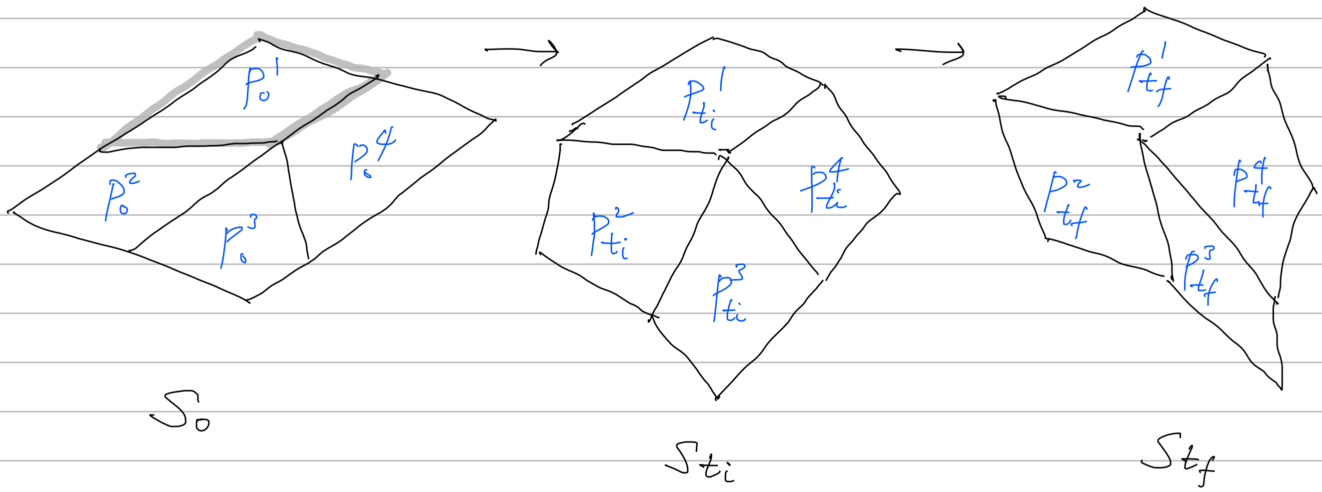


$t \in [0, t_f]$ time parameter

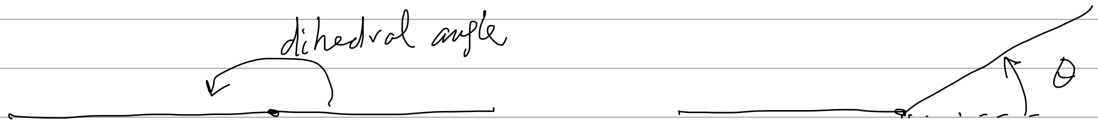
S_t : current configuration of a sheet at t

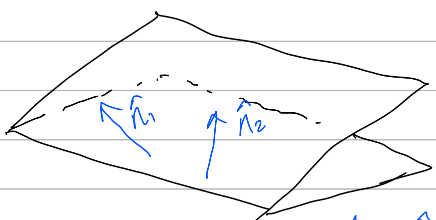
$P_t^1, P_t^2, \dots, P_t^{N_p} \subset S_t$: configurations of the faces

N_p : number of faces in the sheet

Assumption

1. The faces have undergone only rigid deformations
   (no stretch or bend)
2. The sheet is not torn
3. The sheet does not self-intersect.





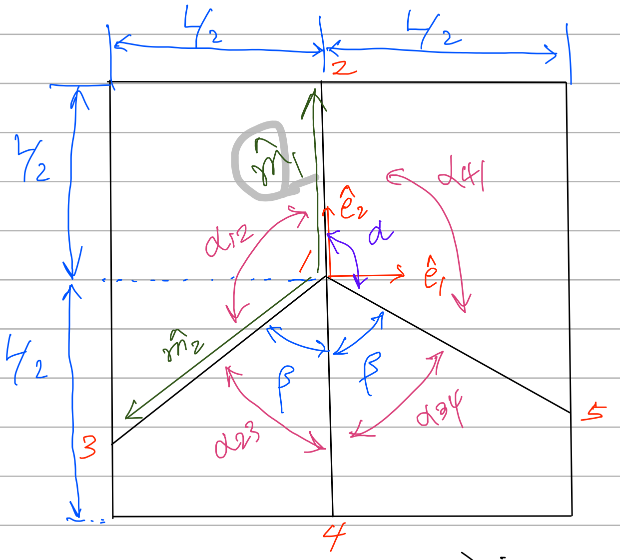
$$\cos\theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1| \cdot |\hat{n}_2|}$$

