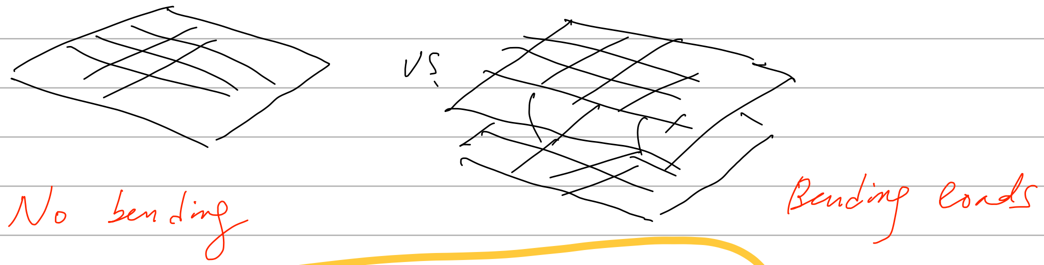
## 2.5. Continuum models for double-layer space frames



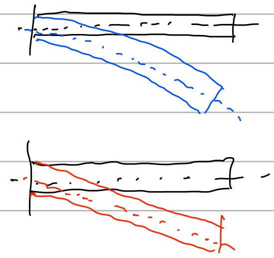

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Homogenized stiffness matrix

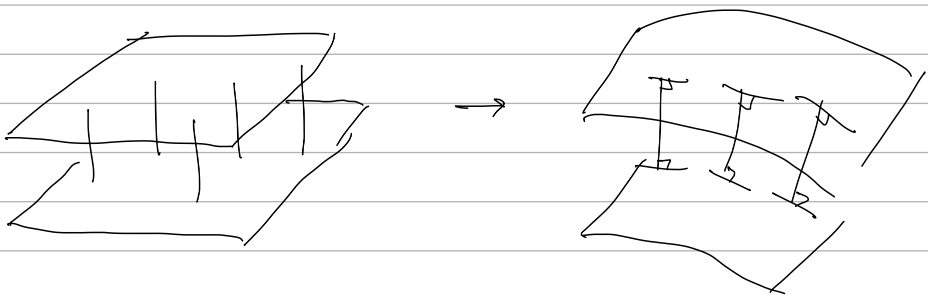
Assumptions

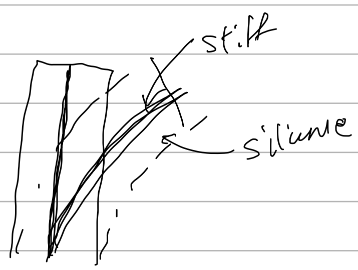
By connecting the two single layers

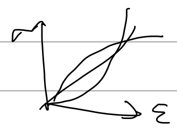
- rigid in shear (equivalent to Kirchhoff-Love plate model)



- compliant in bending and stretching





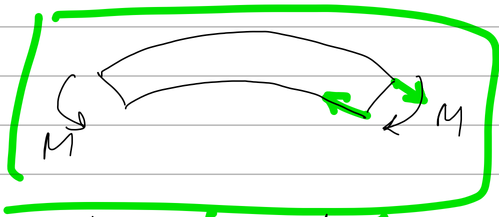

$M = EI\kappa$ or $\kappa = \dfrac{M}{EI}$

$E = \dfrac{1}{2}\dfrac{M^2}{EI}$

- In-plane loading $\qquad A = 2A_{SL}$

- General inextensional deformation

$$\begin{Bmatrix} \varepsilon_x^b \\ \varepsilon_y^b \\ \gamma_{xy}^b \end{Bmatrix} = - \begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_{xy}^t \end{Bmatrix}$$

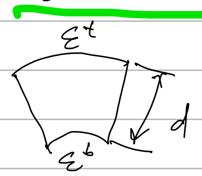



$$\boxed{\begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix}} = A_{SL} \begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_{xy}^t \end{Bmatrix} = -A_{SL} \begin{Bmatrix} \varepsilon_x^b \\ \varepsilon_y^b \\ \gamma_{xy}^b \end{Bmatrix} = \boxed{- \begin{Bmatrix} N_x^b \\ N_y^b \\ N_{xy}^b \end{Bmatrix}}$$

Mid-plane strain

$$\varepsilon_x = \frac{\varepsilon_x^t + \varepsilon_x^b}{2} = 0 \qquad \text{likewise} \quad \varepsilon_y = \gamma_{xy} = 0$$

$$\hat{\kappa}_x = \frac{\varepsilon_x^t - \varepsilon_x^b}{d} = 2\frac{\varepsilon_x^t}{d}$$




$$\hat{\kappa}_y = 2\frac{\varepsilon_y^t}{d}$$

$$\hat{\kappa}_{xy} = 2\frac{\gamma_{xy}^t}{d}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix} + \begin{Bmatrix} N_x^b \\ N_y^b \\ N_{xy}^b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{d}{2} \begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix} - \frac{d}{2} \begin{Bmatrix} N_x^b \\ N_y^b \\ N_{xy}^b \end{Bmatrix} = d \begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ d \begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} 2A_{SL} & B \\ B & D \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{2}{d} \begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_{xy}^t \end{Bmatrix} \end{Bmatrix}$$

