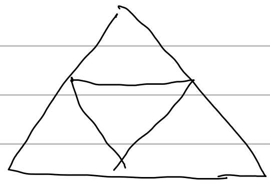
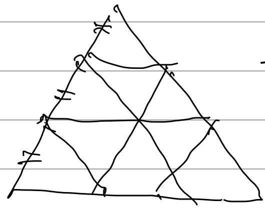
A 2.3. Tessellations of the sphere
(e.g., geodesic dome)

- Start from regular polyhedron (e.g., icosahedron) and subdivide each face into a specific number of equilateral triangles



$b=2$

$2^2=4$



$b=3$

$3^2=9$

Triangles based on the projected vertices on the sphere are no longer identical.

$b=2$: 1, 0.8843

$b=3$:

$b=4$: $\underbrace{1, \cdots, 0.7793}_{\text{6 different sets}}$

These tessellated structures do not make regular polyhedra since the number of edges per vertex varies

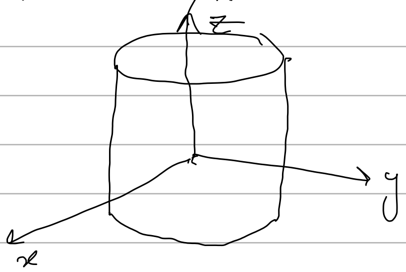
A.3. Surfaces

Surface : 2D locus of points in 3D space
(u, v) (x, y, z)

Developable surface : transformed without stretching

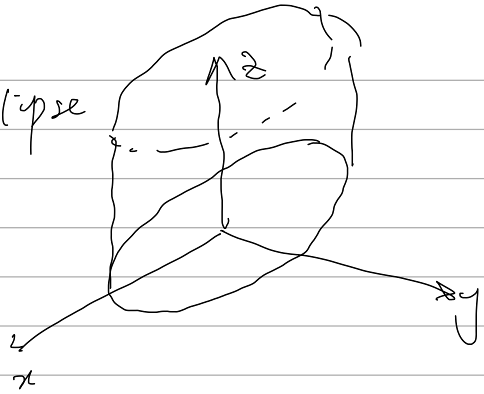
(1) Generalized cylinder

- cylinder



$$\vec{p} = (\cos u, \sin u, v) = (\cos u, \sin u, 0) + (0, 0, 1)\, v = \vec{f}(u) + \vec{g}\, v$$

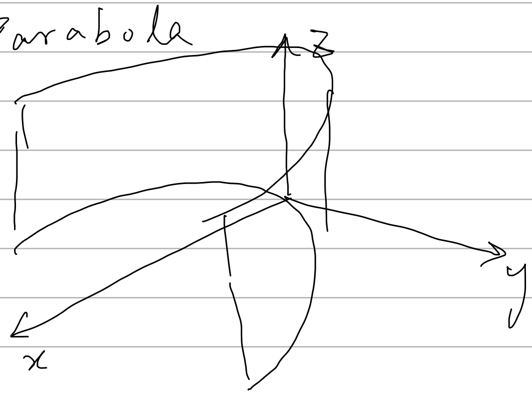
- Ellipse



$$\vec{p} = (a\cos u,\ b\sin u,\ v)$$

$\vdots$

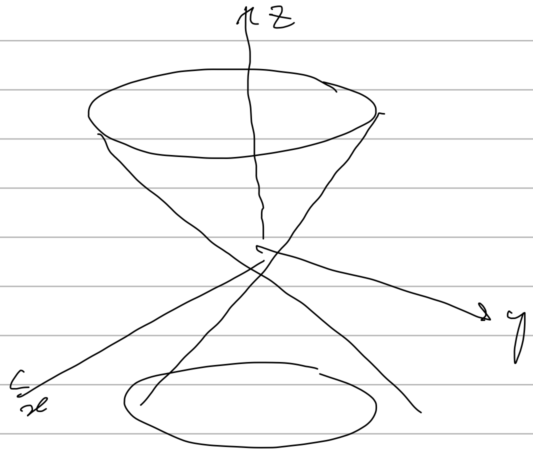
- Parabola



$$\begin{aligned}\vec{p} &= (au^2,\ u,\ v)\\ &= (au^2,\ u,\ 0) + (0,\ 0,\ 1)\,v\\ &= \underbrace{\vec{f}(u) + \vec{g}\cdot v}\end{aligned}$$