Fold angles

$\theta_1(t), \dots \theta_{N_F}(t)$

N_F : number of crease folds

Example : Miura-ori



Vertex $\vec{v}^j \in \mathrm{span}(\hat{e}_1, \hat{e}_2)$

$$\vec{v}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ L/2 \\ 0 \end{bmatrix} \qquad \vec{v}^3 = \begin{bmatrix} -L/2 \\ -L/2 \, \frac{1}{\tan\beta} \\ 0 \end{bmatrix}$$

$$\vec{v}^4 = \begin{bmatrix} 0 \\ -L/2 \\ 0 \end{bmatrix} \qquad \vec{v}^5 = \begin{bmatrix} L/2 \\ -L/2 \, \frac{1}{\tan\beta} \\ 0 \end{bmatrix}$$

N_I : # of vertices located at the interior (=1)

N_B : " " " " boundary (=4)

Fold connectivity matrix $C^F \in R^{N_F \times 2}$

$$C^F = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$$

C^F_{i2} : the same to the end point

C^F_{i1} : index of the vertex corresponding to the start point of the i-th fold line

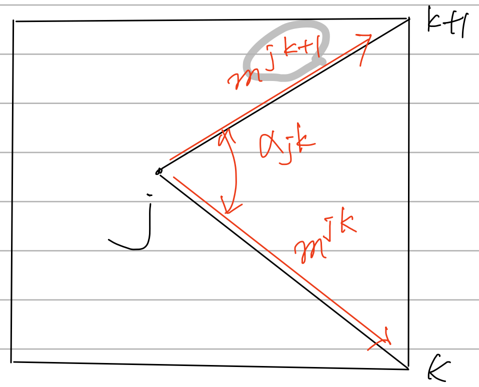
Fold vectors

$$\hat{m}_1 = \hat{v}^{12} - \hat{v}^{11} = \begin{bmatrix} 0 \\ L/2 \\ 0 \end{bmatrix}$$

$$\hat{m}_2 = \hat{v}^{22} - \hat{v}^{21} = \begin{bmatrix} -L/2 \\ -L/2 \, \frac{1}{\tan\beta} \\ 0 \end{bmatrix}$$

Crease angles

$$\alpha_{jk} = \begin{cases} \cos^{-1} \dfrac{\hat{m}^{jk+1} \cdot \hat{m}^{jk}}{\| \hat{m}^{jk+1} \| \, \| \hat{m}^{jk} \|} & ; \ (\hat{e}_3 \times \hat{m}^{jk}) \cdot \hat{m}^{jk+1} \geq 0 \\ 2\pi - \quad '' & ; \quad '' \quad < 0 \end{cases}$$



Input parameters

- position vectors of vertices
- fold connectivity matrix

Developability constraint

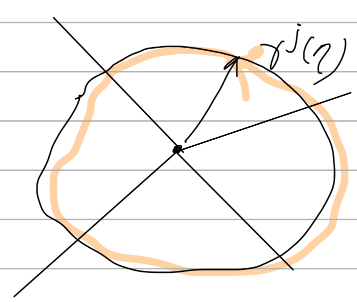
: For any interior fold intersection in the sheet, the discrete Gaussian curvature must be zero for it to be developable

