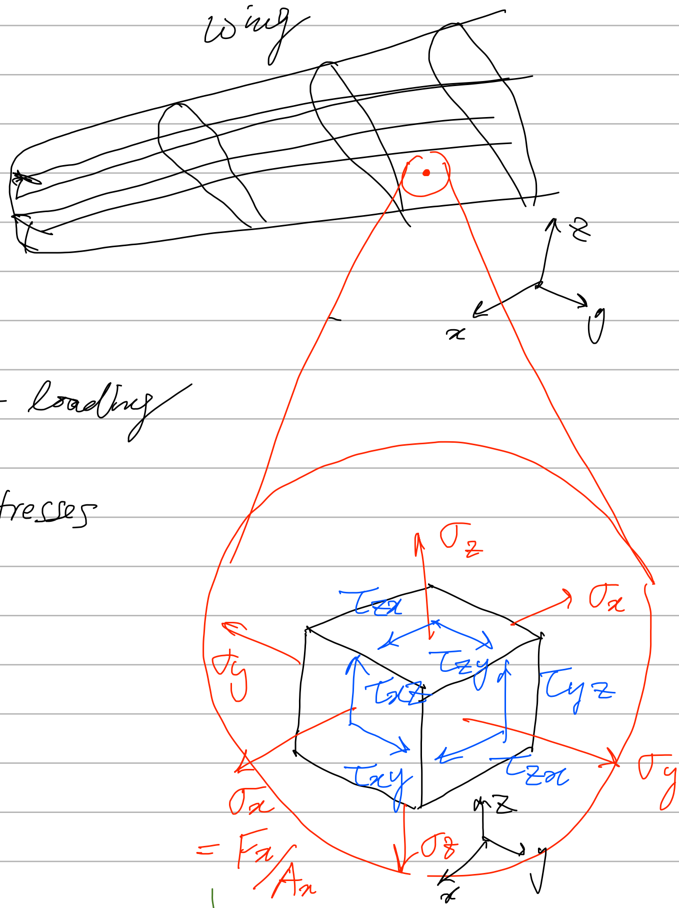
General approach of structural analysis

↳ supports loading

(1) Loading condition

$\vec{F}(x, y, z)$

- body force
  : gravity, magnetic force. ...
- surface force
  : drag, lift



(2) Stresses : intensity of loading **(6)**

$\sigma_x \quad \sigma_y \quad \sigma_z$: normal stresses

direction

$\tau_{xy} \qquad \tau_{xz} \qquad \tau_{yz}$

area of shear stress acting on

$\parallel \qquad \parallel \qquad \parallel$

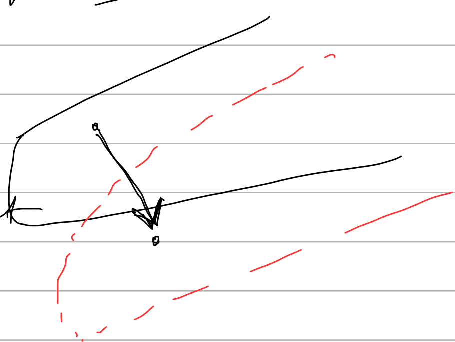
$\tau_{yx} \qquad \tau_{zx} \qquad \tau_{zy}$

(3) Strains (6)

$\varepsilon_x, \ \varepsilon_y, \ \varepsilon_z$: normal strains

$\gamma_{xy}, \ \gamma_{xz}, \ \gamma_{yz}$: shear strains

(4) Displacements (3)