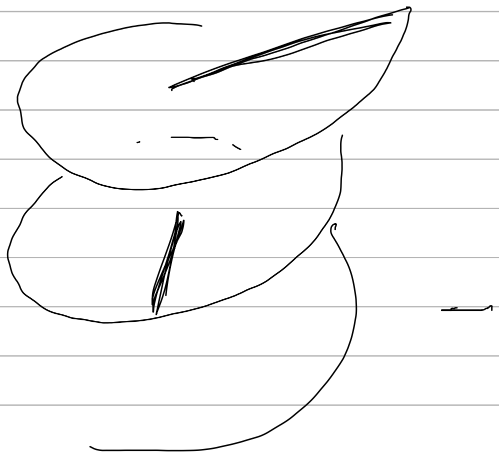
Translating the general 2D curve $\vec{f}(u)$ in the direction of $\vec{g}$

(2) Generalized cone



$$\begin{aligned}\vec{p} &= (v\cos u,\ v\sin u,\ 1-v)\\ &= v[(\cos u,\ \sin u,\ 0) - (0,\ 0,\ 1)]\\ &\quad + (0,\ 0,\ 1)\\ &= v(\vec{f}(u) - \vec{g}) + \vec{g}\end{aligned}$$

(3) Tangent surface



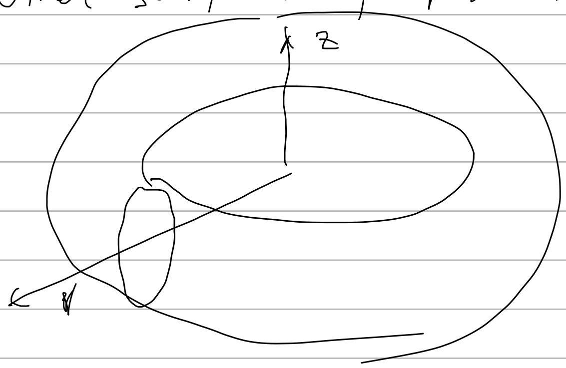
$$\begin{aligned}\vec{p} &= (\cos u - v\sin u,\\ &\qquad \sin u + v\cos u,\\ &\qquad u+v)\\ &= (\cos u,\ \sin u,\ u)\\ &\quad + v(-\sin u,\ \cos u,\ 1)\\ &= \vec{g}(u) + v\,\vec{g}'(u)\end{aligned}$$

Quadrics : quadratic equations
(but not cylinders or cones)

- Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2z = 0$
- Hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z = 0$
- Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

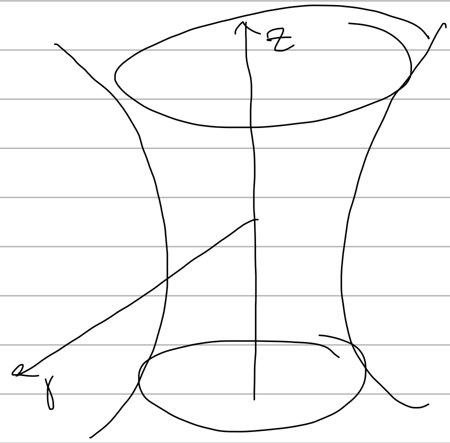
Quadrics of revolution $(a=b)$ are axi-symmetric