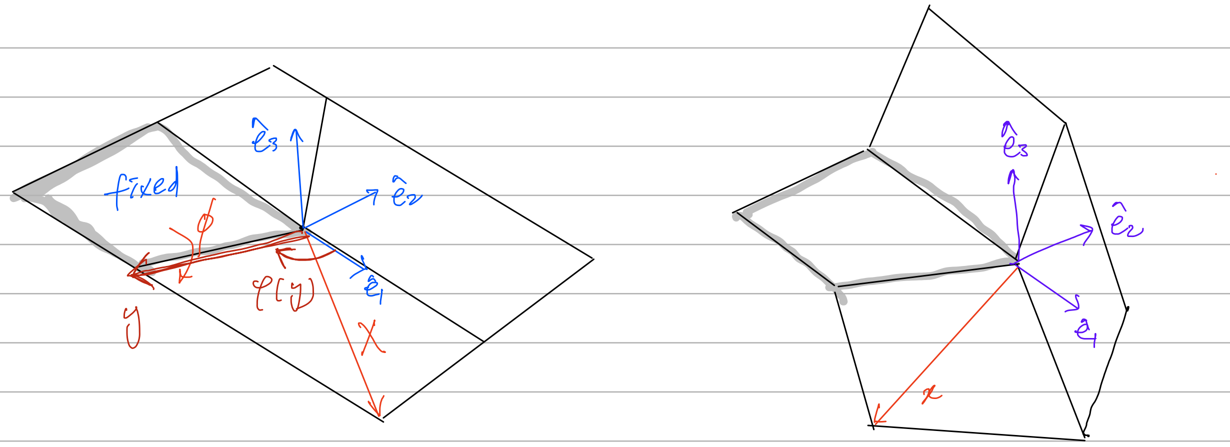
$$K_j = 2\pi - \sum_{k=1}^{n_j} \alpha_{jk} = 0 \ , \ j = 1, \ldots, N_I$$

Loop closure constraint

Let $\gamma^j(\eta) : [0, 1]$

$\rightarrow \gamma^j(0) = \gamma^j(1)$





$$X \in \operatorname{span}(\hat{e}_1, \hat{e}_2)$$

$$x \in R^3$$

$R_1(\phi) \in R^{3\times3}$: transformation matrix associated with a rotation by ϕ about an axis of $\hat{e}_1$

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Likewise

$$R_3(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

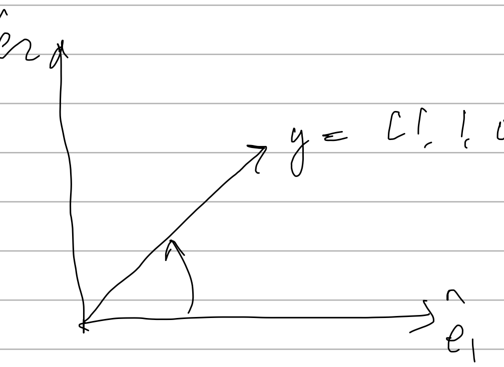
$$x = R_3(\varphi(y))\, R_1(\phi)\, R_3^{-1}(\varphi(y))\, X$$

$R_3^{-1}(\varphi(y))$: The axis of rotation aligned with y is first aligned to $\hat{e}_1$

$R_1(\phi)$: rotation of ϕ about e_1 is performed

$R_3(\varphi(y))$: The axis of rotation is aligned back to its original orientation

Example. Matrix associated with a rotation about an axis $y = [1, 1, 0]$



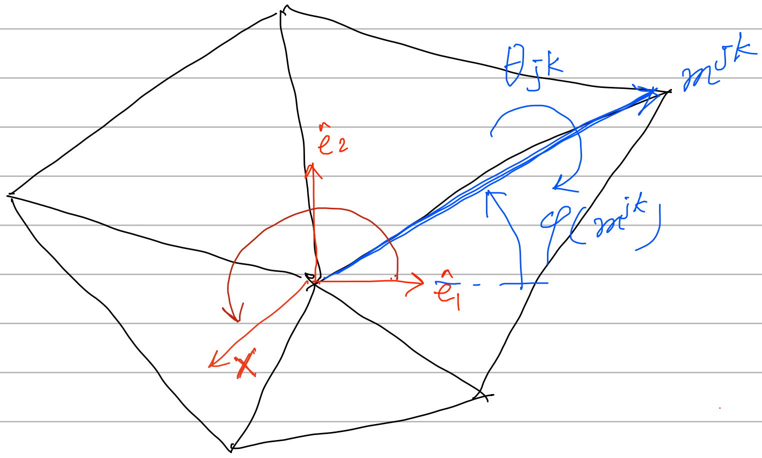
$$R_3(45^\circ) R_1(\phi) R_3^{-1}(45^\circ)$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 + 1/2\, c\phi & 1/2 - 1/2\, c\phi & \frac{\sqrt{2}}{2} s\phi \\ 1/2 - 1/2\, c\phi & 1/2 + 1/2\, c\phi & -\sqrt{2}/2\, s\phi \\ -\sqrt{2}/2\, s\phi & \sqrt{2}/2\, s\phi & c\phi \end{bmatrix}$$