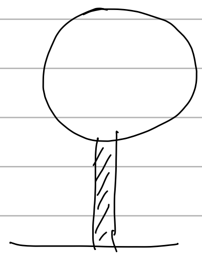


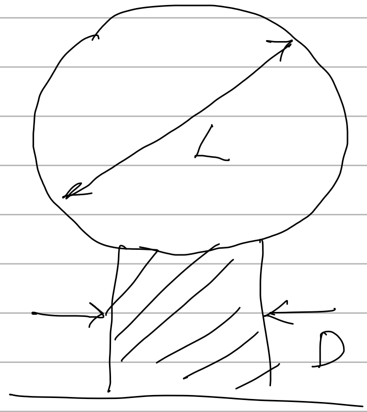
u : x-directional displacement

v : y- 〃

w : z- 〃

e.g.) elephant vs. bird





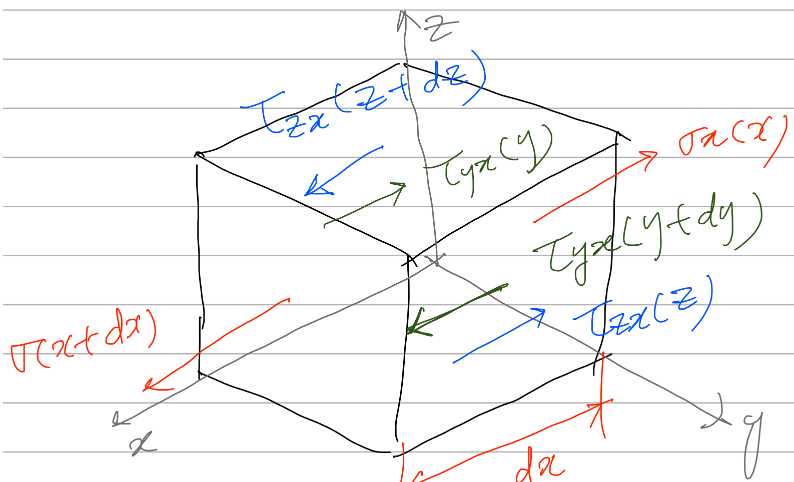
$$W \propto \rho \cdot L^3$$

$$\sigma = \frac{W}{A} \propto \frac{L^3}{D^2} \simeq c$$

$$D \propto L^{3/2}$$

Structural analysis

: process of figuring out (a part of) 15 unknowns from (a part or all of) 15 equations



Force equilibrium in x-direction

$$\sigma(x+dx)\cdot dydz - \sigma(x)\, dy\, dz$$
$$+ \tau_{yx}(y+dy)\cdot dxdz - \tau_{yx}(y)\, dxdz$$
$$+ \tau_{zx}(z+dz)\, dydz - \tau_{zx}(z)\, dydz$$
$$+ F_x \cdot \underbrace{\rho\, dxdydz}_{dm} = \rho\, dxdydz \cdot a_x$$

$$\frac{\sigma_x(x+dx)-\sigma_x(x)}{dx}+\frac{\tau_{yx}(y+dy)-\tau_{yx}(y)}{dy}+\frac{\tau_{zx}(z+dz)-\tau_{zx}(z)}{dz}$$

$$+\rho F_x=\rho a_x$$

body force per mass in x-direction

(1) Equilibrium equations (i.e., force – stress relationship)

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\rho F_x=\rho a_x$$

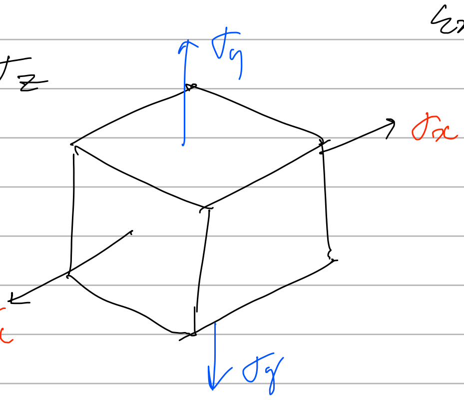
$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\rho F_y=\rho a_y$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+\rho F_z=\rho a_z$$

(2) Constitutive equations (stress – strain relationship)

$\varepsilon_x = ?$

$$\varepsilon_x=\frac{1}{E}\sigma_x-\frac{\nu}{E}\sigma_y-\frac{\nu}{E}\sigma_z$$



$$\nu_{yx}=-\frac{\varepsilon_x}{\varepsilon_y}$$

$$\varepsilon_y=\frac{\sigma_y}{E}-\nu\frac{\sigma_x}{E}-\nu\frac{\sigma_z}{E}$$

$$\varepsilon_z=\frac{\sigma_z}{E}-\nu\frac{\sigma_x}{E}-\nu\frac{\sigma_y}{E}$$

1) Geometry
2) Material
3) Loading

$$\gamma_{xy}=\frac{\tau_{xy}}{G}$$

$$\gamma_{xz}=\frac{\tau_{xz}}{G}$$

$$\gamma_{xy}=\frac{\tau_{xy}}{E}$$

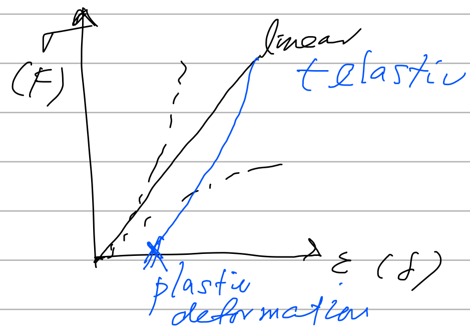
$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/G & & \\ & 0 & & & 1/G & \\ & & & & & 1/G\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix} \qquad (6)$$

