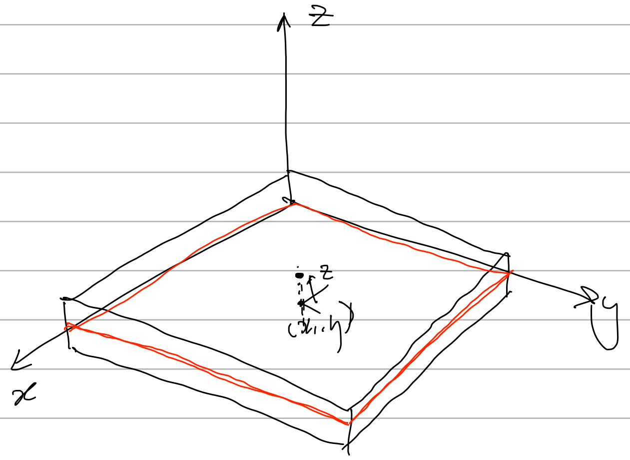
B.4. Plates and shells

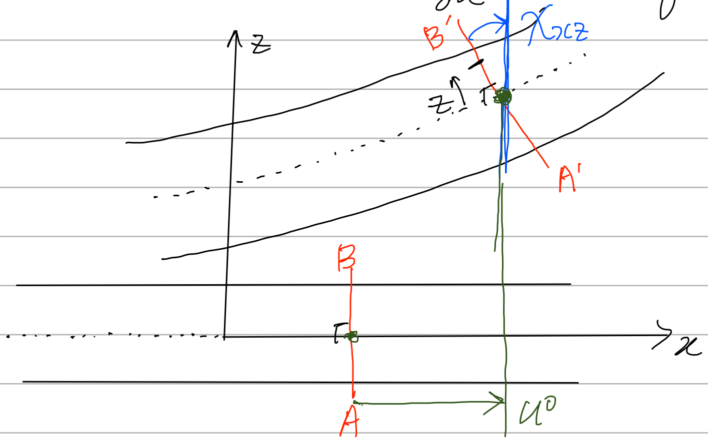
Axial loads : bars, Columbs
Torsional " : shafts
Bending " : beams



Reference plane ($z=0$)
→ Plate's mid-plane

Strains in the reference plane

$$\varepsilon_x^0 = \frac{\partial u^0}{\partial x} \qquad \varepsilon_y^0 = \frac{\partial v^0}{\partial y}, \qquad \gamma_{xy}^0 = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}$$



Kirchhoff hypothesis : normal to the reference surface remain normal and straight

$$\chi_{xz} = \frac{\partial w^0}{\partial x}$$

$$u = u^0 - z\,\chi_{xz}$$
$$= u^0 - z\frac{\partial w^0}{\partial x}$$

Similarly

$$v = v^0 - z\frac{\partial w^0}{\partial y}$$

Strain fields

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u^\circ}{\partial x} - z\,\frac{\partial^2 w^\circ}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v^\circ}{\partial y} - z\,\frac{\partial^2 w^\circ}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} - z\cdot\frac{2\,\partial^2 w^\circ}{\partial x \partial y}$$

$$\boxed{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^\circ \\ \varepsilon_y^\circ \\ \gamma_{xy}^\circ \end{Bmatrix} + z \begin{Bmatrix} \hat{k}_x \\ \hat{k}_y \\ \hat{k}_{xy} \end{Bmatrix}}$$

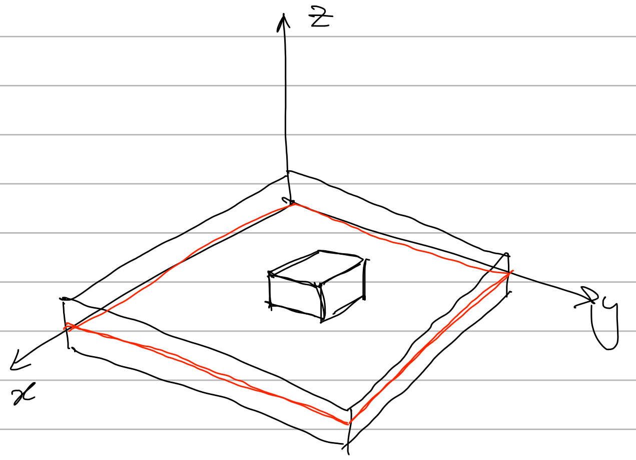
$$\hat{k}_x = -\frac{\partial^2 w^\circ}{\partial x^2} = -k_x \quad \left(= \frac{\partial^2 w^2}{\partial x^2}\right)$$

