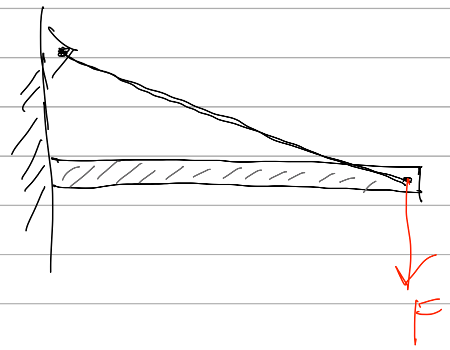
1. Truss structures
(pin-jointed structures)



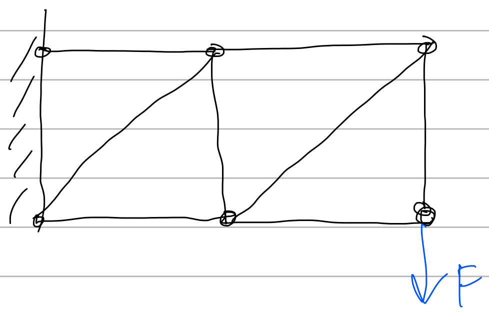
[bars
  joints

1.2. Rigidity theory

Maxwell's equation

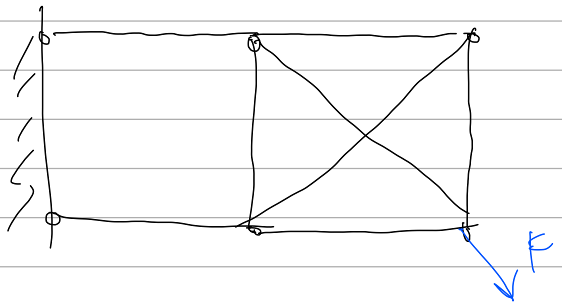
$$2 \times j - k - b \leq 0$$: Necessary condition but not sufficient condition

j : number of joints
k : 〃 of kinematic constraints
b : 〃 〃 pin-jointed bars



$j = 6$
$k = 4$
$b = 8$
$2 \times 6 - 4 - 8 = 0$

Kinematically determinate
: admits no mechanism
Statically determinate
: a structure that admits no state of self-stress



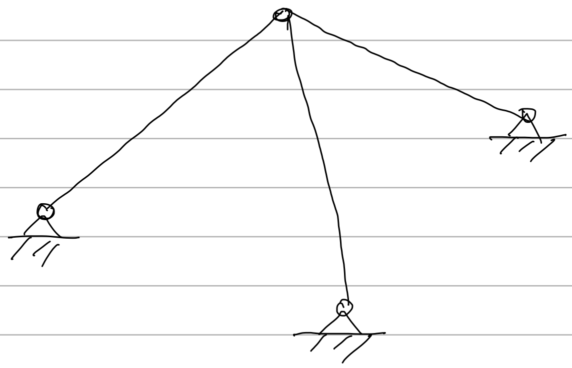
- mechanism
- admits a state of self-stress