Compliance matrix : E, ν, G

→ linear elastic behavior

→ isotropic

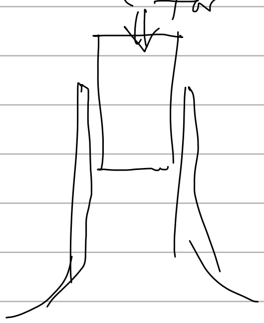
$$G = \frac{E}{2(1+\nu)}$$



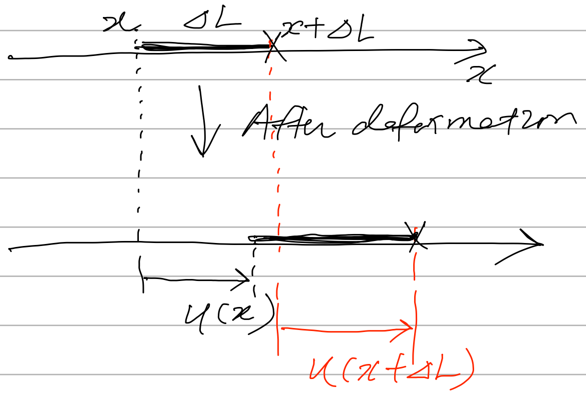
$\nu \simeq 0.3$

$-1 < \nu < 0.5$

(for natural materials $0 < \nu < 0.5$)



(3) Strain – displacement relationship



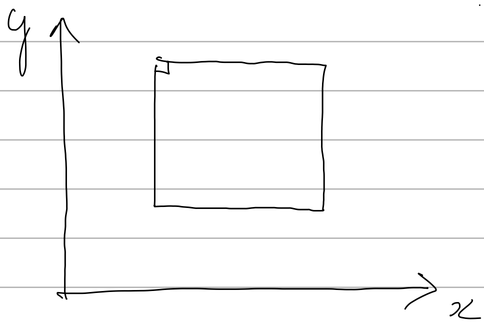
$$\varepsilon_x = \frac{u(x+\Delta L) - u(x) + \Delta L - \Delta L}{\Delta L}$$

$$= \frac{u(x+\Delta L) - u(x)}{\Delta L}$$

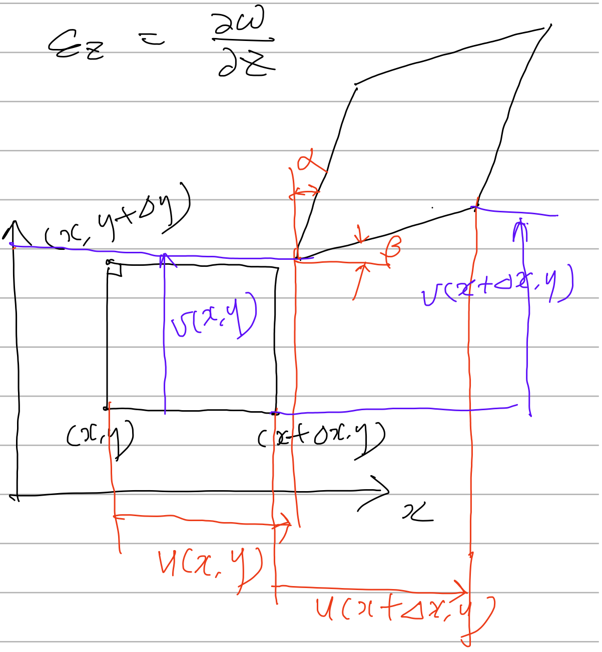
$$= \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$