Other surfaces of revolution



Toroid

$(r-a)^2 + z^2 = b^2$

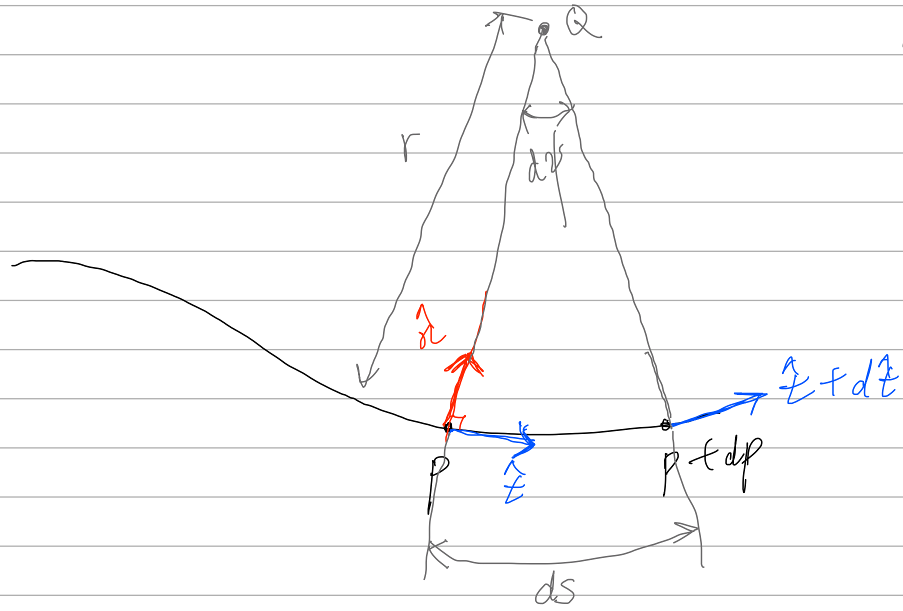


Catenoid (3.4)

$r = a \cosh \frac{z}{a}$

A.4. Curvature of surfaces

$ds = r \cdot d\phi$



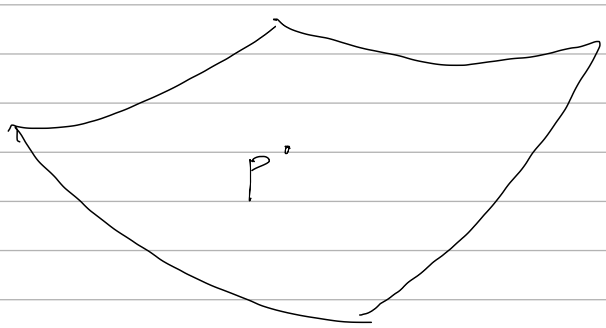
Radius of curvature $r = \overline{QP} = \frac{ds}{d\phi}$

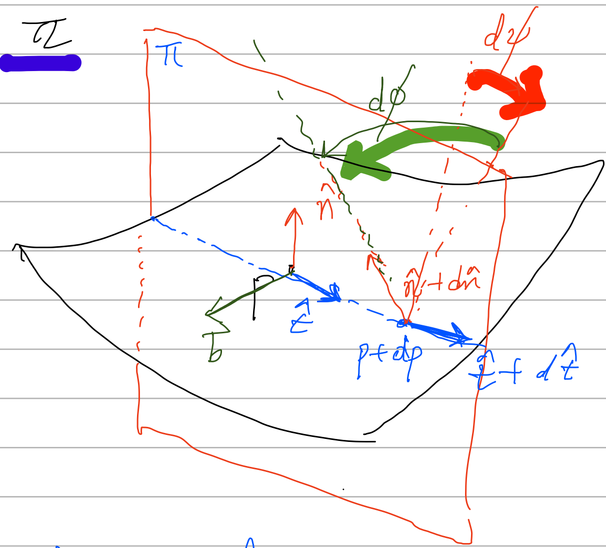
curvature $\kappa = \frac{1}{r} = \frac{d\phi}{ds}$

$$\frac{d\hat{t}}{ds} = \kappa \hat{n}$$

κ is positive when Q is in the direction of $\hat{n}$

Curvature of a surface π





π : plane that contains the normal to the surface at P

$\kappa = \frac{d\phi}{ds}$: curvature

$\tau = \dfrac{d\phi}{ds}$: twist

$\vec{b} = \hat{t} \times \hat{n}$

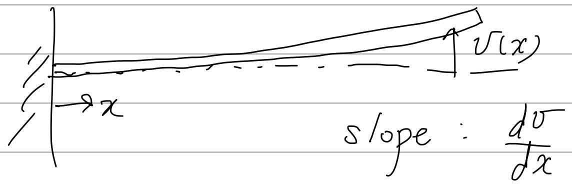
$d\psi$: in-plane rotation with respect to $\vec{b}$