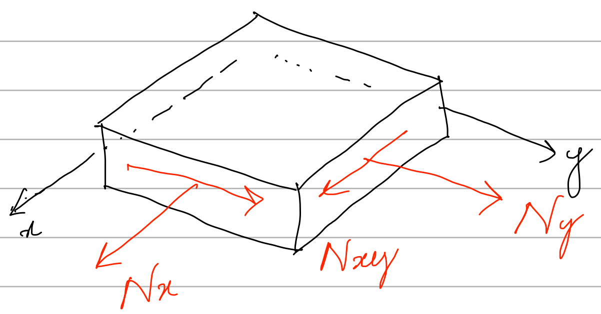
$$\hat{k}_y = -\frac{\partial^2 w^\circ}{\partial y^2} = -k_y$$

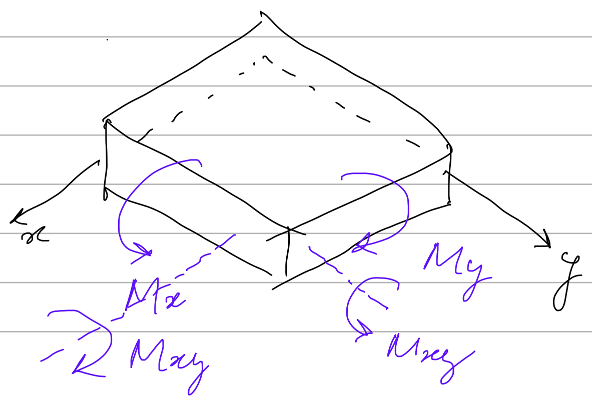
$$\hat{k}_{xy} = -\frac{2\,\partial^2 w^\circ}{\partial x \partial y} = -2k_{xy}$$

curvatures of plate

curvatures of surface







$$\boxed{N_x = \int_{-t/2}^{t/2} \sigma_x \, dz}$$

$$\left.\begin{aligned} N_y &= \int_{-t/2}^{t/2} \sigma_y \, dz \\ N_{xy} &= \int_{-t/2}^{t/2} \tau_{xy} \, dz \end{aligned}\right\}$$

in-plane forces per unit length

$$\left.\begin{aligned} M_x &= \int_{-t/2}^{t/2} z \cdot \sigma_x \, dz \\ M_y &= \int_{-t/2}^{t/2} z \, \sigma_y \, dz \\ M_{xy} &= \int_{-t/2}^{t/2} z \, \tau_{xy} \, dz \end{aligned}\right\}$$

in-plane moments per unit length

Plate theory

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \hat{\kappa}_x \\ \hat{\kappa}_y \\ \hat{\kappa}_{xy} \end{Bmatrix}$$

[A] : in-plane stiffness

[D] : bending stiffness

[B] : in-plane & out-of-plane coupling stiffness
(bending-extension coupling stiffness)

B.5.1. Isotropic plate under plane stress

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & E/G\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

compliance matrix

$E/G \rightarrow 2(1+\nu)$

$$G = \frac{E}{2(1+\nu)}$$

$$\nu\,\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\,\sigma_y)\cdot\nu$$

