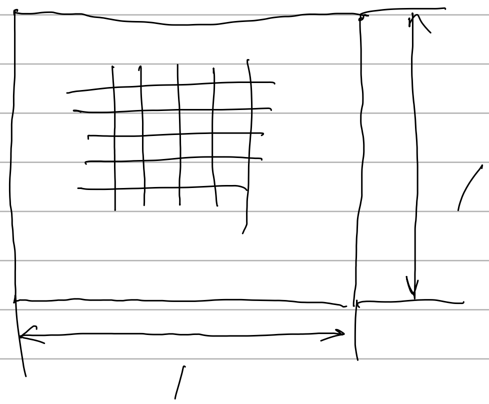
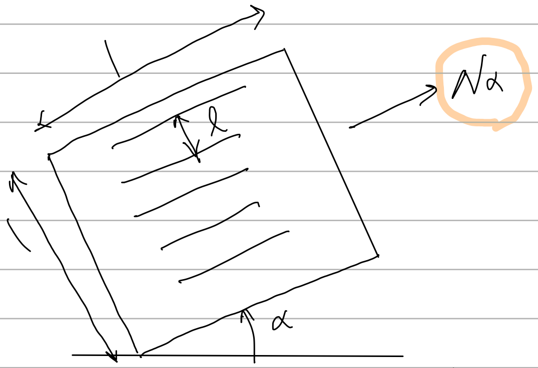
Chapter 2. Space frames

2.3. Continuum models for single layer space frames



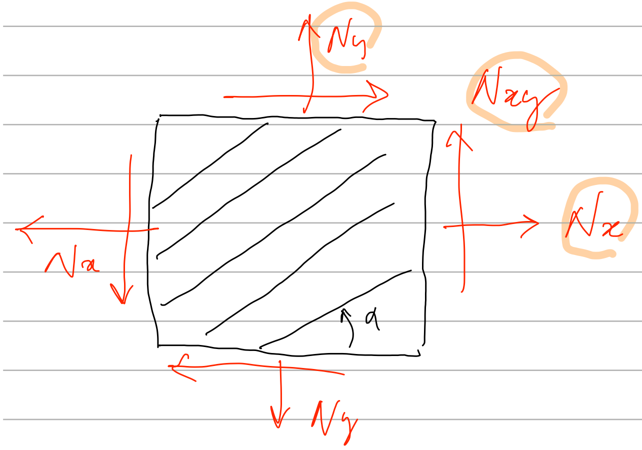
bars : A, E



N_α : sum of the bar axial forces in unit width of space frame

$$\boxed{\frac{N_\alpha \cdot l}{A} = E \cdot \varepsilon_\alpha}$$

ε_α : normal strain



$$\boxed{\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} \cos^2\alpha \\ \sin^2\alpha \\ \sin\alpha\cos\alpha \end{Bmatrix} N_\alpha}$$

Global coordinate

Local coordinate

c.f. Transformation matrices

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

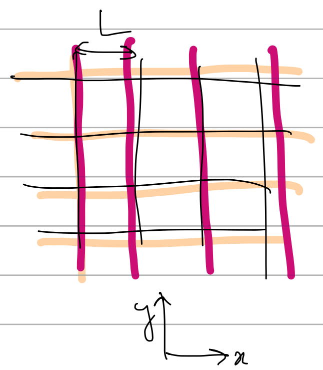
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -sc \\ s^2 & c^2 & sc \\ 2sc & -2sc & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} \cos^2\alpha \\ \sin^2\alpha \\ \sin\alpha\cos\alpha \end{Bmatrix} [N_\alpha]$$

$$= \begin{Bmatrix} \cos^2\alpha \\ \sin^2\alpha \\ \sin\alpha\cos\alpha \end{Bmatrix} \frac{AE}{l} \left[\cos^2\alpha \;\; \sin^2\alpha \;\; \sin\alpha\cos\alpha\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \frac{AE}{l} \underbrace{\begin{bmatrix} \cos^4\alpha & \sin^2\alpha\cos^2\alpha & \sin\alpha\cos^3\alpha \\ \sin^2\alpha\cos^2\alpha & \sin^4\alpha & \sin^3\alpha\cos\alpha \\ \sin\alpha\cos^3\alpha & \sin^3\alpha\cos\alpha & \sin^2\alpha\cos^2\alpha \end{bmatrix}}_{\text{Homogenized stiffness matrix}} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