$d \times j - b - k = m - s$

d : 2 or 3 depending on dimensions

m : number of independent mechanisms

s : " " states of self-stress



$d = 3$

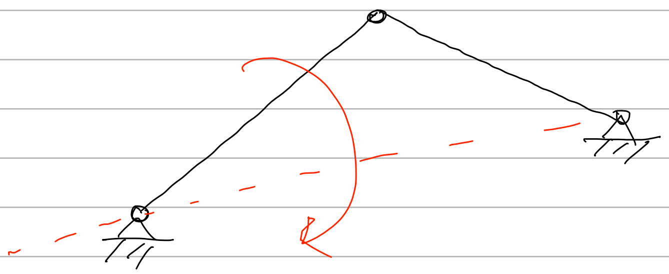
$j = 4$

$b = 3$

$k = 3 \times 3 = 9$

$3 \times 4 - 3 - 9 = 0$

$m = 0, \quad s = 0$



$d = 3$

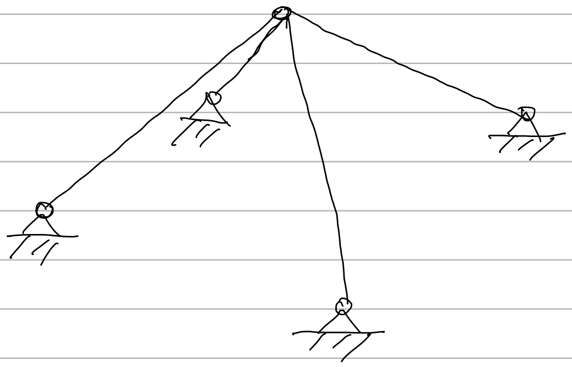
$j = 3$

$b = 2$

$k = 6$

$3 \times 3 - 2 - 6 = 1 = m - s$

$m = 1, \quad s = 0$



$d = 3$

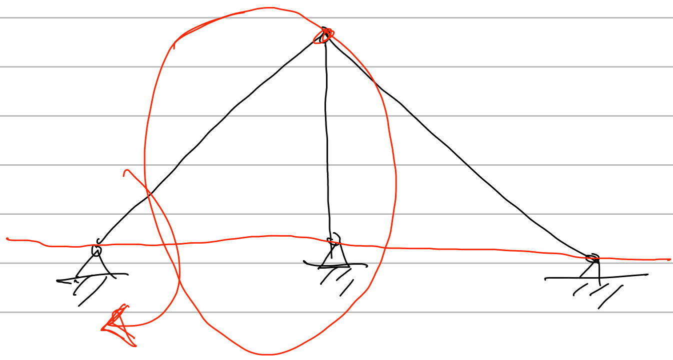
$j = 5$

$b = 4$

$k = 3 \times 4 = 12$

$3 \times 5 - 4 - 12 = -1$

$m = 0, \quad s = 1$



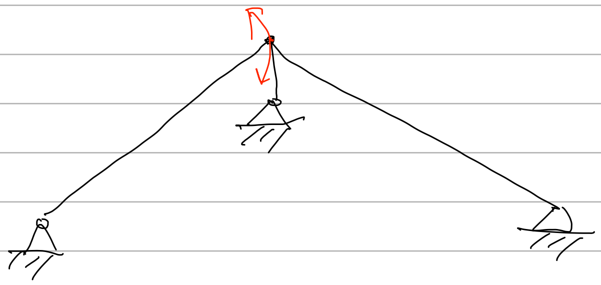
co-planar

$0 = m - s$

$m = 1$

$s = 1$

→ finite mechanism



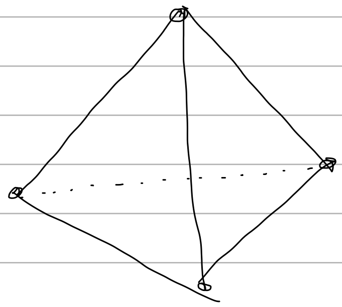
On the same plane

→ infinitesimal mechanism

Rigid structure

- kinematically determinate ($m = 0$)
- " indeterminate ($m \neq 0$) but supports infinitesimal mechanism

1.2.1 Polyhedral Trusses



Tetrahedron

$d = 3$

$j = 4$

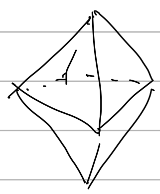
$b = 6$

$k = 0$

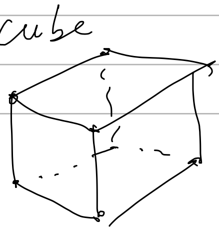
$3 \times 4 - 6 - 0 = 6$
- 3 translations
- 3 rotations

$m - 6 = m'$: number of independent "internal" mechanisms

Same $m' = 0$ for octahedron (8) and icosahedron (20)



cube, dodecahedron $\Rightarrow m' \neq 0$



Cauchy's theorem

→ Every convex polyhedral surface is rigid if all of its faces are triangles.

1.3. Rod-like trusses

$m = 0$: kinematically determinate (i.e., rigid)
→ can carry loads → low mass structure
$s = 0$: statically determinate
→ Uniquely determined by equilibrium consideration
(Disturbances, e.g., thermal gradients, manufacturing imperfections will not lead to over stressing)