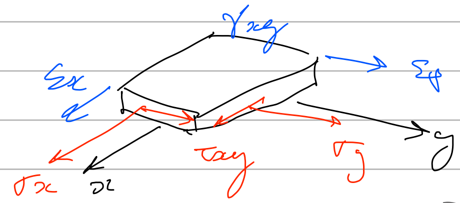
$$+\quad \varepsilon_y = \frac{1}{E}(-\nu\,\sigma_x + \sigma_y)$$

$$\nu\,\varepsilon_x + \varepsilon_y = \frac{1-\nu^2}{E}\,\sigma_y$$

$$\rightarrow \quad \sigma_y = \frac{E}{1-\nu^2}(\nu\,\varepsilon_x + \varepsilon_y)$$

stiffness matrix

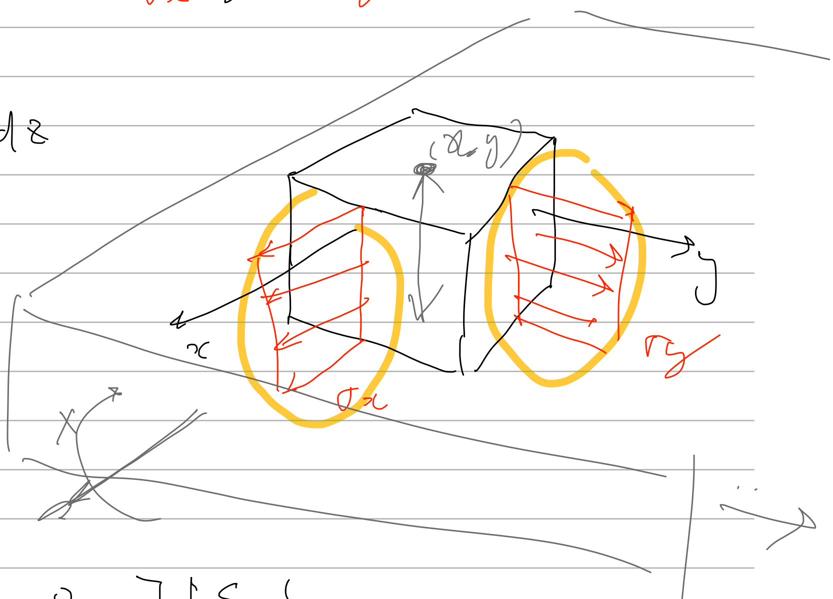
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$



$$N_x = \int_{-t/2}^{t/2} \sigma_x \, dz$$

$$= \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}(\varepsilon_x + \nu\,\varepsilon_y)\,dz$$

$$= \frac{Et}{1-\nu^2}(\varepsilon_x + \nu\,\varepsilon_y)$$



$N_y =$ —

$N_{xy} =$ —

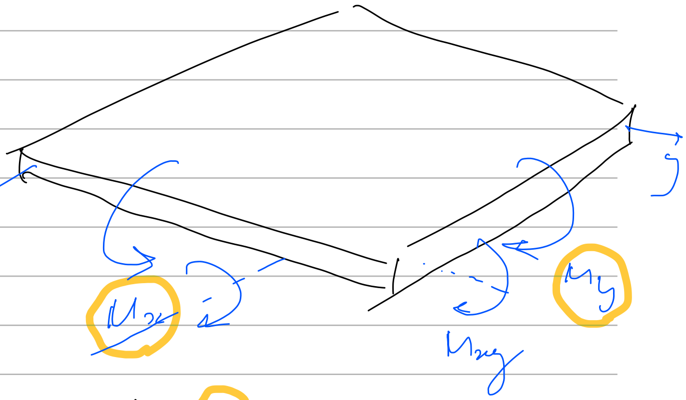
$$\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix} = \frac{Et}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

if $\nu = 1/3$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{9}{8} E t \overbrace{\begin{bmatrix} 1 & 1/3 & 0 \\ 1/3 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}^{[A] \text{ matrix}} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$M_x = \int_{-t/2}^{t/2} z \, \sigma_x \, dz$$

$$= \int_{-t/2}^{t/2} z \frac{E}{1-\nu^2} \left(z \hat{k}_x + \nu z \hat{k}_y \right) dz$$



K_x, K_y, K_{xy} ?