e.g. $\phi = 90^\circ$ $\quad X = [1, 0, 0]$

$$x' = \begin{bmatrix} 1/2 \\ 1/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

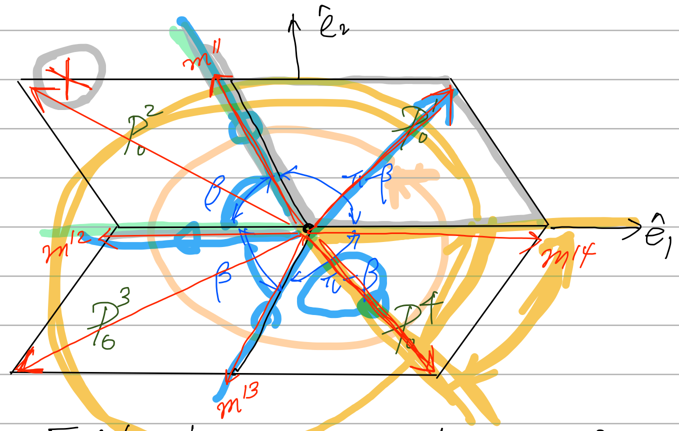


$$R_3(\varphi(m^{jk}))\ R_1(\theta_{jk})\ R_3^{-1}(\varphi(m^{jk}))$$

The mapping $X \to x$ along path $\gamma^j(\eta)$ that connects $\gamma^j(0)$ to the face connecting the point with initial position X

$$x = \left[\prod_{k=1}^{n_r} R_3(\varphi(m^{jk}))\ R_1(\theta_{jk})\ R_3^{-1}(\varphi(m^{jk}))\right] X$$

n_r: number of folds crossed



Fold between P_0^1 and P_0^2 (along m^{11})

$$R_3(\pi-\beta)\, R_1(\theta_{11})\, R_3^{-1}(\pi-\beta)$$

Fold between P_0^2 and P_0^3 (along m^{12})

$$R_3(\pi)\, R_1(\theta_{12})\, R_3^{-1}(\pi)$$

Fold between P_0^3 and P_0^4 (along m^{13})

$$R_3(\pi+\beta)\, R_1(\theta_{13})\, R_3^{-1}(\pi+\beta)$$

Fold between P_0^4 and P_0^1 (along m^{14})

$$R_3(0)\, R_1(\theta_{14})\, R_3^{-1}(0)$$

Mapping

Mapping for $X \in P_0^2$

$$x = R_3(\pi-\beta)\, R_1(\theta_{11})\, R_3^{-1}(\pi-\beta)$$

Mapping for $X \in P_0^3$

$$x = R_3(\pi-\beta) \cdot R_1(\theta_{11})\, \underbrace{R_3^{-1}(\pi-\beta) \cdot R_3(\pi)}_{-(\pi-\beta)+\pi=\beta}\, R_1(\theta_{12})\, R_3^{-1}(\pi)\, X$$

$$= R_3(\pi-\beta) \cdot R_1(\theta_{11})\, R_3(\beta)\, R_1(\theta_{12})\, R_3^{-1}(\pi)\, X$$