→ After deformation



$$\gamma_{xy} = \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\beta \doteq \tan\beta = \frac{v(x+\Delta x, y) - v(x,y)}{\cancel{u(x+\Delta x, y) - u(x,y)} + \Delta x}$$

$$= \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

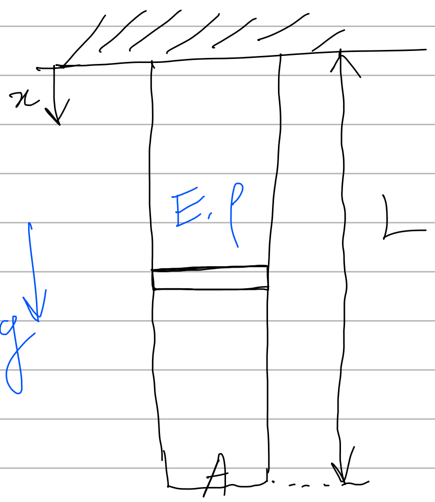
engineering strain

: infinitesimal strain definition

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad : \text{true strain}$$

$$\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = \varepsilon_x$$

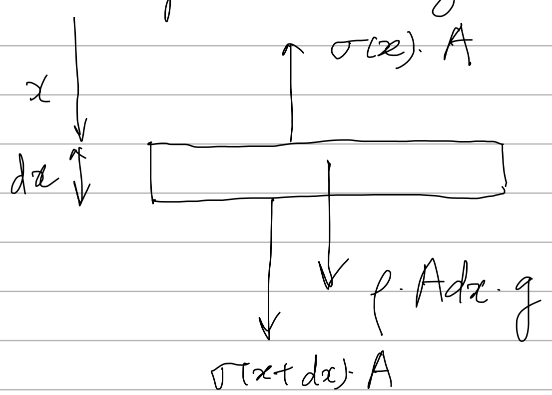
Exercise : A bar under gravity



1) Geometry : L, A
2) Material properties : E, ρ
3) Loading : g

Q. What is a tip displacement?

1) Equilibrium equation



$$\sigma(x+dx)A - \sigma(x)A + \rho A dx g = 0$$

$$\frac{\sigma(x+dx) - \sigma(x)}{dx} + \rho g = 0$$

$$\frac{d\sigma}{dx} + \rho g = 0$$

$$\int d\sigma = \int -\rho g \, dx \rightarrow \sigma(x) = -\rho g x + C$$

$$\text{B.C.} \quad \sigma(L) = -\rho g L + C = 0$$

$$\sigma(x) = \rho g (L - x)$$

(2) Stress - strain relationship

$$\varepsilon(x) = \frac{\sigma(x)}{E} = \frac{\rho g}{E}(L - x)$$

(3) Strain - displacement relationship

$$\frac{du}{dx} = \varepsilon_x = \frac{\rho g}{E}(L - x)$$

$$\int du = \int \frac{\rho g}{E}(L - x)\, dx$$

$$u(x) = \frac{\rho g}{E}\left(Lx - \frac{x^2}{2}\right) + C$$

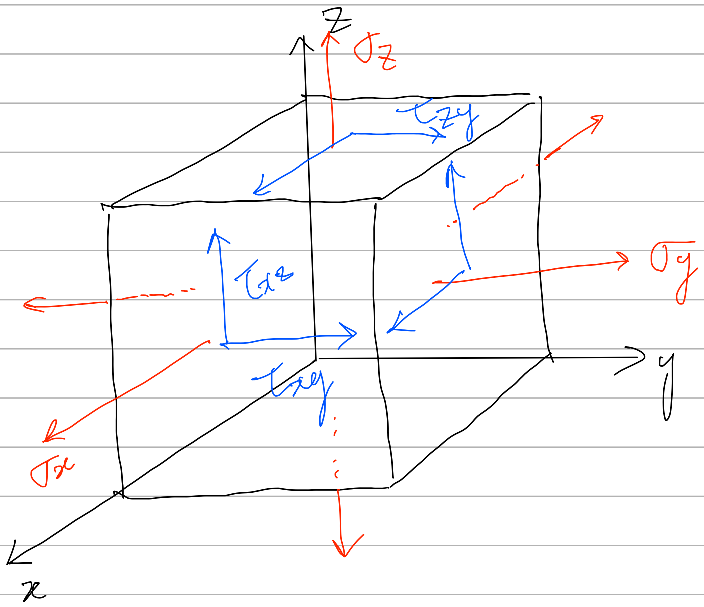
$$u(x=0) = C = 0$$

$$u(x=L) = \frac{\rho g}{E}\left(L^2 - \frac{L^2}{2}\right) = \frac{\rho g}{2E} L^2$$