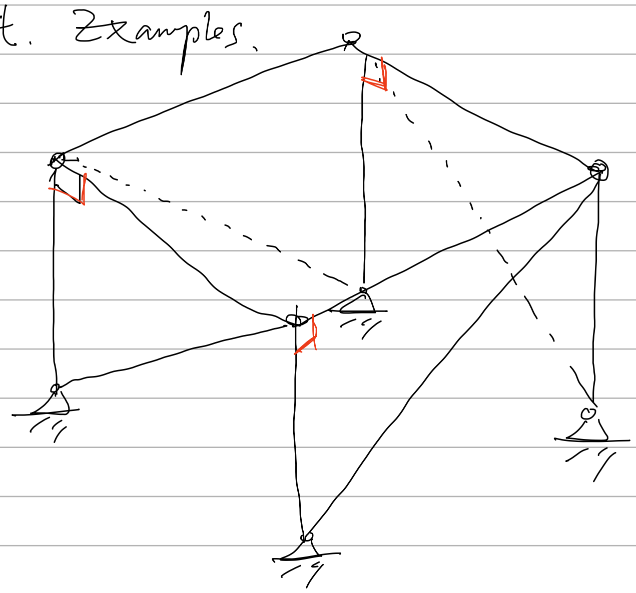
→ Easy to assemble
→ Shape can be adjusted by the length change of members

Maxwell's equation

$$3j - b - 6 = 0$$

$$j \doteq b/3$$

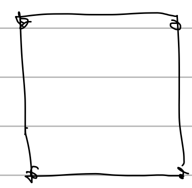
1.4. Examples.

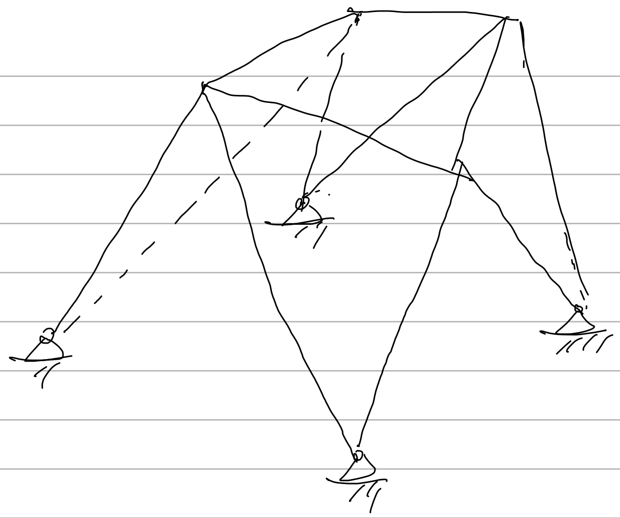


$j = 8$
$b = 12$
$k = 12$
$3 \times 8 - 12 - 12 = 0$

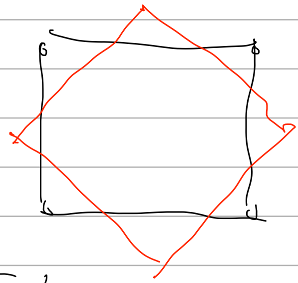
statically
kinematically) determinate

top view





top view



$m = s = 1$

: Allows a finite amplitude distortion of the structure (i.e., supports finite mechanism)

1.5. Rigidity computations

d : number of dimensions

j : " " joints

k : " " kinematic constraints

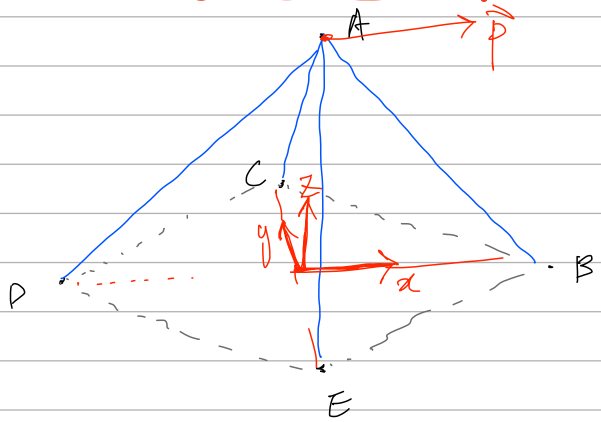
$n_r = d \times j - k$: number of linear equations

$n_c = b$: " bars

$\vec{t}$: axial force vector

$\vec{p}$: external load components

$$\boxed{A\vec{t} = \vec{p}}$$



$$\vec{t} = \begin{Bmatrix} t_{AB} \\ t_{AC} \\ t_{AD} \\ t_{AE} \end{Bmatrix}$$

x-direction : $\frac{t_{AB}}{\sqrt{2}} - \frac{t_{AD}}{\sqrt{2}} + P_x = 0$

y- " : $-\frac{t_{AE}}{\sqrt{2}} + \frac{t_{AC}}{\sqrt{2}} + P_y = 0$

z- " : $-\frac{1}{\sqrt{2}}(t_{AB} + t_{AC} + t_{AD} + t_{AE}) + P_z = 0$

$$\frac{1}{\sqrt{2}}\boxed{\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}\begin{Bmatrix} t_{AB} \\ t_{AC} \\ t_{AD} \\ t_{AE} \end{Bmatrix} = \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}$$