Mapping for $X \in P_0^4$

$$x = R_3(\pi-\beta)\, R_1(\theta_{11})\, R_3(\beta)\, R_1(\theta_{12})\, \underbrace{R_3^{-1}(\pi) \cdot R_3(\pi+\beta)}_{R_3(\beta)} \cdot R_1(\theta_{13})\, R_3^{-1}(\pi+\beta)\, X$$

$$= R_3(\pi-\beta)\, R_1(\theta_{11})\, R_3(\beta)\, R_1(\theta_{12})\, R_3(\beta)\, R_1(\theta_{13})\, R_3^{-1}(\pi+\beta)\, X$$

Mapping for $X \in P_0^1$

$$x = R_3(\pi-\beta)\, \overbrace{R_1(\theta_{11})\, R_3(\beta)\, R_1(\theta_{12})\, R_3(\beta)\, R_1(\theta_{13})\, R_3^{-1}(\pi+\beta)\, R_3(0)}^{I}\, R_1(\theta_{14})\, R_3^{-1}(0)\, X$$

$$R_3^{-1}(\pi+\beta) = R_3(\pi-\beta)$$

$$= R_3(\pi-\beta) \cdot R_1(\theta_{11}) R_3(\beta) \cdot R_1(\theta_{12}) R_3(\beta) \cdot R_1(\theta_{13}) R_3(\pi-\beta) \cdot R_3^{-1}(0)\, X$$

$$= I_3 X \qquad (x = X)$$

$$\prod_{k=1}^{n_j} R_3(\varphi(m^{jk}))\, R_1(\theta_{jk})\, R_3^{-1}(\varphi(m^{jk})) = I_3$$

$$= R_3(\varphi(m^{j1})) \left[\prod_{k=1}^{n_j-1} R_1(\theta_{jk})\, R_3(\alpha_{jk}) \right] R_1(\theta_{j n_j})\, R_3^{-1}(\varphi(m^{j n_j}))$$

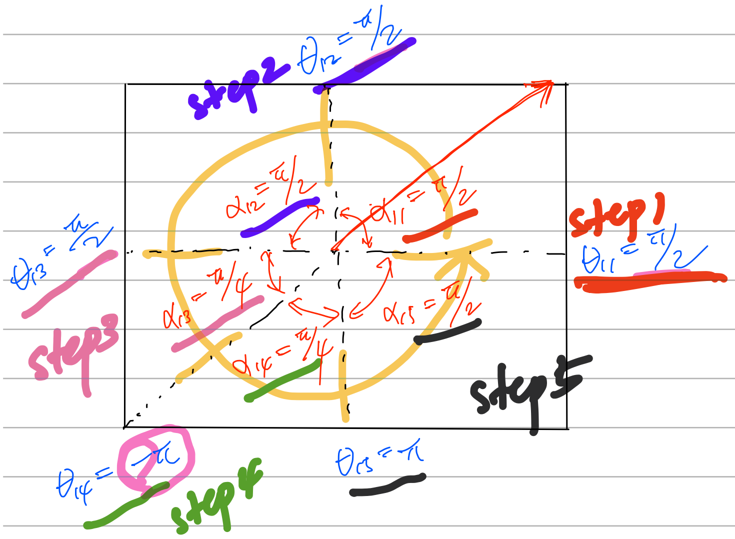
$$I_3 = R_3^{-1}(\varphi(m^{j1}))\, R_3(\varphi(m^{j1}))$$

$$= \left[\prod_{k=1}^{n_j-1} R_1(\theta_{jk})\, R_3(\alpha_{jk}) \right] \cdot R_1(\theta_{j n_j})\, \underbrace{R_3^{-1}(\varphi(m^{j n_j})) \cdot R_3(\varphi(m^{j1}))}_{R_3(\alpha_{j n_j})}$$

$$\boxed{I_3 = \prod_{k=1}^{n_j} R_1(\theta_{jk})\, R_3(\alpha_{jk})}$$

Example: Folding the corner of a cube

: verify the kinematic constraint is satisfied for the fold angles shown below.