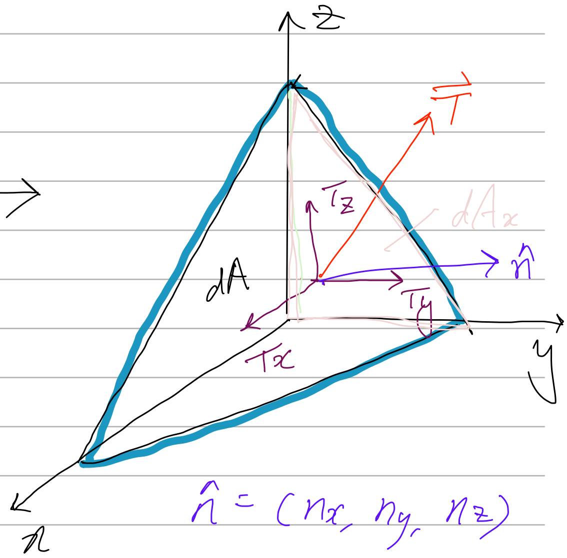
Stresses

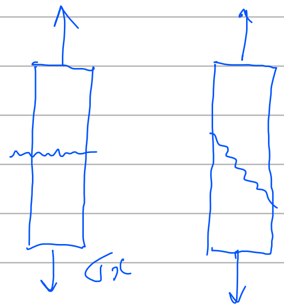
We expressed stresses in a form: $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$

$\rightarrow \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$: Cauchy stress tensor



$\longrightarrow$





$\vec{T}$: traction (force/area)

$$\vec{T}\,dA = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} dA$$

Force equilibrium in x-direction

$$T_x\,dA - \sigma_x\,dA_x - \tau_{yx}\,dA_y - \tau_{zx}\,dA_z = 0$$

$$T_x = \sigma_x \underbrace{\frac{dA_x}{dA}}_{n_x} + \tau_{yx} \underbrace{\frac{dA_y}{dA}}_{n_y} + \tau_{zx} \underbrace{\frac{dA_z}{dA}}_{n_z}$$

$$T_y = \tau_{xy}\,n_x + \sigma_y\,n_y + \tau_{zy}\,n_z$$

$$T_z = \tau_{xz}\,n_x + \tau_{yz}\,n_y + \sigma_z\,n_z$$

$$\vec{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = [\sigma]\,\hat{n}$$

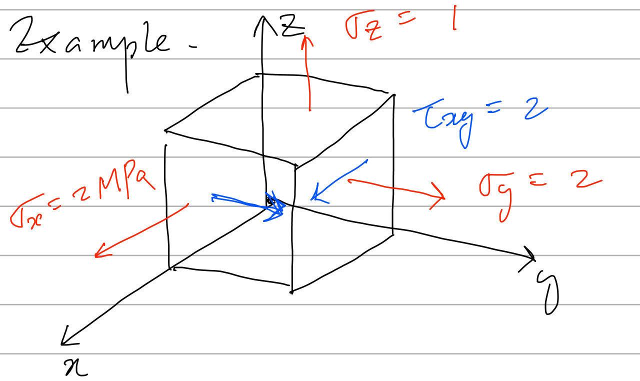
$$\begin{aligned} \sigma_n &= \vec{T} \cdot \hat{n} \\ &= T_x n_x + T_y n_y + T_z n_z \\ &= (\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z) n_x + (\qquad\qquad) n_y + (\quad) n_z \\ &= \sigma_x n_x^2 + \sigma_y n_y^2 + \sigma_z n_z^2 \\ &\quad + 2\tau_{xy} n_x n_y + 2\tau_{yz} n_y n_z + 2\tau_{zx} n_z n_x \end{aligned}$$

$$\tau = \left| \vec{T} - (\vec{T} \cdot \hat{n})\,\hat{n} \right|$$

$$[\sigma]\,\hat{n} = \underbrace{\sigma_p}_{\text{scaler}}\,\hat{n} \quad \rightarrow \quad ([\sigma] - \sigma_p I)\,\hat{n} = 0$$

$$\det([\sigma] - \sigma_p I) = 0 \quad : \text{eigen-value problem}$$

Example.



$$[\sigma] = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

principal stresses ?