$d\phi$: out-of-plane rotation w.r.t. $\vec{t}$

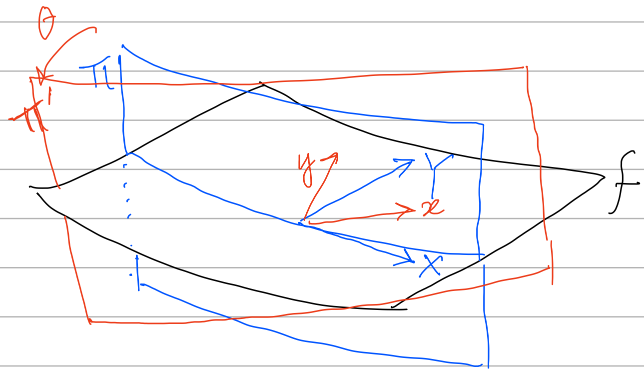
Equation $f(X, Y)$ of the surface

$$f \cong \frac{1}{2}\left(\frac{\partial^2 f}{\partial X^2}\right)_p X^2 + \left(\frac{\partial^2 f}{\partial X \partial Y}\right)_p XY + \frac{1}{2}\left(\frac{\partial^2 f}{\partial Y^2}\right)_p Y^2$$

$$\kappa = \frac{d^2 v / dx^2}{\left[1 + \left(dv/dx\right)^2\right]^{3/2}} \simeq d^2 v / dx^2$$



$$= \frac{1}{2} K_X X^2 + K_{XY} XY + \frac{1}{2} K_Y Y^2$$



(X, Y) coordinate (π-plane) at point P, K_X, K_Y, K_{XY}

$\Downarrow$

(x, y) coordinate (π'-plane) at point P, K_x, K_y, K_{xy}

Coordinate transformation between π and π'

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\begin{aligned} f(x,y) = {} & \frac{1}{2} K_X (\cos\theta\, x - \sin\theta\, y)^2 \\ & + K_{XY} (\cos\theta\, x - \sin\theta\, y)(\sin\theta\, x + \cos\theta\, y) \\ & + \frac{1}{2} K_Y (\sin\theta\, x + \cos\theta\, y)^2 \end{aligned}$$

$$= \frac{1}{2} K_x \left(c^2 x^2 - 2cs\, xy + s^2 y^2 \right)$$

$$+ K_{xy} \left(cs\, x^2 + c^2 xy - s^2 xy - sc\, y^2 \right)$$

$$+ \frac{1}{2} K_y \left(s^2 x^2 + 2sc\, xy + c^2 y^2 \right)$$

$$= \frac{1}{2} \left[K_x c^2 + 2 K_{xy} sc + K_y s^2 \right] x^2$$

$$+ \left[- K_x sc + K_{xy} (c^2 - s^2) + K_y sc \right] xy$$

$$+ \frac{1}{2} \left(K_x s^2 - 2 K_{xy} sc + K_y c^2 \right) y^2$$

$$= \frac{1}{2} K_{xx} x^2 + K_{xy} xy + \frac{1}{2} K_{yy} y^2$$

$$\therefore K_x = \frac{\partial^2 f}{\partial x^2} = K_x \cdot \frac{\cos 2\theta + 1}{2} + K_{xy} \sin 2\theta + K_y \frac{1 - \cos 2\theta}{2}$$

$$\left(\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \right)$$

$$= \frac{1}{2}(K_x + K_y) + \frac{1}{2}(K_x - K_y)\cos 2\theta + K_{xy} \sin 2\theta$$

$$K_y = \frac{\partial^2 f}{\partial y^2} = \frac{1}{2}(K_x + K_y) - \frac{1}{2}(K_x - K_y)\cos 2\theta - K_{xy} \sin 2\theta$$