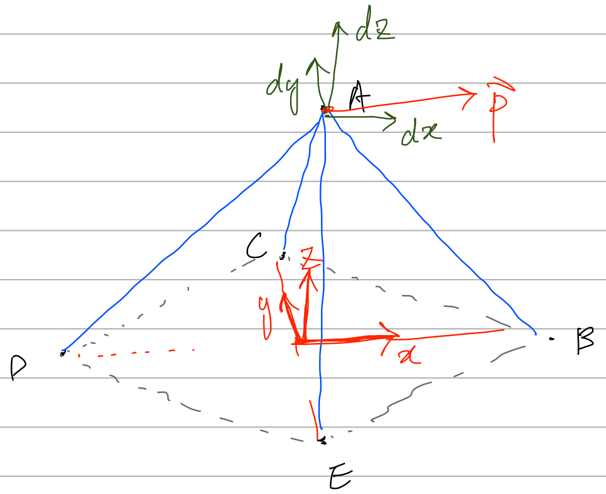
$n_r = 3\times 5 - 3\times 4 = 3$

$n_c = 4$

$A\vec{t} = \vec{p}$ $\qquad A: \; n_r \times n_c$

$B\vec{d} = \vec{e} \Leftarrow$ extension of the bars

$\uparrow$ displacement components of joints



$$\vec{d}_A = \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix}$$

$$\vec{e} = \begin{Bmatrix} e_{AB} \\ e_{AC} \\ e_{AD} \\ e_{AE} \end{Bmatrix}$$

$$e_{AB} = -\frac{1}{\sqrt{2}}dx + \frac{1}{\sqrt{2}}dz$$

$$e_{AC} = -\frac{1}{\sqrt{2}}dy + \frac{1}{\sqrt{2}}dz$$

$$e_{AD} = \frac{1}{\sqrt{2}}dx + \frac{1}{\sqrt{2}}dz$$

$$e_{AE} = \frac{1}{\sqrt{2}}dy + \frac{1}{\sqrt{2}}dz$$

$$\boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{Bmatrix} e_{AB} \\ e_{AC} \\ e_{AD} \\ e_{AE} \end{Bmatrix}$$

$B\vec{d} = \vec{e}$

$B = A^T$

r : rank of the matrix

(equal to the number of linearly independent rows or columns)

$$n_r \rightarrow d \times j - k$$

$$m = n_r - r$$

$$s = n_c - r$$

$$m - s = (n_r - r) - (n_c - r) = d \times j - k - b$$

$m = 0$: kinematically determinate (i.e. rigid)

$s = 0$: statically " (i.e. bar forces depend only on the external forces)