〃 directions ?

$$[\sigma]\,\hat{n}_p = \sigma_p \cdot \hat{n}_p$$

$$\boxed{([\sigma] - \sigma_p I)\,\hat{n}_p = 0}$$

$$\det\left(\begin{bmatrix} 2-\sigma_p & 2 & 0 \\ 2 & 2-\sigma_p & 0 \\ 0 & 0 & 1-\sigma_p \end{bmatrix} \right) = 0$$

$$\begin{aligned} (1-\sigma_p) \cdot \det\left(\begin{bmatrix} 2-\sigma_p & 2 \\ 2 & 2-\sigma_p \end{bmatrix} \right) &= (1-\sigma_p)\{(2-\sigma_p)^2 - 4\} \\ &= (1-\sigma_p)\,\sigma_p\,(\sigma_p - 4) = 0 \end{aligned}$$

1) $\sigma_p = 4$

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \hat{n}_p = 0 \qquad \hat{n}_p = \frac{1}{\sqrt{2}}(1, 1, 0)$$

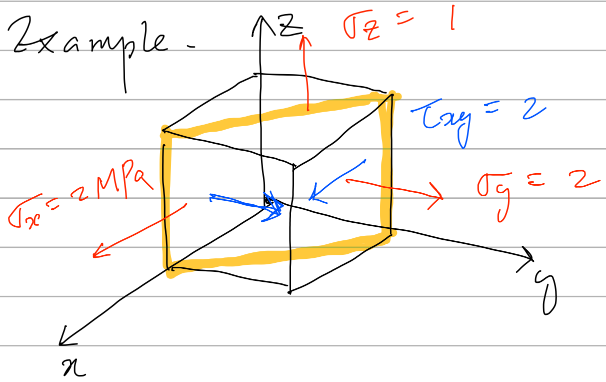
2) $\sigma_p = 1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{n}_p = 0 \qquad \hat{n}_p = (0, 0, 1)$$

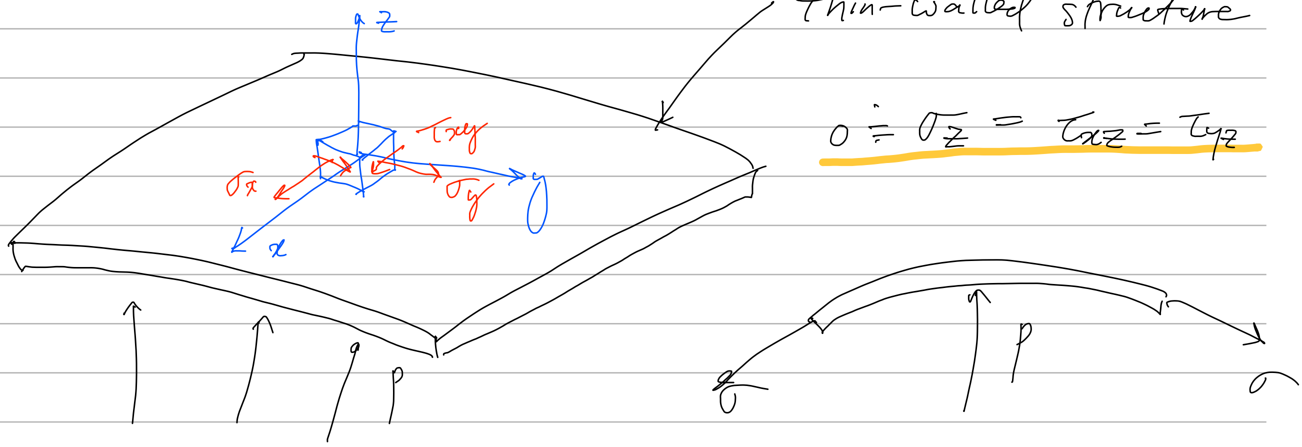
3) $\sigma_p = 0$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{n}_p = 0 \qquad \hat{n}_p = \frac{1}{\sqrt{2}}(1, -1, 0)$$

Example.



