,$ $\boldsymbol{T_B \sim 0.036}\,\mathrm{K}$

18.3 Bose Einstein Condensation

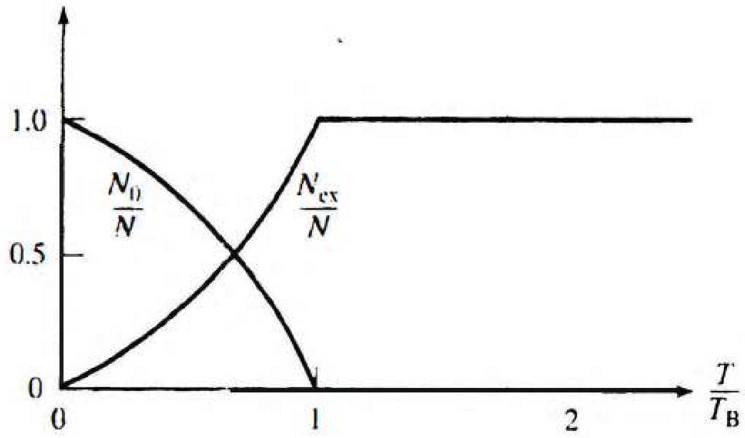


Fig. Variation with temperature of N_0/N and N_{ex}/N for a boson gas

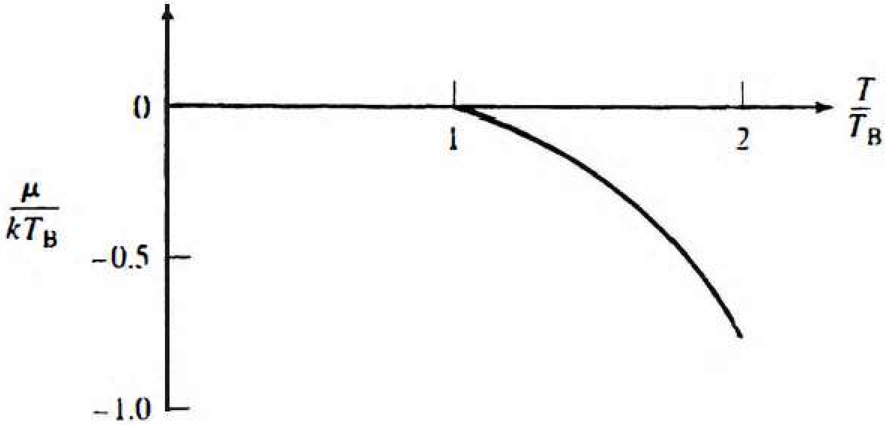


Fig. Variation with temperature of μ/kT_B versus T/T_B.

18.4 Properties of a Boson Gas

- The internal energy

$$U = \sum N_j \varepsilon_j = N_0 \varepsilon_0 + N_{ex} \varepsilon_{ex}$$

$$\text{For } T > T_B, \qquad N_0 = 0 \qquad U = \frac{3}{2} NkT$$

$$T_B = \frac{h^2}{2\pi mk} \left(\frac{N}{2.612V} \right)^{\frac{2}{3}}$$

$$\text{For } T < T_B, \qquad N_0 \gg 1, \varepsilon_0 = 0 \qquad \frac{N_{ex}}{N} = \left(\frac{T}{T_B} \right)^{\frac{3}{2}} \qquad U = ?$$

18.4 Properties of a Boson Gas

- For $T < T_B$,

$$U = \int_0^{\varepsilon} \varepsilon N(\varepsilon) d\varepsilon$$

$$= \int_0^{\varepsilon} \varepsilon \cdot \frac{1}{e^{(\varepsilon-\mu)/kT} - 1} \cdot \underbrace{\frac{4\sqrt{2}\pi V}{h^3} m^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon}_{= g(\varepsilon)d\varepsilon}$$

$$= 2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^{\infty} \frac{\varepsilon^{\frac{3}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}$$

$\boldsymbol{\mu = 0 \ (T < T_B), x = \varepsilon/kT}$

$$= -2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \underbrace{\int_0^{\infty} \frac{x^{\frac{3}{2}} dx}{e^x - 1}}_{= \frac{3\sqrt{\pi}}{4} \times 1.34} \cdot (kT)^{\frac{5}{2}}$$

$$= \frac{3}{2} \times 1.33kT \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} V$$

$$= \boldsymbol{0.770NkT \left(\frac{T}{T_B}\right)^{\frac{3}{2}}} \qquad T_B = \frac{h^2}{2\pi mk} \left(\frac{N}{2.612V}\right)^{\frac{2}{3}}$$

18.4 Properties of a Boson Gas

- The heat capacity

$$C_V = \frac{dU}{dT} = 1.92Nk\left(\frac{T}{T_B}\right)^{\frac{3}{2}}$$

- The absolute entropy

$$S = \int_0^T \frac{C_V dT}{T} = 1.28Nk\left(\frac{T}{T_B}\right)^{\frac{3}{2}} \qquad T \to 0, S \to 0$$

- The Helmholtz function $F = U - TS$

$$F = U - TS = -0.51NkT\left(\frac{T}{T_B}\right)^{\frac{3}{2}} \qquad T < T_B$$

$$= -1.33kT\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} V$$

$$P = 1.33kT\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} \qquad T < T_B$$

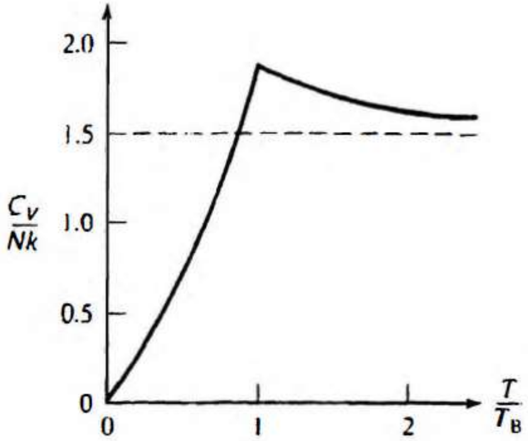


Fig. Variation with temperature of the heat capacity of a boson gas.