$$\left(\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} z \hat{k}_x \\ z \hat{k}_y \\ z \hat{k}_{xy} \end{Bmatrix} \right)$$

$$= \frac{E}{1-\nu^2} \left[\left(\hat{k}_x + \nu \hat{k}_y \right) \int_{-t/2}^{t/2} z^2 \, dz \right]$$

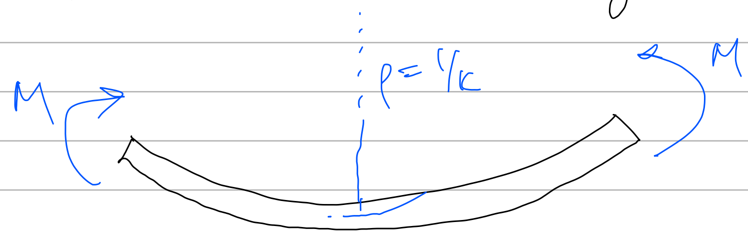
$$= \frac{E}{1-\nu^2} \left[\left(\hat{k}_x + \nu \hat{k}_y \right) \cdot \frac{z^3}{3} \Big|_{-t/2}^{t/2} \right]$$

$$M_x = \frac{E t^3}{12(1-\nu^2)} \left(\hat{k}_x + \nu \hat{k}_y \right)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{E t^3}{12(1-\nu^2)} \overbrace{\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}^{[D] \text{ matrix}} \begin{Bmatrix} \hat{k}_x \\ \hat{k}_y \\ \hat{k}_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} & 0 \\ 0 & \frac{E t^3}{12(1-\nu^2)} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \hat{k}_x \\ \hat{k}_y \\ \hat{k}_{xy} \end{Bmatrix}$$

c.f. Euler beam theory



$$\kappa = \frac{M}{E\,I}$$

loading

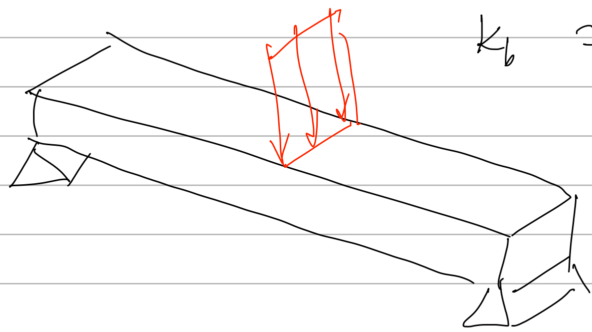
geometry

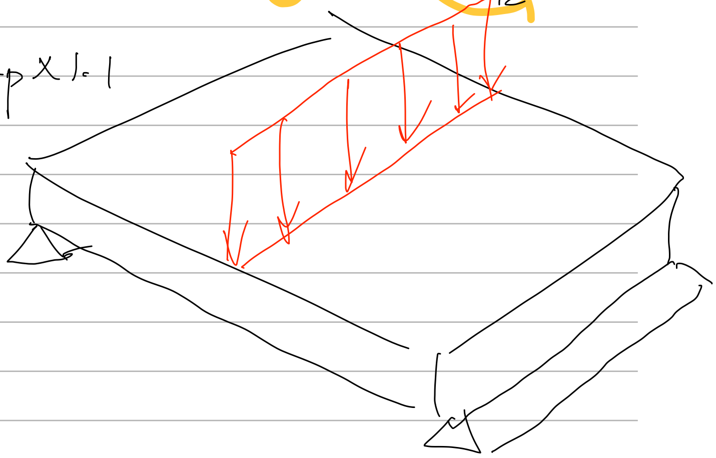
material

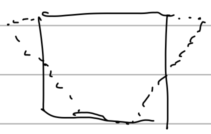
$$M = \frac{E t^3}{12(1-\nu^2)}\hat{\kappa}_x$$