e.g.,



Horizontal : $\alpha = 0$, $l = L$

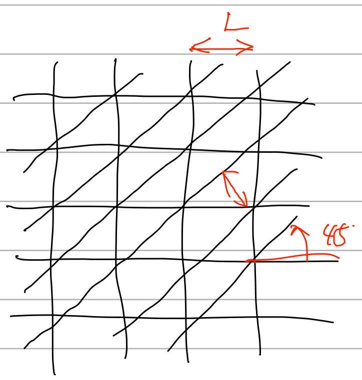
$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Vertical : $\alpha = 90^\circ$: $l = L$

$$\frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{— (*)}$$

Rank 2 : lack of shearing stress
(exists zero-stiffness deformation mode)



Add diagonal stiffness term

$\alpha = 45^\circ$ $\quad l = \frac{\sqrt{2}}{2}L$

Diagonal

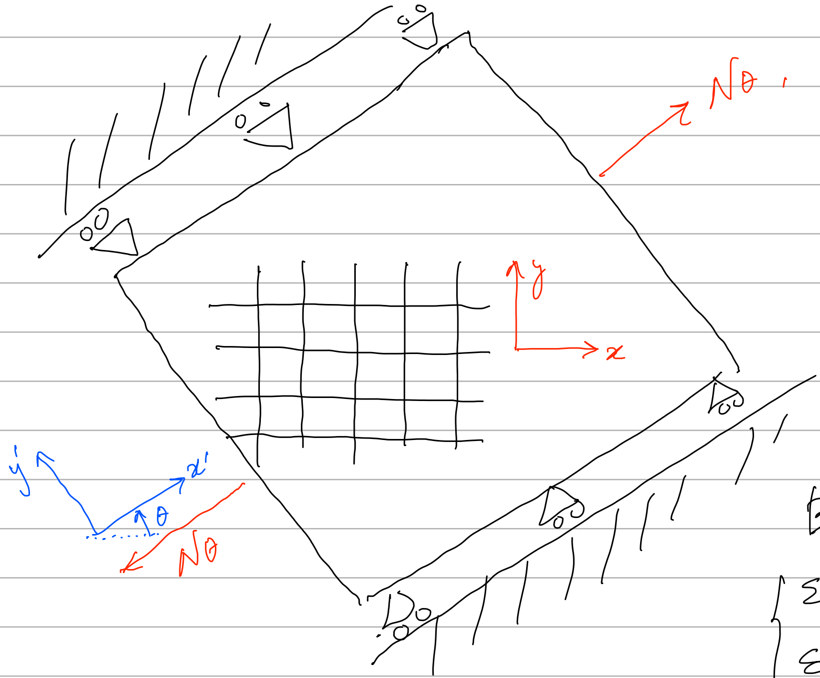
$$\frac{AE}{\frac{1}{\sqrt{2}}L}\begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Summing up with (*)

$$\frac{AE}{L}\begin{bmatrix} 1+a & a & a \\ a & 1+a & a \\ a & a & a \end{bmatrix} \qquad a = \frac{1}{2\sqrt{2}}$$

Rank 3 (full rank)

2.3.2 Stiffness in direction θ



Stiffness of the space frame in θ-direction

$$E_\theta^* = \frac{N_\theta}{\varepsilon_\theta}$$

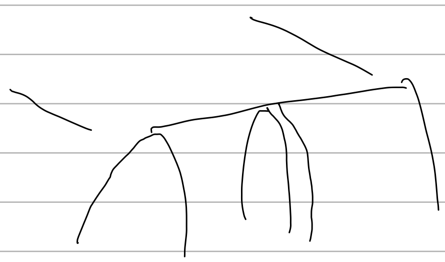
Boundary condition

$$\begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_\theta \\ 0 \\ 0 \end{Bmatrix}$$