$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

$$R_3 = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1(\theta_{11})\,R_3(\alpha_{11}) = R_1(\pi/2)\cdot R_3(\pi/2)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_1(\theta_{12})\cdot R_3(\alpha_{12}) = R_1(\pi/2)\cdot R_3(\pi/2)$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_1(\theta_{13})\,R_3(\alpha_{13}) = R_1(\pi/2)\cdot R_3(\pi/4)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ +\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & -1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix}$$

$$R_1(\theta_{14})\,R_3(\alpha_{14}) = R_1(-\pi)\;R_3(\pi/4)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1(\theta_{15})\;R_3(\alpha_{15}) = R_1(\pi)\,R_3(\pi/2)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\prod_{k=1}^{5} R_1(\theta_{jk})\,R_3(\alpha_{jk}) = I_3$$

$(j=1)$

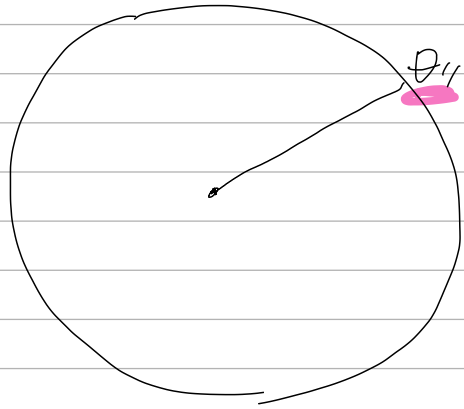
Example.

1 crease

$n_j = 1$

$$\boxed{I_3 = R_1(\theta_{11})}\;\underbrace{R_3(2\pi)}_{I}$$

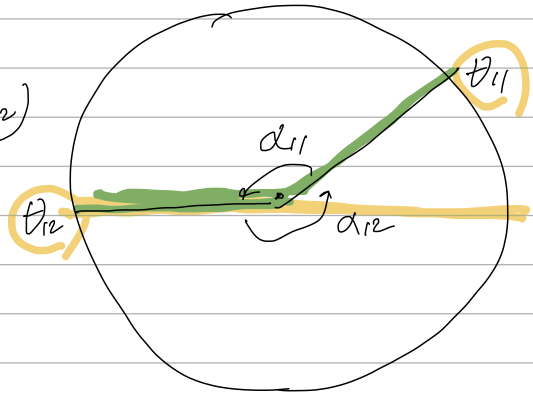
$\theta_{11} = 0$



2 creases

$$I_3 = R_1(\theta_{11}) R_3(\alpha_{11}) R_1(\theta_{12}) R_3(\alpha_{12})$$

$$\begin{cases} \theta_{11} = \theta_{12} = 0 & \text{for } \alpha_{11} \neq \pi \\ \theta_{11} = \theta_{12} & \text{for } \alpha_{11} = \pi \end{cases}$$



3 creases

$\theta_{11} = \theta_{12} = \theta_{13} = 0$

$;\ \alpha_{11} \neq \pi,\ \alpha_{12} \neq \pi,\ \alpha_{13} \neq \pi$

$\theta_{11} = \theta_{12},\quad \theta_{13} = 0$

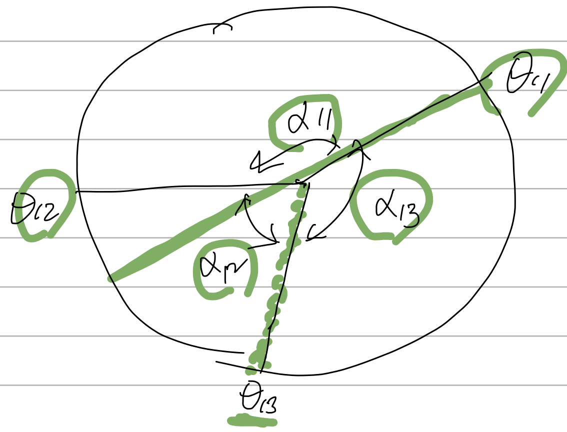
$:\ \alpha_{11} = \pi$

$\theta_{12} = \theta_{13},\quad \theta_{11} = 0$

$:\ \alpha_{12} = \pi$

$\theta_{13} = \theta_{11},\quad \theta_{12} = 0$

$:\ \alpha_{13} = \pi$



For non-trivial folding motion, any interior fold intersection must have at least four crease folds.