$B = 0$ : No in-plane & out-of-plane coupling

$$\begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix} = \frac{2}{d^2} D \begin{Bmatrix} \varepsilon_x^t \\ \varepsilon_y^t \\ \gamma_{xy}^t \end{Bmatrix}$$

$$\frac{2}{d^2} D = A_{SL} \rightarrow D = \frac{d^2}{2} A_{SL} = \frac{d^2}{4} A \quad (A = 2A_{SL})$$

$$ABD = \begin{bmatrix} A & 0 \\ 0 & \frac{d^2}{4} A \end{bmatrix}$$

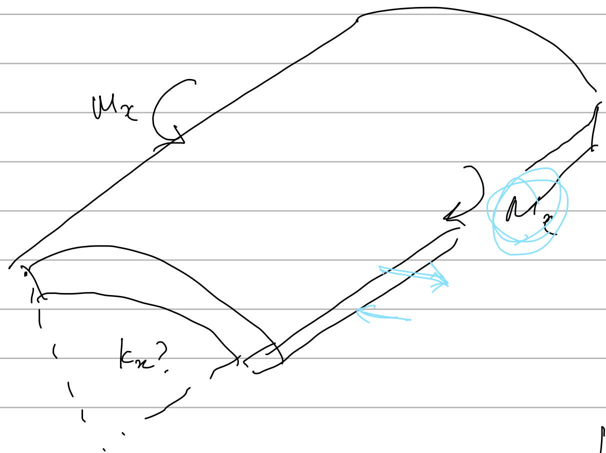
## 2.6. Bending efficiency of double-layer space frames

$$\begin{Bmatrix} \hat{k}_x \\ \hat{k}_y \\ \hat{k}_{xy} \end{Bmatrix} = \begin{Bmatrix} \neq 0 \\ 0 \\ 0 \end{Bmatrix}$$ : Boundary condition

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{Bmatrix} \hat{k}_x \\ \hat{k}_y \\ \hat{k}_{xy} \end{Bmatrix}$$

$$\boxed{\begin{aligned} M_x &= D_{11} \hat{k}_x \\ D_0^* &= \frac{M_x}{\hat{k}_x} = D_{11} \end{aligned}}$$

Bending of thin flat plates




Plane stress condition

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\varepsilon_x = \kappa_x \cdot z = -\frac{\partial^2 w}{\partial x^2}\cdot z$$

$$\left( \text{c.f.} \quad \varepsilon_x = \varepsilon_0 - \frac{\partial^2 w}{\partial x^2}\cdot z \rightarrow \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}\hat{\kappa}_x\\ \hat{\kappa}_y\\ \hat{\kappa}_{xy}\end{Bmatrix} \right)$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\,\varepsilon_y\right)$$

$$= \boxed{\frac{E}{1-\nu^2}\cdot\left(-\frac{\partial^2 w}{\partial x^2}\cdot z\right.} + \underbrace{\nu\left(-\frac{\partial^2 w}{\partial y^2}\, z\right)}_{\to 0}\Bigg)$$

$$M_x = \int_{-t/2}^{t/2} z\cdot \sigma_x \, dz$$

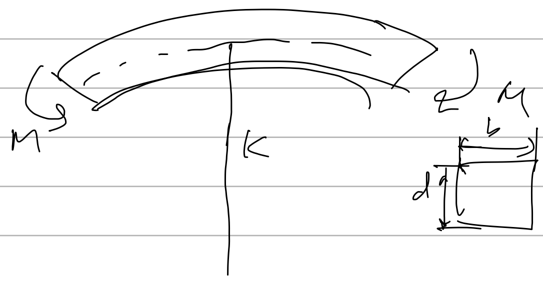
$$= \int_{-t/2}^{t/2} z^2\cdot\frac{E}{1-\nu^2}\cdot\left(-\frac{\partial^2 w}{\partial x^2}\right) dz$$

$$= -\frac{E t^3}{12(1-\nu^2)}\cdot\frac{\partial^2 w}{\partial x^2} = \frac{E t^3}{12(1-\nu^2)}\cdot \kappa_x$$

Plugging in $\nu = 1/3$

$$\begin{Bmatrix}M_x\\ M_y\\ M_{xy}\end{Bmatrix} = \frac{3}{32} E t^3 \begin{bmatrix}1 & 1/3 & 0\\ 1/3 & 1 & 0\\ 0 & 0 & 1/3\end{bmatrix}\begin{Bmatrix}\hat{\kappa}_x\\ \hat{\kappa}_y\\ \hat{\kappa}_{xy}\end{Bmatrix}$$

$$D_0 = \frac{M_x}{\hat{\kappa}_x} = \frac{3}{32} E d^3$$




$$\frac{M}{EI} = K$$

$$\frac{M}{K} = EI = E\,\frac{1}{12}\, b d^3$$

$$\rightarrow \frac{M/b}{K} = \frac{E d^3}{12}$$

example. Square lattice

$$D = \frac{d^2}{2} A_{SL} = \frac{d^2}{2} \cdot \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_0^* = D_{11} \quad \frac{AEd^2}{2L}$$

Bending efficiency of double layer square lattice

$$\mu_0' = \frac{D_0^*}{\alpha D_0}$$

$$= \frac{\cancel{AE}\,\cancel{d^2}}{2\cancel{L}} \cdot \frac{3\cancel{2}^{4}}{3} \cdot \frac{1}{\cancel{E}\,d^3} \cdot \frac{1}{8\sqrt{\cancel{2}}} \frac{\cancel{L^2}}{\cancel{A}} = \frac{2}{3}$$

$$\left( \alpha = \frac{(2+2+4) \cdot AL}{L^2 \; L/\sqrt{2}} = 8\sqrt{2}\frac{A}{L^2} \right)$$

In-plane efficiency

$$\frac{E}{\alpha E_0} = \frac{L^2}{\cancel{8}\sqrt{2} \cdot \cancel{A}} \cdot \frac{\cancel{AE}/L \cdot 2}{9/\cancel{8}\,Et} = \frac{\sqrt{2}}{9} \cdot \frac{L}{L/\sqrt{2}} = \frac{2}{9}$$