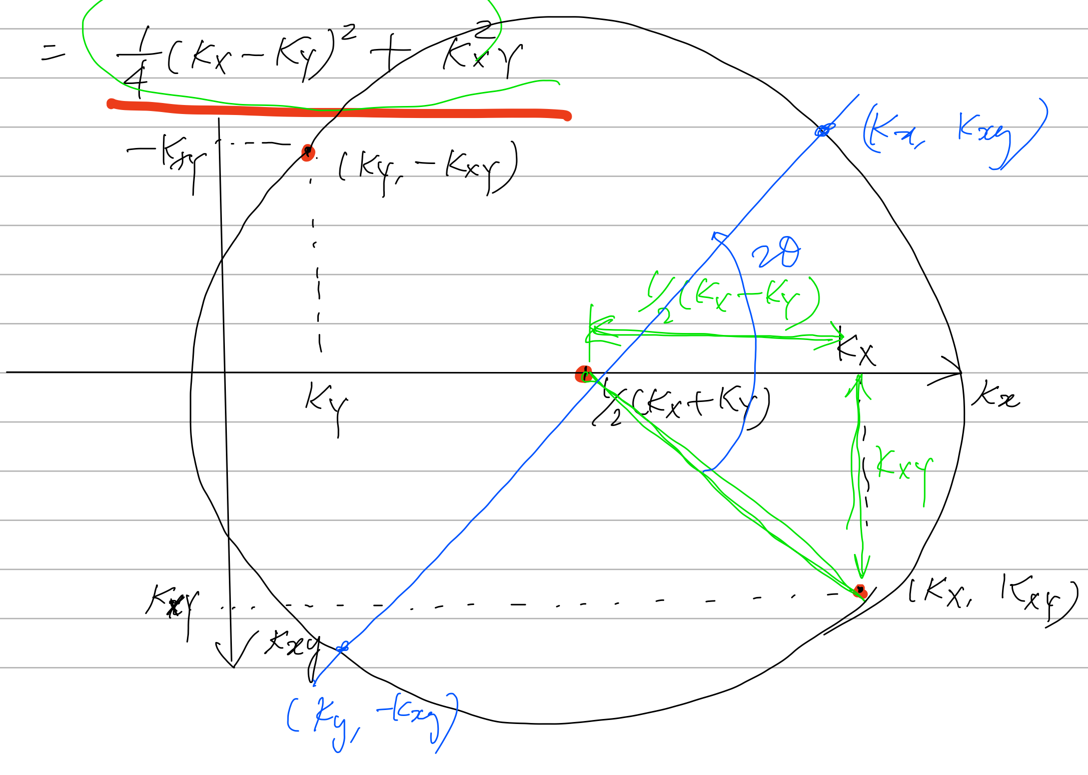
$$K_{xy} = \frac{\partial^2 f}{\partial x \partial y} = -K_x \cdot \frac{\sin 2\theta}{2} + K_{xy} \cos 2\theta + K_y \cdot \frac{\sin 2\theta}{2}$$

$$= -\frac{1}{2}(K_x - K_y) \sin 2\theta + K_{xy} \cos 2$$

$$\left[K_x - \frac{1}{2}(K_x + K_y) \right]^2 + K_{xy}^2$$

$$= \left[\frac{1}{2}(K_x - K_y)\cos 2\theta + K_{xy} \sin 2\theta \right]^2 + \left[-\frac{1}{2}(K_x - K_y)\sin 2\theta + K_{xy} \cos 2\theta \right]^2$$

$$= \frac{1}{4}(K_x - K_y)^2 + K_{xy}^2$$



Scalar : $[\ ]$

vector : $\begin{bmatrix} - \\ - \\ - \end{bmatrix}$

tensor : $\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$

Principal curvature

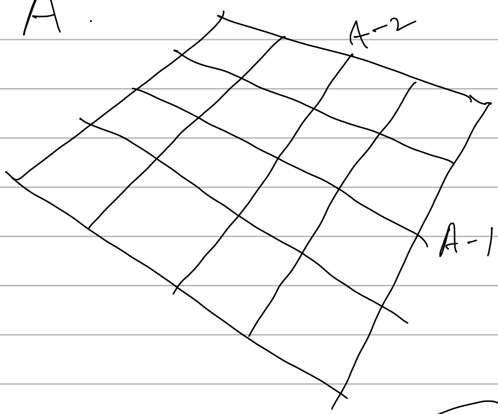
$$K_1 = \frac{1}{2}(K_x + K_y) + \sqrt{\frac{1}{4}(K_x - K_y)^2 + K_{xy}^2}$$