- Plane stress condition



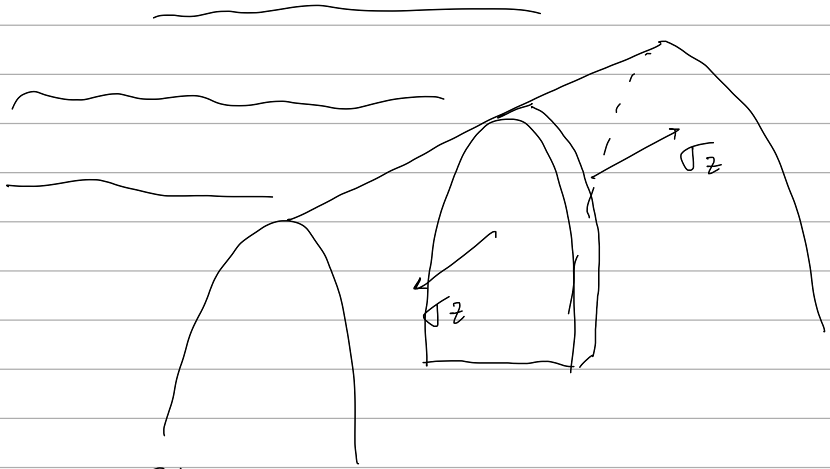
$0 \doteq \sigma_z = \tau_{xz} = \tau_{yz}$

Stresses in the out-of-plane direction can be neglected.

(Note $\varepsilon_z \neq 0$)

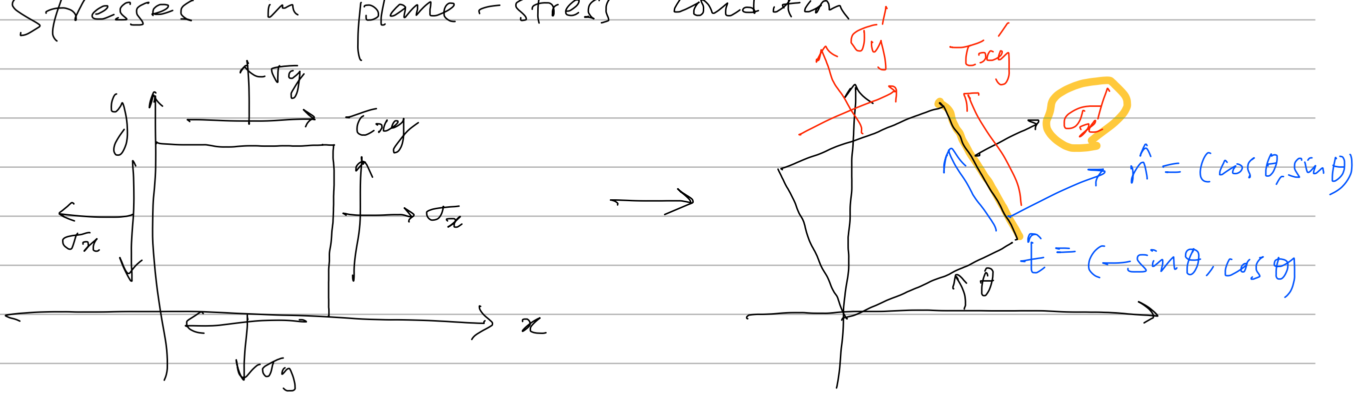
- Plane strain condition

$$0 = \varepsilon_{zz} = \gamma_{xz} = \gamma_{yz}$$



Strains in the out-of-plane direction can be neglected.
(Note $\sigma_z \neq 0$)

Stresses in plane-stress condition



$$\sigma = \begin{bmatrix} \boxed{\begin{matrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{matrix}} & \begin{matrix} \cancel{\tau_{xz}} \\ \cancel{\tau_{yz}} \end{matrix} \\ & \cancel{\sigma_z} \end{bmatrix}$$

$$\vec{T} = \sigma \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} = \begin{Bmatrix} \sigma_x \cos\theta + \tau_{xy} \sin\theta \\ \tau_{xy} \cos\theta + \sigma_y \sin\theta \end{Bmatrix}$$

$$\sigma_n = \vec{T} \cdot \hat{n} = \sigma_x \cos^2\theta + 2\tau_{xy} \sin\theta\cos\theta + \sigma_y \sin^2\theta = \sigma_x'$$

$$\tau = \vec{T} \cdot \hat{t} = -\sigma_x \sin\theta\cos\theta - \tau_{xy}\sin^2\theta + \tau_{xy}\cos^2\theta + \sigma_y \sin\theta\cos\theta$$

$$= \tau_{xy}'$$