→ plane strain condition

From strain transformation matrix

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -sc \\ s^2 & c^2 & sc \\ 2sc & -2sc & c^2-s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_\theta \\ 0 \\ 0 \end{Bmatrix}$$



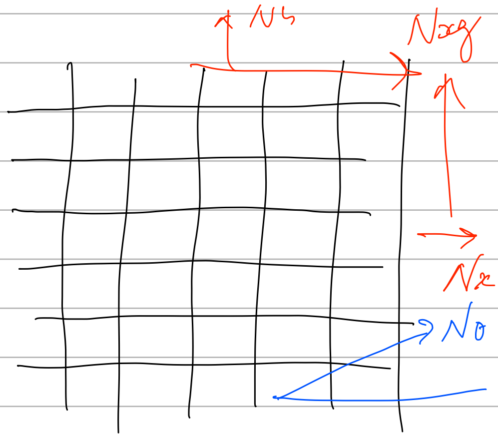
$$= \begin{Bmatrix} c^2 \\ s^2 \\ 2sc \end{Bmatrix} \varepsilon_\theta$$

Likewise

$$\begin{Bmatrix} N_\theta \\ \vdots \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}$$

$$N_\theta = N_x \cos^2\theta + N_y \sin^2\theta + 2 N_{xy} \sin\theta \cos\theta$$

2.4.1. Square lattice



Recall

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$N_\theta = \frac{AE}{L} \varepsilon_x \cdot \cos^2\theta + \frac{AE}{L} \varepsilon_y \cdot \sin^2\theta$$

$$= \frac{AE}{L} \cdot \cos^4\theta \, \varepsilon_\theta + \frac{AE}{L} \sin^4\theta \cdot \varepsilon_\theta$$

$$= \frac{AE}{L} (\cos^4\theta + \sin^4\theta) \cdot \varepsilon_\theta$$

$$E_\theta^* = \frac{N_\theta}{\varepsilon_\theta} = \frac{AE}{L} (\cos^4\theta + \sin^4\theta)$$

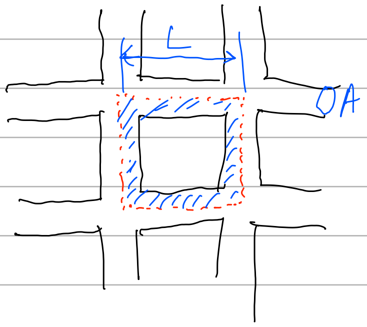
2.4. In-plane efficiency of single-layer space frames

: Stiffness is compared to that of a homogeneous, continuous plate of arbitrary thickness t and made from the same material as the bars of our space frame

$$\mu_\theta = \frac{E_\theta^* / \rho^*}{E_\theta / \rho} = \frac{E_\theta^*}{\alpha E_\theta}$$

$$\rho^* = \frac{\text{mass of bars in repeating unit}}{\text{Volume of repeating unit}}$$

$$\alpha = \frac{\rho^*}{\rho} \; : \text{ relative density (i.e., volume fraction)}$$



$$\alpha = \frac{2A \cdot L}{L^2 \cdot t}$$

Recall B.5.1. Plate under plane stress

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \cdot t = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

if $\nu = 1/3$
$$= \frac{9}{8} Et \begin{bmatrix} 1 & 1/3 & 0 \\ 1/3 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Under $\varepsilon_y = \gamma_{xy} = 0$

$$N_x = \frac{9}{8} E t \, \varepsilon_x$$

$$\therefore E_\theta = \frac{N_x}{\varepsilon_x} = \underline{\frac{9}{8} E t}$$

$$M_\theta = \frac{E_\theta^*}{\alpha E_\theta} = \frac{\frac{\cancel{A}\cancel{E}}{\cancel{L}}(\cos^4\theta + \sin^4\theta)}{\frac{9}{8} \cancel{E}\cancel{t}} \cdot \frac{\cancel{L^2}\cancel{t}}{2\cancel{A}\cancel{L}}$$

$$\left(\alpha = \frac{2A \cdot L}{L^2 \cdot t}\right) \qquad = \frac{4}{9}(\cos^4\theta + \sin^4\theta)$$