$$K_2 = \frac{1}{2}(K_x + K_y) - \sqrt{\frac{1}{4}(K_x - K_y)^2 + K_{xy}^2}$$

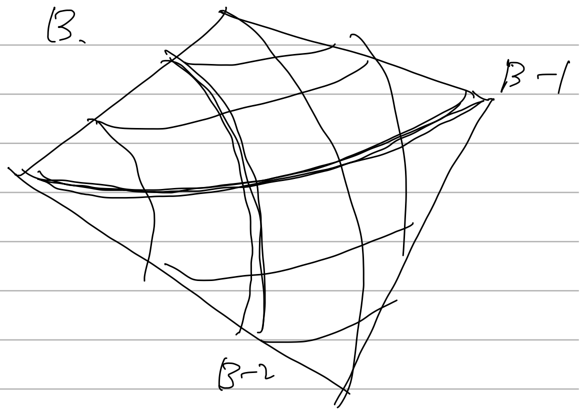
Principal curvature directions

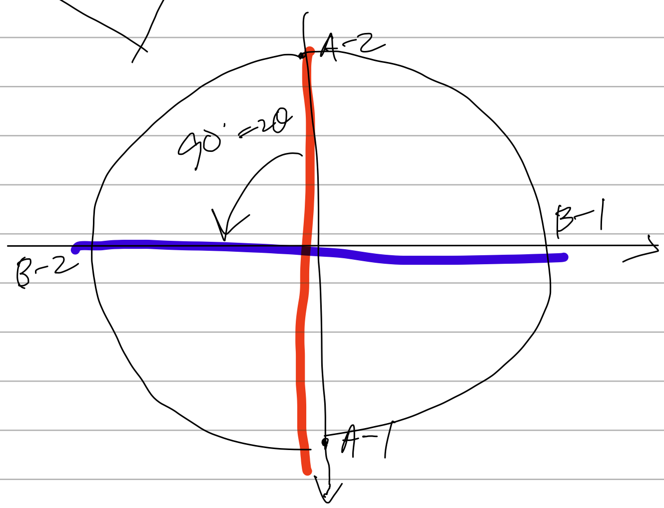
: No twist along this direction

A.

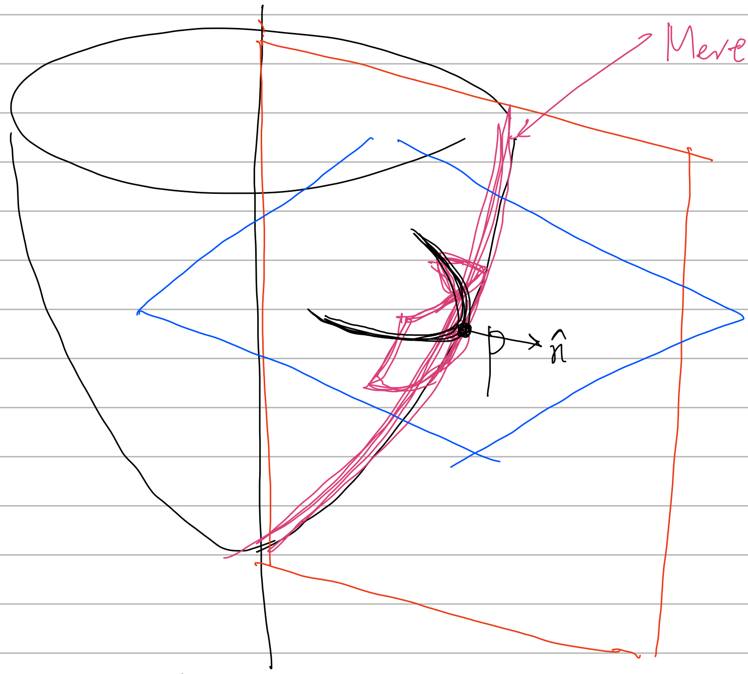


B.