$$\hat{\kappa}_x = \frac{M \cdot b\,(1-\nu^2)}{E \cdot \frac{b t^3}{12}}$$

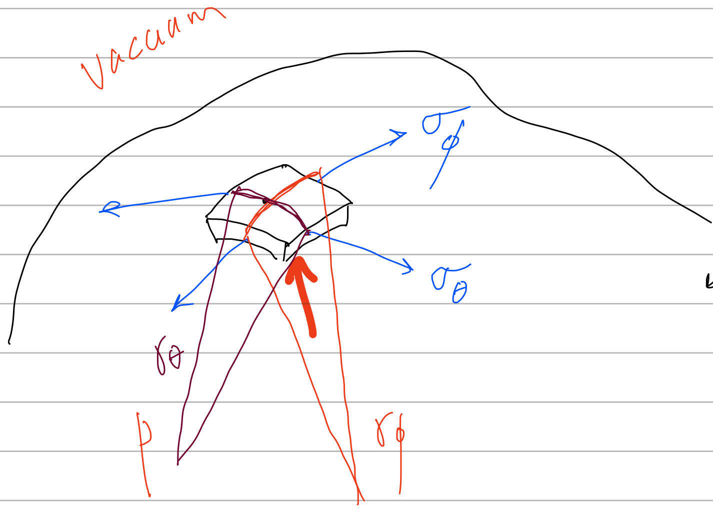
$$k_b \simeq k_p \times 1.1$$

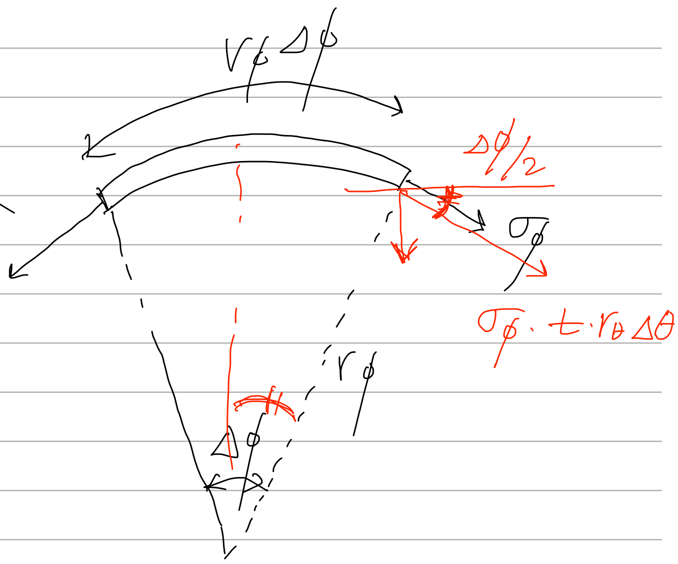






B.6. Equilibrium equations for shells





Assume $t \ll r_\theta, r_\phi$

$$p \cdot r_\phi \Delta\phi \cdot r_\theta \Delta\theta - \left[\sigma_\phi t\, r_\theta \Delta\theta \cdot \sin\frac{\Delta\phi}{2}\right] \times 2 - \left[\sigma_\theta t\, r_\phi \Delta\phi \cdot \sin\frac{\Delta\theta}{2}\right] \times 2 = 0$$

$\Delta\phi,\ \Delta\theta \simeq 0 \rightarrow \sin\frac{\Delta\phi}{2} \simeq \frac{\Delta\phi}{2} \quad \sin\frac{\Delta\theta}{2} \simeq \frac{\Delta\theta}{2}$

$$p\, r_\phi r_\theta \cancel{\Delta\phi\,\Delta\theta} = \sigma_\phi t\, r_\theta \cancel{\Delta\theta\,\Delta\phi} + \sigma_\theta \cdot t\, r_\phi \cancel{\Delta\theta\,\Delta\phi}$$

$$\frac{p}{t} = \frac{\sigma_\phi}{r_\phi} + \frac{\sigma_\theta}{r_\theta}.$$: Membrane equation

$\left(N_\phi = \int \sigma_\phi\, dz = \sigma_\phi \cdot t, \quad N_\theta = \sim \right)$

$$p = \frac{N_\theta}{r_\theta} + \frac{N_\phi}{r_\phi}$$

$\rightarrow \quad p \ll \sigma_\phi, \sigma_\theta$