1.5.1. SVD of Equilibrium matrix

A : dimension of $n_r \times n_c$, rank r

Singular value decomposition

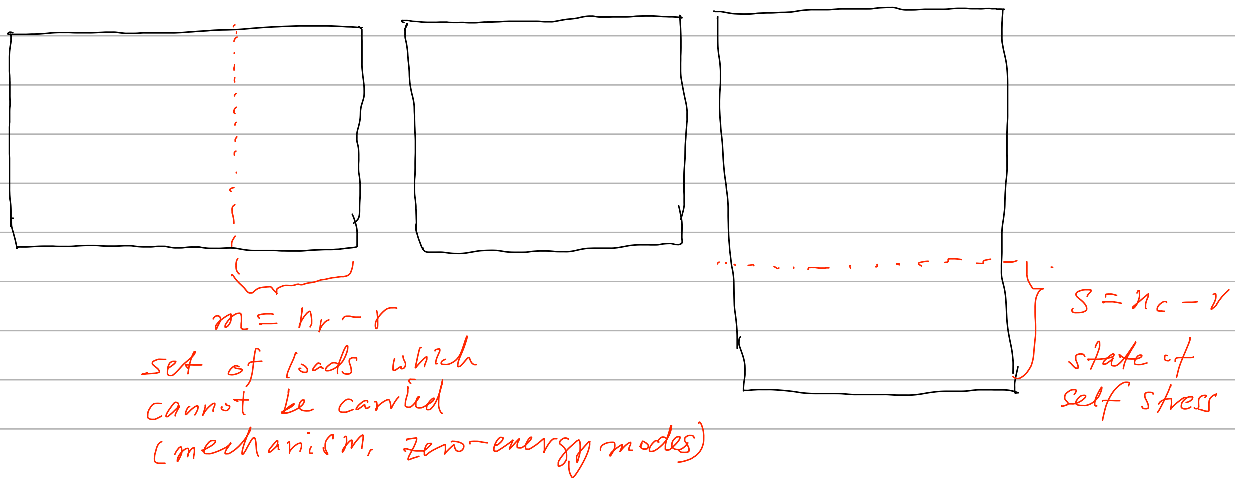
$$A = U V W^T$$

$$U \cdot U^T = I$$

$U = [\vec{u}_1, \cdots \vec{u}_{n_r}]$: $n_r \times n_r$ orthogonal matrix

$W = [\vec{w}_1, \cdots \vec{w}_{n_c}]$: $n_c \times n_c$ orthogonal matrix

V : r positive elements on the leading diagonal



Back to example (Case - I)

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}$$

① ② ③

state of self-stress

rank of A: $r = 3$

$m = 3 - 3 = 0$

$s = 4 - 3 = 1$

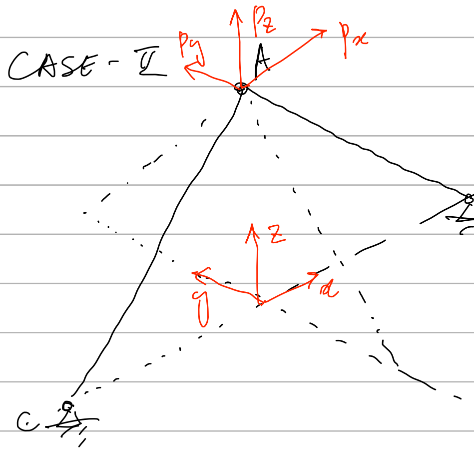
① × ② → ③

loading ; bar force

① × $\frac{1}{②}$ → ③

joint displacement ; bar extension

CASE - II



Equilibrium

$$\frac{1}{\sqrt{2}} t_{AB} - \frac{1}{\sqrt{2}} t_{AC} + P_x = 0$$

$$P_y = 0$$

$$-\frac{1}{\sqrt{2}} t_{AB} - \frac{1}{\sqrt{2}} t_{AC} + P_z = 0$$

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{Bmatrix} t_{AB} \\ t_{AC} \end{Bmatrix} = \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}$$

$d=3, \quad j=3, \quad k=6, \quad b=2, \quad r=2$

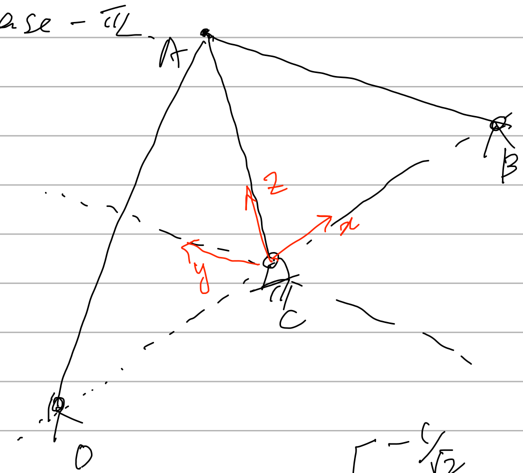
$$m = n_r - r = (3\times 3 - 6) - 2 = 1$$

$$s = n_c - r = 2 - 2 = 0$$

$$U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

mechanism

case - III



Equilibrium

$$\frac{1}{\sqrt{2}} t_{AB} - \frac{1}{\sqrt{2}} t_{AD} + p_x = 0$$

$$p_y = 0$$

$$-\frac{1}{\sqrt{2}} t_{AB} - \frac{1}{\sqrt{2}} t_{AD} - t_{AC} + p_z = 0$$

$$\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix} \begin{Bmatrix} t_{AB} \\ t_{AC} \\ t_{AD} \end{Bmatrix} = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

$d=3, \quad j=4, \quad k=9, \quad b=3, \quad r=2$

$$m = n_r - r = (3\times 4 - 9) - 2 = 1$$

$$s = n_c - r = 3 - 2 = 1$$

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad V = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1/2 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & -1/2 \end{bmatrix}$$

$\boxed{\text{mechanism}}$

self stress