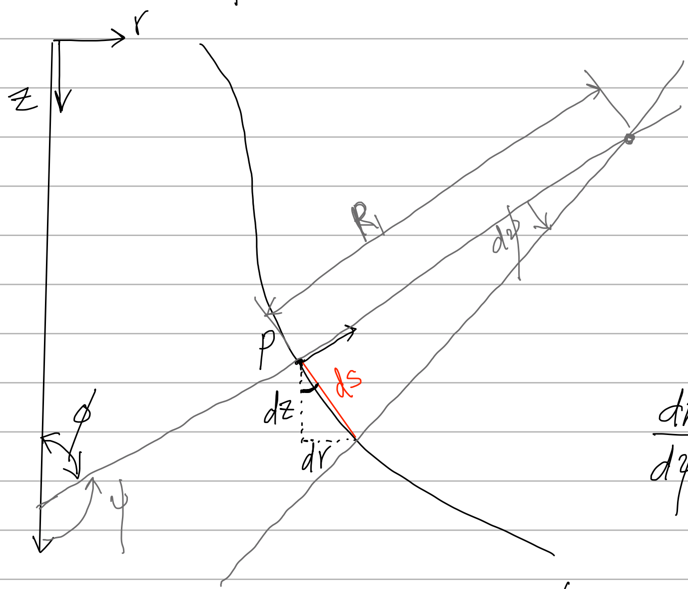


A4.2. Principal curvatures of axi-symmetric surfaces



Meredians : no twist
→ principal direction of curvature

R_1, R_2 in axi-symmetric case



$$ds = R_1 d\psi$$

$$dr = ds \sin(\psi - 90^\circ)$$

$$= -ds \cos\psi$$

$$\frac{dr}{d\psi} = \frac{-ds \cdot \cos\psi}{ds/R_1} = -R_1 \cos\psi$$

$$\frac{1}{R_1} = -\cos\psi \frac{d\psi}{dr} = -\cos(\pi - \phi) \cdot \frac{-d\phi}{dr}$$

$$(\phi = \pi - \psi, \quad d\phi = -d\psi)$$

$$= \cos\phi \cdot \frac{d\phi}{dr}$$

$$\boxed{\frac{1}{R_1} = \frac{d \sin\phi}{dr}}$$

$$R_2 = \frac{r}{\sin\psi}$$

$$\frac{1}{R_2} = \frac{\sin(\pi - \phi)}{r} = \boxed{\frac{\sin\phi}{r}}$$

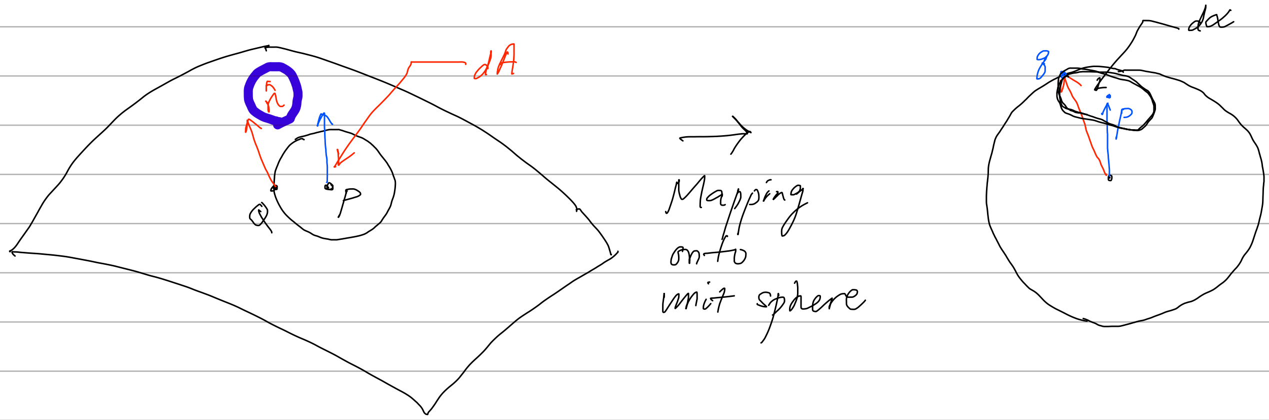
Gaussian curvature

$$K = k_1 k_2 = \frac{1}{R_1}\frac{1}{R_2}$$

$$\underset{\text{(kaai)}}{\chi} = \frac{1}{2\pi}\int_S K\,dA = 2-2g$$

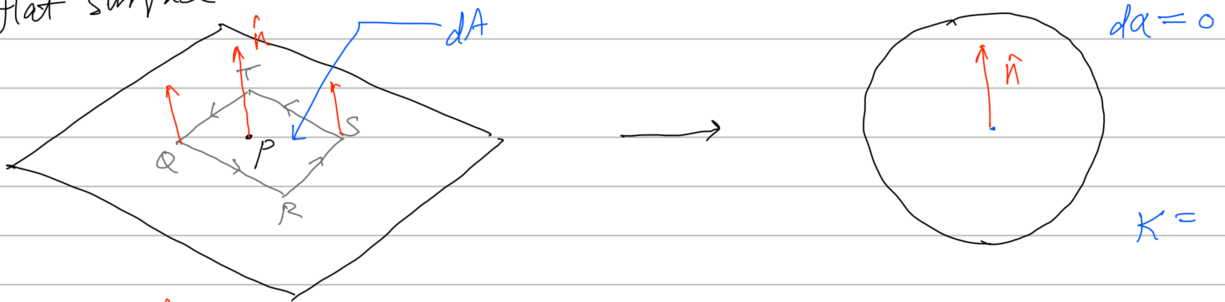
g : genus number (# of holes)

$$\begin{cases} K > 0 & : \text{synclastic or oval} \\ K < 0 & : \text{anticlastic or saddle shaped} \\ K = 0 & : \text{developable (e.g. cones, cylinders)} \end{cases}$$



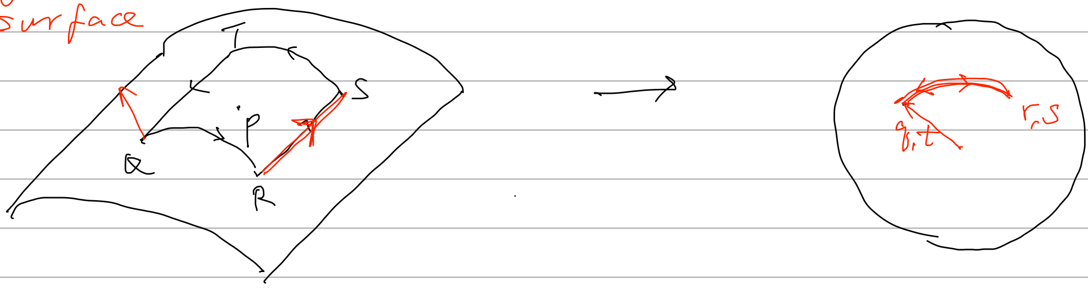
$$K = \frac{d\alpha}{dA}$$

flat surface

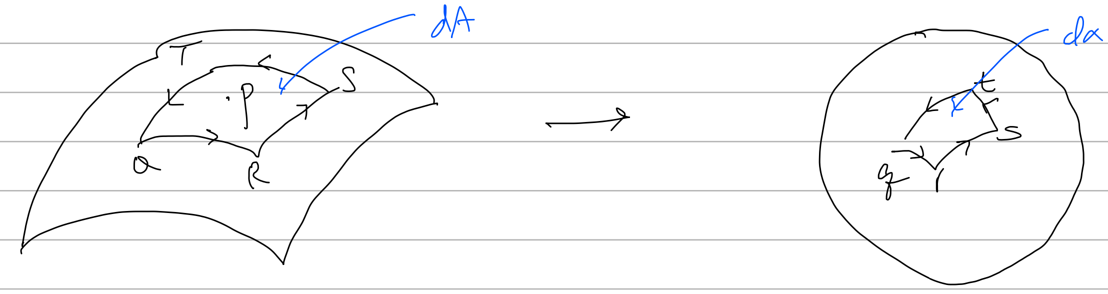


$$K = \frac{d\alpha}{dA} = 0$$

Cylindrical surface

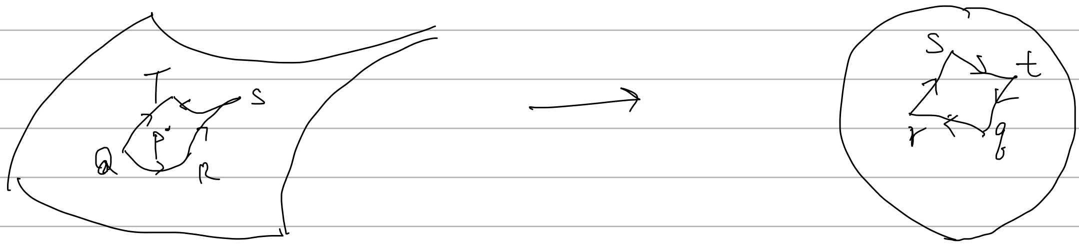


$$K = \frac{d\alpha}{dA} = 0$$



$K = \dfrac{d\alpha}{dA} \neq 0$

> 0



$K = \dfrac{d\alpha}{dA} < 0$